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Abstract

We examine the stability of equilibrium in sunspot-driven real business cycle (RBC) models under adaptive learning. We show that the general reduced form of this class of models can admit rational expectations equilibria that are both indeterminate and stable under adaptive learning. Indeterminacy of equilibrium allows for the possibility that non-fundamental "sunspot" variable realizations can serve as the main driving force of the model, and several researchers have put forward calibrated structural models where sunspot shocks play such a role. We show analytically how the structural restrictions that researchers have imposed on this type of model lead to reduced form systems where equilibrium is indeterminate but always unstable under adaptive learning. We thereby resolve a "stability puzzle" identified by Evans and McGough (2002).

1 Introduction

It is now well known that dynamic general equilibrium real business cycle models with production externalities and other types of nonconvexities may admit equilibria that are locally non-unique or indeterminate. Some researchers, following the lead of Farmer and Guo (1994) have exploited this possibility to derive models where realizations of non-fundamental "sunspot" variables play a prominent role in driving business cycle fluctuations.[1] One critique of this approach has been that the calibrations of the structural models necessary to obtain indeterminacy are empirically implausible.[2] However, a more recent generation of RBC models with a variety of different non-convexities has been successful at delivering indeterminate equilibria using empirically plausible calibrations of the structural model. Furthermore, sunspot-shock processes in these models can explain a variety of features of the macroeconomic data at business cycle frequencies that RBC models with determinate equilibria and technology shocks have a difficult time explaining. Consequently, many have come to view these models of sunspot-driven business cycles as quite promising.

A second critique of sunspot equilibria in RBC models with nonconvexities is that these equilibria are not stable under adaptive learning dynamics. In assessing whether an equilibrium is stable under learning, one typically posits that agents have the correct reduced form equations of the model but must learn the true (i.e. rational expectations equilibrium) parameterization of the model using some kind of adaptive inference technique such as recursive least squares. Thus, stability under learning, or "expectational stability" (E-stability) provides an important robustness check on the plausibility of rational expectations equilibria (REE). [3] For the Farmer-Guo model, Packalén (1999) and Evans and Honkapohja (2001) report that the sunspot equilibrium is *unstable* under adaptive learning when the model is calibrated according to Farmer and Guo's own specification. Evans and McGough (2002) look at the Farmer-Guo model as well as some more recent models due to Benhabib and Farmer (1996) and to Schmitt-Grohé and Uribe (1997) that yield indeterminate equilibria for more empirically plausible calibrations. They identify what they call a "stability puzzle". For a general reduced form system of equations that includes all three of

[1]See, e.g. Farmer (1999) or Benhabib and Farmer (1999) for discussions of this literature.

[2]See, e.g. Aiyagari (1995).

[3]See, e.g. Evans and Honkapohja (2001) for an introduction to the stability of REE under adaptive learning. The stability of sunspot equilibria under adaptive learning dynamics has been demonstrated e.g. by Woodford (1990), Duffy (1994) and Evans and Honkapohja (1994), in the context of simple, dynamic nonlinear models, e.g. overlapping generations models. Demonstrating the stability of sunspot equilibria in multivariate, RBC–type models has proved to be more elusive.

the models they examine as special cases, they can find parameter regions for which the rational expectations equilibrium is both indeterminate and stable under adaptive learning. However, when they restrict attention to versions of the reduced form model consistent with calibrations of the three structural models, they find that the sunspot equilibria are always *unstable* under learning. These puzzling findings are based on a numerical analysis of the equilibria of all three models using the same calibrations adopted by the researchers who developed those models.

In this paper we resolve this stability puzzle. In particular, we provide *analytic conditions* for indeterminacy and stability under learning for a general reduced form system of equations that is consistent with a number of one–sector RBC business cycle models. We then show precisely why the conditions for stability under learning will always be violated when the general reduced form system is restricted to be consistent with calibrations of the structural model that give rise to indeterminacy. Finally we show how three RBC models with nonconvexities that have appeared in the literature on sunspot-driven business cycles – models due to Farmer and Guo (1994), Schmitt-Grohé and Uribe (1997) and Wen (1998) – have reduced forms that map into our general reduced form system. We then show how the structural parameter restrictions imposed on these models prevent equilibria from being simultaneously indeterminate and stable under adaptive learning behavior.

2 General conditions for E-stability and Indeterminacy

We begin by presenting a general reduced form system of equations that characterizes equilibria in a variety of different one-sector RBC models. We derive our main findings using this general reduced form. In particular, we provide conditions under which the rational expectations equilibrium is 1) "learnable" or *expectationally stable* (E–stable) under adaptive learning behavior and 2) *indeterminate*, thereby allowing non–fundamental sunspot variable realizations to drive the business cycle, either together with fundamental technology shocks, or, as in the analysis here, in the absence of technology shocks.[4] We then combine the two sets of conditions to yield necessary conditions for both E–stability and indeterminacy of equilibrium.

[4] If the REE of an RBC model of the class we examine is *determinate* or unique, then expectational errors must be a unique function of fundamentals (e.g. technology shocks) alone; sunspots variable realizations cannot matter.

