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Analogy Between Linear Optical Systems and Linear Two-Port Electrical Networks

R. M. A. Azzam and N. M. Bashara

Attention is called to the analogy between linear optical systems and linear two-port electrical networks. For both, the transformation of a pair of oscillating quantities between input and output is of interest. The mapping of polarization by an optical system and of impedance (admittance) by a two-port network is described by a bilinear transformation. Therefore for each transfer property of a system of one type, there is a similar property for the system of the other type. Two-port electrical networks are synthesized whose impedance-(or admittance-) mapping properties are the same as the polarization-mapping properties of a given optical system. The opposite problem of finding the optical analogs of two-port networks is also considered. Besides unifying the methods of handling these two different kinds of systems, the analogy appears fruitful if used reciprocally to simulate electrical networks by optical systems, and vice versa. Linear mechoacoustic systems have optical analogs besides their well-known electrical analogs.

I. Introduction

The advent of microwaves and more recently of lasers has stimulated the interest of electrical engineers in optics and optical systems. Establishing analogies between electrical networks and optical systems would be useful in (1) unifying the methods of treating both kinds of systems and (2) the reciprocal simulation of systems of one type by systems of the other.

II. Basis of the Analogy

The basis of the analogy between optical systems and two-port electrical networks is explained with reference to Fig. 1. For the optical system $S$, shown in Fig. 1(a), nonlinear optical effects and other frequency-changing phenomena (such as Brillouin or Raman scattering) are assumed absent. Also, incoherent scattering processes are excluded. Under these conditions the incident and outgoing waves are monochromatic and of the same frequency, and both are totally polarized. The polarization forms (or the ellipses of polarization) of the incident and outgoing waves are determined by the complex ratios $\chi$ and $\chi'$, respectively, where $\chi$ refers to the ordinary Cartesian Jones matrix and the elements of the vector $E$ correspond to the Cartesian components of the electric vector of the light wave.

In many cases we are interested in the polarization-mapping properties of the system $S$ overlooking any over-all amplitude or phase changes between input and output. The polarization forms (or the ellipses of polarization) of the incident and outgoing waves are determined by the complex ratios $\chi$ and $\chi'$, respectively,
According to Eq. (1) \( x' \) and \( x \) are interrelated by
\[
x' = (T_{ax} + T_{a0})/(T_{ax} + T_{a1}),
\]
which is a bilinear transformation. The result in Eq. (4) has been utilized to investigate various aspects of polarization transfer by optical systems and to provide a procedure (generalized ellipsometry) to measure the matrix \( T \).

The two-port network shown in Fig. 1(b) is assumed linear and may contain active elements operating in the linear domain. The voltage and current \( V' \) and \( I' \) at one port are related to the voltage and current \( V \) and \( I \) at the other port by a linear transformation
\[
\begin{pmatrix} I' \\ V' \end{pmatrix} = \frac{1}{W_{11} W_{22} - W_{12} W_{21}} \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix},
\]
where \( W_{ij} \) are the complex elements of the network transfer matrix. The impedances at the two ports are defined by
\[
Z = V/I, \quad Z' = V'/I',
\]
and according to Eq. (5) they are related by
\[
Z' = (W_{22} Z + W_{21})/(W_{12} Z + W_{11}),
\]
which is a bilinear transformation. Obviously, the transformation of admittance between the two ports is also bilinear,
\[
Y' = (W_{11} Y + W_{12})/(W_{21} Y + W_{22}).
\]

The analogy between optical systems and two-port networks is clearly demonstrated by the similarity of Eqs. (1) and (5) and Eqs. (4) and (7) or (8). In both cases we deal with the transformation of (the complex amplitudes of) a pair of oscillating quantities between the input and output. These are the two oscillating components of the light vector along the two chosen basis states in the case of optical systems and the voltage and current oscillations in the case of two-port networks.

Rumsey has previously shown the similarity of the definition of impedance and polarization ratios and suggested the use of impedance charts (Smith or Carter) to represent elliptical polarizations. Deschamps briefly mentioned the similarity between the mapping of polarization by optical systems and of impedance by electrical networks.

**III. Analogous Properties of Optical Systems and Two-Port Networks**

From the previous section we have seen that the mapping of polarization by an optical system and of impedance (or admittance) by a two-port electrical network between input and output is described by a bilinear transformation. It follows that for each terminal characteristic of a system of one type there is a similar characteristic for the system of the other type.

First we note that a bilinear transformation can always be