2.1 Reduced form model

The general reduced form of a sunspot-driven RBC model can be characterized by the following system of 2 equations:

$$k_{t+1} = d_k k_t + d_c c_t \tag{1}$$

$$c_t = b_k E_t k_{t+1} + b_c E_t c_{t+1} \tag{2}$$

where, for simplicity, we assume there are no fundamental shocks. The equilibrium is found by assuming that agents use these equations to form rational expectations of $E_t k_{t+1}$ and $E_t c_{t+1}$. Since we are interested in the stability of this equilibrium under adaptive learning, we instead assume that while agents possess knowledge of the functional form of these equations, they are initially uniformed as to the correct parameter values in these equations. Specifically, they have a "perceived law of motion" (PLM) of the form:

$$y_t = a_1 + a_y y_{t-1} + a_s s_t + \epsilon_t$$

where y_t' denotes the 2×1 vector of endogenous variables, (k_t, c_t), s_t represents a vector of non-fundamental expectation errors or sunspot variables and ϵ_t is a vector of random variables with 0 mean. This perceived law of motion corresponds to a particular, minimal state variable (MSV), AR(1) solution class. In particular, it is assumed that agents cannot observe current consumption and capital, though capital is predetermined, and so is known at time t. It is possible to relax this assumption for capital, as we show below, without changing any of our results.

As equation (1) is already in this AR(1) form, does not depend on any expectations and therefore on any expectation errors or sunspots, we can assume that agents know the coefficients of equation (1), d_k, d_c.[5] Alternatively, the coefficients of this equation could be learned as well, though this will not change any of our results. Hence the relevant perceived law of motion consists of the single equation for c_t which we write as

$$c_t = a_1 + a_k k_{t-1} + a_c c_{t-1} + a_f f_t + \varepsilon_t, \tag{3}$$

where f_t is a sunspot variable, and ε is a noise variable with 0 mean.

[5] Packalén (1999) and Evans and McGough (2002) make the same simplifying assumption.

2.2 REE Solution

Given the PLM, agents form expectations (in lieu of rational expectations) as follows:

$$\begin{aligned} E_t c_t &= c_t = a_1 + a_k k_{t-1} + a_c c_{t-1} + a_f f_t && (4) \\ E_t k_t &= d_k k_{t-1} + d_c c_{t-1} && (5) \\ E_t k_{t+1} &= d_k k_t + d_c E_t c_t && (6) \\ E_t c_{t+1} &= a_1 + a_k E_t k_t + a_c E_t c_t && (7) \end{aligned}$$

Substituting (4) - (5) into (6) and (7) and collecting terms, we get

$$\begin{aligned} E_t c_{t+1} &= a_1(1+a_c) + a_k(d_k + a_c)k_{t-1} + (a_c^2 + a_k d_c)c_{t-1} + a_c a_f f_t && (8) \\ E_t k_{t+1} &= a_1 d_c + (d_k^2 + d_c a_k)k_{t-1} + d_c(d_k + a_c)c_{t-1} + d_c a_f f_t && (9) \end{aligned}$$

Next, substitute (8) and (9) into (2), to get a mapping, T, between the perceived law of motion and the actual law of motion.

$$\begin{aligned} T(a_1) &= a_1[d_c b_k + b_c(1+a_c)] \\ T(a_k) &= b_k(d_k^2 + d_c a_k) + b_c(a_k d_k + a_c a_k) \\ T(a_c) &= b_k d_c(d_k + a_c) + b_c(a_c^2 + a_k d_c) \\ T(a_f) &= a_f(b_k d_c + b_c a_c) \end{aligned}$$

The actual law of motion for consumption is

$$c_t = T(a_1) + T(a_k)k_t + T(a_c)c_{t-1} + T(a_f)f_t + \varepsilon_t, \quad (10)$$

This actual law of motion (10) together with (1) comprise the data generating process for the economy under adaptive learning.

The rational expectations solution is just a fixed point of this T-mapping and is found by application of the method of undetermined coefficients, i.e. by setting the coefficients of (2) equal to their T-map coefficients in (10). This yields the RE solution:

$$a_c = \frac{1 - b_k d_c}{b_c}, a_k = -\frac{b_k d_k}{b_c}, \ a_1 = 0, \text{with } a_f \text{ indeterminate.} \quad (11)$$

2.3 Conditions for E-stability

We now examine the stability of the RE solution under adaptive learning. The specific question we seek to address is whether the RE solution is stable under some unspecified adaptive adjustment process by which the parameters of the perceived law of motion are slowly adjusted toward the actual law of motion parameters. Specifically, let a be the vector of parameters in the perceived law of motion and $T(a)$ the vector of parameters in the actual law of motion. The rational expectations solution is said to be expectationally stable, or E-stable if it is locally asymptotically stable under the equation

$$\frac{da}{d\tau} = T(a) - a.$$

That is, if this differential equation evaluated at the RE solution values for a is locally stable. The time variable τ in this equation refers to notional time.[6] Intuitively, we are checking whether the adjustment of the PLM parameters toward the ALM parameters is leading agents toward the RE solution, and not away form it, within a small neighborhood of the RE solution.

The RE solution is said to be E-stable if all eigenvalues of $\frac{d(T(a)-a)}{da}$, when evaluated at the RE solution, have negative real parts. It is easy to derive the expression for $\frac{d(T(a)-a)}{da}$ (evaluated at the REE values) as

$$\begin{bmatrix} b_c & 0 & 0 & 0 \\ 0 & b_c d_k & -b_k d_k & 0 \\ 0 & b_c d_c & 1 - b_k d_c & 0 \\ 0 & 0 & b_c \overline{a_f} & 0 \end{bmatrix}, \tag{12}$$

where $\overline{a_f}$ is the REE value of a_f. The eigenvalues of this matrix are determined by the equation

$$-\lambda(b_c - \lambda)[\lambda^2 + (b_k d_c - b_c d_k - 1)\lambda + b_c d_k] = 0$$

Obviously,

$$\lambda_1 = 0 \tag{13}$$

$$\lambda_2 = b_c \tag{14}$$

The other two eigenvalues are determined by the quadratic formula

$$\frac{1 - b_k d_c + b_c d_k \pm \sqrt{(1 - b_k d_c + b_c d_k)^2 - 4 b_c d_k}}{2}$$

[6]It turns out that there is a deep connection between the stability of the RE solution under this differential equation, and the stability of the RE solution under a real-time adaptive learning algorithm such as recursive least squares learning. See Evans and Honkapohja (2001) for details.

The necessary conditions for both of these roots to be negative are

$$\lambda_3 \lambda_4 = b_c d_k > 0 \tag{15}$$

$$\lambda_3 + \lambda_4 = 1 - b_k d_c + b_c d_k < 0 \tag{16}$$

The presence of a zero eigenvalue can be problematic in assessing the stability of a system under adaptive learning. The zero eigenvalue is clearly due to the presence of the sunspot variable f_t in the perceived law of motion. As it turns out, the differential equation for a_f is given by

$$\frac{da_f}{d\tau} = a_f(b_k d_c + b_c a_c - 1),$$

which is a separable equation that can be directly integrated as:

$$a_f(\tau) = a_f(0) \exp\left\{\int_0^\tau (b_k d_c + b_c a_c(u) - 1) du\right\}.$$

So long as $a_c \to \frac{1-b_k d_c}{b_c}$ exponentially as $\tau \to +\infty$, a_f will also converge to a finite value, so the zero eigenvalue need not hinder our analysis of the stability of the system under adaptive learning.

Proposition 1 *The necessary conditions for the system (1) and (2) to be E-stable are (15), (16) and*

$$b_c < 0 \tag{17}$$

As noted above, the perceived law of motion (3) assumes that k_t is not known at time t, when in fact it is predetermined by decisions made in period $t-1$. Thus it is might be more reasonable to assume that agents use the alternative perceived law of motion,

$$c_t = a_1 + a_k k_t + a_c c_{t-1} + a_f f_t + \varepsilon_t \tag{18}$$

in place of (3). Following the same steps as outlined above, one can show that if agents use (18) as their perceived law of motion, the actual law of motion will be given by:

$$c_t = T(a_1) + T(a_k) k_t + T(a_c) c_{t-1} + T(a_f) f_t + \varepsilon_t,$$

where

$$\begin{aligned} T(a_1) &= a_1[d_c b_k + b_c(1 + a_k d_c + a_c)] \\ T(a_k) &= b_k d_k + b_k d_c a_k + b_c a_k(d_k + a_k d_c + a_c) \\ T(a_c) &= b_k d_c a_c + b_c a_c(a_c + a_k d_c) \\ T(a_f) &= a_f[b_k d_c + b_c(a_k d_c + a_c)] \end{aligned}$$

The matrix $\frac{d(T(a)-a)}{da}$ (evaluated at the REE values) then becomes

$$\begin{bmatrix} b_c & 0 & 0 & 0 \\ 0 & b_c d_k - b_k d_c & -b_k & 0 \\ 0 & d_c & 1 & 0 \\ 0 & -b_c d_c \overline{a_f} & b_c \overline{a_f} & 0 \end{bmatrix}, \tag{19}$$

where $\overline{a_f}$ is again the REE value of a_f (c.f. (12)). The eigenvalues of (19) are determined by the equation

$$\lambda(b_c - \lambda)[\lambda^2 + (b_k d_c - b_c d_k - 1)\lambda + b_c d_k] = 0,$$

which is precisely the same characteristic equation we obtained for the matrix (12). It follows that Proposition 1 also holds in the case where agents use the alternative perceived law of motion (3).

2.4 Conditions for indeterminacy

Indeterminacy refers to local nonuniqueness of the solution paths leading to a RE solution. To assess whether the RE solution is indeterminate, let us rewrite the general reduced form system (1-2) as

$$\begin{bmatrix} 1 & 0 \\ b_k & b_c \end{bmatrix} \begin{bmatrix} k_{t+1} \\ c_{t+1} \end{bmatrix} = \begin{bmatrix} d_k & d_c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_t \\ c_t \end{bmatrix}$$

or equivalently,

$$\begin{bmatrix} k_{t+1} \\ c_{t+1} \end{bmatrix} = J \begin{bmatrix} k_t \\ c_t \end{bmatrix}, \tag{20}$$

$$J = \begin{bmatrix} d_k & d_c \\ -\frac{b_k d_k}{b_c} & \frac{1-b_k d_c}{b_c} \end{bmatrix} \tag{21}$$

where the determinant and trace of the Jacobian can be obtained as

$$\det(J) = \frac{d_k}{b_c}$$

$$tr(J) = \frac{1 - b_k d_c + d_k b_c}{b_c}$$

Indeterminacy in this model requires that both eigenvalues of J lie inside the unit circle. Since the trace of the Jacobian measures the sum of the roots and the determinant measures the product, the necessary conditions for indeterminacy are

$$-1 < \det(J) < 1 \tag{22}$$

$$-1 - \det(J) < tr(J) < 1 + \det(J) \tag{23}$$

Proposition 2 *The necessary conditions for the system (1) and (2) to have stationary sunspot equilibria are*

$$-1 \quad < \quad \det(J) = \frac{d_k}{b_c} < 1 \tag{24}$$

$$-1 - \det(J) \quad < \quad tr(J) = \frac{1 - b_k d_c + d_k b_c}{b_c} < 1 + \det(J) \tag{25}$$

2.5 Necessary conditions for both E-stability and indeterminacy

If the reduced form model has indeterminate equilibria that are also E-stable, conditions (15) - (17), (24) and (25) must be satisfied simultaneously. Consider the condition (15). It implies that $\det(J) = d_k/b_c > 0$. With (24) this requires that the determinant of the Jacobian must satisfy

$$0 < \det(J) = \frac{d_k}{b_c} < 1 \tag{26}$$

Furthermore, (17) and (26) imply that

$$d_k < 0.$$

Similarly, the combination of (16), (17) and (25) requires that

$$0 < tr(J) = \frac{1 - b_k d_c + d_k b_c}{b_c} < 1 + \det(J) \tag{27}$$

Proposition 3 *The E-stability requirement imposes further restrictions on the parameters of the sunspot model (RBC model with indeterminate equilibria). It requires that conditions (17), (26), and (27) hold simultaneously. In particular, (17) states that*

$$b_c < 0$$

Note that a positive determinant implies that both roots of J have the same sign, and a positive trace implies that the sign of the roots is positive. Hence we have the following corollary.

Corollary 1 *The necessary conditions for the stationary sunspot equilibria in (1) and (2) to be E-stable are that both roots of the Jacobian matrix (21) have positive real parts and* $b_c < 0$.

2.6 Further modifying the PLM

A closer look at the matrix (12) (or 19) reveals that the eigenvalue b_c comes solely from the T-map of the parameter a_1, the constant term in the agent's perceived law of motion. One could argue that the variables in the reduced form model are all deviations from the steady states, therefore it is not necessary to incorporate a constant term in the PLM. Indeed in all RBC models the reduced form equations do not involve constant terms.

On the other hand, the presence of the constant term can be regarded as a slight model misspecification; agents could, after all, learn that coefficient on this constant term is zero in the rational expectations equilibrium.

Suppose however, that we eliminate the constant term. The PLM (3) now becomes

$$c_t = a_k k_{t-1} + a_c c_{t-1} + a_f f_t + \varepsilon_t,$$

With this modification, the necessary conditions for indeterminacy and E-stability need to be rederived. With $b_c < 0$, the necessary conditions will be the same, so we focus on the case where $b_c > 0$.

Note that $b_c > 0$ and (16) immediately imply that

$$tr(J) = \frac{1 - b_k d_c + d_k b_c}{b_c} < 0 \tag{28}$$

(15) again implies that

$$\det(J) = \frac{d_k}{b_c} > 0 \tag{29}$$

A negative trace and a positive determinant imply that both eigenvalues of J have negative real parts. This is problematic because all sunspot models rely on eigenvalues that have *positive real parts* to generate dampened cycles. If J has negative eigenvalues, the system oscillates around the steady state at each period, which is empirically implausible.

Proposition 4 *When the PLM has no constant term, the REE of calibrated sunspot models are not E-stable because they would necessarily violate (28) and (29).*

In the next three sections, we examine three real business cycle models that have appeared in the literature and show how they map into the reduced form system (1-2) that we have examined in this section. We then demonstrate that under the parameter restrictions placed on the structural models, the REE cannot be simultaneously indeterminate and stable under adaptive learning.

3 The Wen (1998) model

The economy in Wen's (1998) model consists of a large number of identical consumer-producer households who solve:

$$\max_{\{c_t, n_t, k_t, u_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \left(\log c_t - \frac{n_t^{1+\gamma}}{1+\gamma} \right) \tag{30}$$

subject to:

$$c_t + x_t = \overline{e}_t \left(u_t k_t^{\alpha} \right) n_t^{1-\alpha} \tag{31}$$

$$k_{t+1} = x_t + (1 - \delta_t) k_t \tag{32}$$

$$\overline{e}_t = (\overline{u}_t \overline{k}_t)^{\alpha\eta} \overline{n}_t^{(1-\alpha)\eta} \tag{33}$$

$$\delta_t = \frac{1}{\theta} u_t^{\theta} \tag{34}$$

for a given initial stock of capital, $k_0 > 0$. We adopt Wen's (1998) notation. The choice variables are consumption, c_t, the number of hours worked, n_t, the capital stock, k_t, and the rate of capacity utilization, $u_t \in (0, 1)$. The parameters restrictions are: $0 < \alpha < 1$, $0 < \beta < 1$, $\gamma \geq 0$, $\eta > 0$, and $\theta > 1$. The production externality, $\overline{e}_t$, is a function of the mean productive capacity, $\overline{u}_t \overline{k}_t$, and mean labor hours, $\overline{n}_t$. The rate of depreciation of the capital stock, $\delta_t \in (0, 1)$ is an increasing function of the capacity utilization rate, u_t. The restriction that $\theta > 1$ ensures that the optimal capacity utilization rate u_t lies in $(0, 1)$. The restriction that $\eta > 1$ ensures increasing returns to scale in production, which in important both for generating indeterminacy and for allowing capacity utilization to affect the extent of aggregate returns to scale. Indeed, Wen (1998) showed that the addition of variable capital utilization could significantly reduce the degree of increasing returns to scale needed to deliver indeterminate equilibria (e.g. by comparison with Farmer and Guo (1994)), from empirically implausible to empirically plausible levels. This feature of the Wen model has made it an attractive choice for other researchers interested in empirical applications of sunspot-driven RBC models e.g. Harrison and Weder (2002) use the Wen model with sunspot shocks to explain a number of features of the data found over the Great Depression era. Benhabib and Wen (2002) use the Wen model to show how shocks to aggregate demand can explain a number of business cycle anomalies that have eluded standard RBC models (without indeterminate equilibria).

Following Wen (1998), we can first solve for the optimal capacity utilization rate u_t, and using this expression, derive a reduced–form aggregate production function of the form:

$$y_t = k_t^{a^*} n_t^{b^*},$$

where $a^* = \alpha(1+\eta)\tau_k$, $b^* = (1-\alpha)(1+\eta)\tau_n$, and $\tau_k = \frac{\theta-1}{\theta-\alpha(1+\eta)}$, $\tau_n = \frac{\theta}{\theta-\alpha(1+\eta)}$.

3.1 The reduced form

The Wen (1998) model can be written as a reduced form system of 2 equations:

$$\begin{bmatrix} k_{t+1} \\ n_{t+1} \end{bmatrix} = J \begin{bmatrix} k_t \\ n_t \end{bmatrix}, \tag{35}$$

where

$$J = \begin{bmatrix} 1 & (1+\gamma)c/k \\ \frac{(1-\beta)(1-a^*)}{1+\gamma-\beta b^*} & \frac{1+\gamma-b^*+[1+\beta(a^*-1)](1+\gamma)c/k}{1+\gamma-\beta b^*} \end{bmatrix},$$

3.2 Requirements for indeterminacy

For this model to have multiple stationary sunspot equilibria, the conditions (22) and (23) must be satisfied. After lengthy algebra, one can show that

$$\det(J) = \frac{1+\gamma-b^*+a^*(1+\gamma)c/k}{1+\gamma-\beta b^*} \tag{36}$$

$$= \frac{1}{\beta}(1+\frac{\eta(1+\gamma)(1-\beta)\tau_n}{1+\gamma-\beta b^*}) \tag{37}$$

$$tr(J) = 1+\det(J)+\frac{(1+\gamma)(1-\beta)(1-a^*)c/k}{1+\gamma-\beta b^*} \tag{38}$$

The crucial insight of (37) is that when there is no externality ($\eta = 0$), $\det(J) = 1/\beta > 1$, which violates the condition (22). For there to be indeterminacy, therefore, the second term of (37) must become negative as externality becomes positive. Since $\eta(1+\gamma)(1-\beta)\tau_n > 0$, this requires that the denominator be negative:

$$1+\gamma-\beta b^* < 0 \tag{39}$$

Proposition 5 *A necessary condition for the system (35) to possess multiple sunspot equilibria is (39)*

3.3 E-stability

To check the conditions for E-stability, we need to convert the system (35) into the form of (1) and (2). After this is done, the mapping from the parameters of the model to those of (1) and (2) are:

$$b_c = \frac{\beta b^*-(1+\gamma)}{b^*-(1+\gamma)} \tag{40}$$

$$
\begin{aligned}
b_k &= \frac{(\beta-1)[b^* - (1+\gamma)(1-a^*)]}{b^* - (1+\gamma)} && (41)\\
d_k &= 1 - \frac{a^*(1+\gamma)c/k}{b^* - (1+\gamma)} && (42)\\
d_c &= \frac{(1+\gamma)c/k}{b^* - (1+\gamma)} && (43)
\end{aligned}
$$

We can now use the conditions derived in the previous section to examine if the REE of this model is E-stable. Start with (40). In proposition 5 we have shown that $\beta b^* - (1+\gamma) > 0$. Since $0 < \beta < 1$, this implies that $b^* - (1+\gamma) > 0$. It is immediately evident that

$$
b_c = \frac{\beta b^* - (1+\gamma)}{b^* - (1+\gamma)} > 0,
$$

which exactly violates the required condition for E-stability (17).

Proposition 6 *The REE of the Wen (1998) model is not E-stable under adaptive learning since it violates condition (17).*

4 The Farmer and Guo (1994) model

In Farmer and Guo's (1994) model a large number of identical consumer-producer households solve:

$$
\max_{C_t, L_t} E_0 \sum_{t=0}^{\infty} \rho^t \left(\log C_t - A \frac{L_t^{1-\gamma}}{1-\gamma} \right)
$$

subject to:

$$
\begin{aligned}
K_{t+1} &\leq Y_t + (1-\delta)K_t - C_t\\
Y_t &= Z_t K_t^{\alpha} L_t^{\beta}\\
Z_t &= Z_{t-1}^{\theta} \eta_t
\end{aligned}
$$

Here we are using the same notation as in Farmer and Guo (1994): C_t denotes consumption, L_t denotes labor supply, K_t is the capital stock, Y_t is output, Z_t is a productivity shock and η_t is an i.i.d random variable with unit mean. The parameters satisfy $\gamma > 0$, $0 < \rho < 1$, $0 < \delta < 1$, $0 < \theta < 1$ and, most importantly, $\alpha + \beta > 1$, so that the technology exhibits increasing returns.[7] However, from the perspective of individual producers, the production technology is Cobb-Douglas

[7] An alternative interpretation of the latter restriction involving monopolistically competitive firms is also possible – see Farmer and Guo (1994) for the details.

with constant returns, where a and b represent capital and labor's shares of output, respectively, and $a + b = 1$. Farmer and Guo assume that

$$\alpha = a/\lambda, \quad \beta = b/\lambda \tag{44}$$

with $0 < \lambda < 1$ to insure increasing returns to scale.

4.1 The reduced form

Omitting fundamental shocks, Z_t, the model can be reduced to:

$$\begin{aligned} Y_t &= K_t^{\alpha} L_t^{\beta} && (45) \\ AC_t/L_t^{\gamma} &= bY_t/L_t && (46) \\ K_{t+1} &= Y_t + (1-\delta)K_t - C_t && (47) \\ \frac{1}{C_t} &= \rho E_t[\frac{1}{C_{t+1}}(a\frac{Y_{t+1}}{K_{t+1}} + 1 - \delta)], && (48) \end{aligned}$$

where equations (46) and (48) are the first order conditions from the representative agent's problem.

The two dynamic equations can be linearized as

$$\begin{aligned} c_t &= E_t c_{t+1} + \rho\frac{y}{k}(E_t k_{t+1} - E_t y_{t+1}) && (49) \\ k_{t+1} &= \frac{y}{k} y_t + (1-\delta)k_t - \frac{c}{k} c_t && (50) \end{aligned}$$

Combining the linearized versions of (45) and (46)

$$\begin{aligned} y_t &= \alpha k_t + \beta l_t \\ c_t + (1-\gamma) l_t &= y_t \end{aligned}$$

we can get

$$y_t = \frac{\beta}{\beta - 1 + \gamma} c_t - \frac{\alpha(1-\gamma)}{\beta - 1 + \gamma} k_t$$

Plugging this equation into the two (linearized) dynamic equations, we get

$$\begin{aligned} c_t &= (1 + \rho a \frac{y}{k} \frac{\beta}{1-\beta-\gamma}) E_t c_{t+1} + \rho a \frac{y}{k}[1 - \frac{\alpha(1-\gamma)}{1-\beta-\gamma}] E_t k_{t+1} \\ k_{t+1} &= [\frac{y}{k}\frac{\alpha(1-\gamma)}{1-\gamma-\beta} + 1 - \delta] k_t - (\frac{c}{k} + \frac{y}{k}\frac{\beta}{1-\gamma-\beta}) c_t \end{aligned}$$

Mapped into our general representation in equations section 2.1, the first critical parameter is

$$\begin{aligned} b_c &= 1 + \rho a \frac{y}{k} \frac{\beta}{1-\beta-\gamma} \\ &= \frac{1-\gamma-\beta\rho(1-\delta)}{1-\beta-\gamma}, \end{aligned}$$

where the second equality comes from the steady state version of (48): $\rho a \frac{y}{k} = 1 - \rho(1-\delta)$. The second critical parameter is

$$\begin{aligned} d_k &= \frac{y}{k} \frac{\alpha(1-\gamma)}{1-\gamma-\beta} + 1 - \delta \\ &= \frac{\frac{1-\rho(1-\delta)}{\rho a}\alpha(1-\gamma) + (1-\delta)(1-\gamma-\beta)}{1-\gamma-\beta} \end{aligned}$$

4.2 Requirements for indeterminacy

We only need a subset of the necessary conditions for indeterminacy to make our point. As we proved in section 2.4, one necessary condition for indeterminacy is

$$-1 < \frac{d_k}{b_c} < 1$$

In this model we have

$$\frac{d_k}{b_c} = \frac{\frac{1-\rho(1-\delta)}{\rho a}\alpha(1-\gamma) + (1-\delta)(1-\gamma-\beta)}{1-\gamma-\beta\rho(1-\delta)}$$

Plugging equation (44) into the expression and simplify terms, we get

$$\begin{aligned} \frac{d_k}{b_c} &= \frac{\frac{1}{\rho}\{\frac{1}{\lambda}(1-\gamma)[1-\rho(1-\delta)] + \rho(1-\delta)(1-\gamma-\frac{b}{\lambda})\}}{1-\gamma-\frac{b}{\lambda}(1-\delta)\rho} \\ &= \frac{1}{\rho}\{1 + \frac{(1-\frac{1}{\lambda})(1-\gamma)[\rho(1-\delta)-1]}{1-\gamma-\frac{b}{\lambda}(1-\delta)\rho}\} \end{aligned}$$

Suppose the economy has constant returns to scale ($\lambda = 1$), then $\frac{d_k}{b_c} = 1/\rho > 1$, since the discount factor ρ must be less than 1. In this case the condition for indeterminacy is violated. To have indeterminacy, we need increasing returns ($\lambda < 1$), and the second term in the bracket must be negative. It is easy to see that the numerator is positive, given $1 - 1/\lambda < 0, \gamma < 0$, and $\rho(1-\delta) - 1 = -\rho a y/k < 0$. Therefore the denominator must be negative, that is,

$$\beta\rho(1-\delta) > 1 - \gamma \tag{51}$$

Since $0 < \rho < 0$ and $0 < 1 - \delta < 1$, this also implies

$$\beta > 1 - \gamma \tag{52}$$

4.3 E-instability

Now we show that if the Farmer and Guo (1994) model satisfies the above necessary condition for indeterminacy, then it must be E-unstable under adaptive learning. We only have to check the necessary condition

$$b_c < 0$$

In this model

$$b_c = \frac{\beta\rho(1-\delta) - (1-\gamma)}{\beta - (1-\gamma)}$$

If condition (52) holds, the numerator must be positive. Combining condition (51), we have

$$\begin{aligned} b_c &= \frac{\beta\rho(1-\delta) - (1-\gamma)}{\beta - (1-\gamma)} \\ &> \frac{1-\gamma-(1-\gamma)}{\beta - (1-\gamma)} = 0 \end{aligned}$$

It follows that the equilibrium of this model is E-unstable under adaptive learning.

5 The Schmitt-Grohé and Uribe (1997) model

We focus on the simpler version of Schmitt-Grohé and Uribe's (1997) model where there is no capital income tax. We consider a discrete-time version of the model with labor income taxes only and adopt Schmitt-Grohé and Uribe's (1997) notation. A large number of identical consumer-producer households solve:

$$\max_{C_t, H_t} E_0 \sum_{t=0}^{\infty} \beta^t \left(\log C_t - AH_t\right)$$

subject to:

$$\begin{aligned} K_{t+1} &\leq Y_t + (1-\delta)K_t - C_t - G \\ Y_t &= K_t^a L_t^b \\ G &= \tau_t b Y_t \end{aligned}$$

Here, C_t denotes consumption, H_t is hours worked, K_t is the capital stock, and Y_t is output produced according to a Cobb-Douglas technology with constant returns, $a + b = 1$. Government revenue, G, is obtained through taxes on labor income at rate $\tau_t \in (0, 1)$. The discount factor satisfies $0 < \beta < 1$, as does the rate of depreciation of the capital stock, $0 < \delta < 1$, and we assume $A > 0$.

5.1 The reduced form

This model can be reduced to

$$
\begin{aligned}
Y_t &= K_t^a H_t^b & (53)\\
AC_t &= b(1-\tau_t)Y_t/H_t & (54)\\
K_{t+1} &= Y_t + (1-\delta)K_t - C_t - G & (55)\\
\frac{1}{C_t} &= \beta E_t[\frac{1}{C_{t+1}}(a\frac{Y_{t+1}}{K_{t+1}} + 1 - \delta)] & (56)\\
G &= \tau_t b Y_t, & (57)
\end{aligned}
$$

Equations (54) and (56) are the first order conditions from the representative agent's problem. Letting lower case letters denote deviations from steady state values. We can eliminate y_t and τ_t by using the linearized version of (53), (54) and (57):

$$
\begin{aligned}
y_t &= ak_t + bh_t\\
c_t &= y_t - h_t - \frac{\tau}{1-\tau}\tau_t\\
0 &= \tau_t + y_t,
\end{aligned}
$$

which implies

$$
y_t = \frac{(1-\tau)b}{b-1+\tau}c_t - \frac{(1-\tau)a}{b-1+\tau}k_t
$$

. Substituting this equation into the two dynamic equations, we get

$$
\begin{aligned}
c_t &= [1 - \beta a\frac{y}{k}\frac{(1-\tau)b}{b-1+\tau}]E_t c_{t+1} + a\beta\frac{y}{k}[1 + \frac{(1-\tau)a}{b-1+\tau}]E_t k_{t+1}\\
k_{t+1} &= (\frac{y}{k}\frac{b(1-\tau)}{b-1+\tau} - \frac{c}{k})c_t + [\frac{y}{k}\frac{(1-\tau)\alpha}{1-b-\tau} + 1 - \delta]k_t
\end{aligned}
$$

Again, the two critical parameters are

$$
\begin{aligned}
b_c &= 1 - \beta a\frac{y}{k}\frac{(1-\tau)b}{b-1+\tau}\\
&= \frac{b-1+\tau-[1-\beta(1-\delta)]b(1-\tau)}{b-1+\tau}\\
d_k &= \frac{y}{k}\frac{(1-\tau)\alpha}{1-b-\tau} + 1 - \delta\\
&= \frac{[1/\beta-(1-\delta)](1-\tau)+(1-\delta)(1-b-\tau)}{1-b-\tau}
\end{aligned}
$$

5.2 Requirements for indeterminacy

We again check the condition

$$-1 < \frac{d_k}{b_c} < 1$$

In this model

$$\begin{aligned} \frac{d_k}{b_c} &= \frac{1/\beta\{[1-\beta(1-\delta)](\tau-1)+\beta(1-\delta)(\tau+b-1)\}}{b-1+\tau-[1-\beta(1-\delta)]b(1-\tau)} \\ &= \frac{1}{\beta}\{1+\frac{-\tau b[1-\beta(1-\delta)]}{b-1+\tau-[1-\beta(1-\delta)]b(1-\tau)}\} \end{aligned}$$

When there is no labor income tax ($\tau = 0$), this expression is equal to $1/\beta > 1$, the equilibrium is always determinate. To have indeterminacy, we need $\tau > 0$, and the second expression in the bracket must be negative. It is obvious that the numerator is negative, so the denominator must be positive to have indeterminacy. That is,

$$b - 1 + \tau > b(1-\tau)[1-\beta(1-\delta)] \tag{58}$$

Since the right-hand side of the equation is positive, we must also have

$$b - 1 + \tau > 0 \tag{59}$$

5.3 E-instability

Next we show that when indeterminacy holds, the condition for E-stability $b_c < 0$ will be violated. Combining the expression for b_c with (58) and (59), we have

$$\begin{aligned} b_c &= 1-\frac{b(1-\tau)[1-\beta(1-\delta)]}{b-1+\tau} \\ &> 1-\frac{b-1+\tau}{b-1+\tau} = 0 \end{aligned}$$

This equilibrium of this model is therefore E-unstable under adaptive learning.

6 Conclusions

In this paper we have examined the conditions for indeterminacy and stability under adaptive learning for a general reduced form model that characterizes a number of one-sector real business cycle models. We have found simple, analytic conditions under which the equilibrium of this system is both indeterminate and stable under adaptive learning behavior. To our knowledge, such conditions have not previously appeared in the literature. These conditions imply that, in principle, it is possible for agents to learn the REE of sunspot-driven RBC models, giving such models added plausibility as explanations for business cycle fluctuations. Furthermore, these conditions help us to understand why Evans and McGough (2002) can (numerically) find large regions of the parameter space of a general reduced form model (such as the one we consider) where sunspot solutions are stable under learning, but cannot find any regions where sunspot solutions of *structural* RBC models are stable under adaptive learning. Our findings serve to resolve this "stability puzzle." In particular, we show how the parameter restrictions implied by structural models rule out the possibility that REE can be simultaneously indeterminate and learnable. We do this in the context of three RBC models with nonconvexities that allow for the possibility of indeterminate equilibria.

While we have applied our conditions for indeterminacy and stability under learning to just three RBC models, we believe that our instability conclusion is even more general. Consider the perceived law of motion (3). Imagine the case where $b_c < 0$, the necessary condition for E-stability. If the actual law of motion turns out to be the same as the perceived law of motion, as when agents have learned the REE, then we would have:

$$c_t = \overline{a}_1 - \frac{b_k d_k}{b_c} k_{t-1} + \frac{1 - b_k d_k}{b_c} c_{t-1} + \overline{a}_f f_t + \varepsilon_t,$$

where we have substituted in the REE values for the parameters of the PLM. A negative value for b_c implies that the autoregressive coefficient on consumption is *negative*. [8] That is, consumption will oscillate around its steady state period by period. This is obviously empirically implausible. Any structural model calibrated to match the data must have $b_c > 0$. We therefore conclude with

Conjecture 1 *If a structural model is calibrated to match empirical regularities in the data, then its REE are E-unstable under adaptive learning, as condition (17) will necessarily be violated.*

Our findings cast doubt on the plausibility of equilibria in calibrated RBC models where business cycles are driven in whole or in part by non-fundamental sunspot variables, as such equilibria are found to be unstable under adaptive learning even when agents possess knowledge of the correct reduced form. On the other hand, our results only apply to one-sector models; recently researchers have shown that indeterminacy of equilibria is more readily obtained in multi-sector models. Hence, it need not be the case that our findings imply that all sunspot-driven RBC models

[8] The same result holds for the alternative PLM (18).

are unstable under adaptive learning behavior. We leave an analysis of the stability under learning of indeterminate equilibria in multi-sector models to future research.

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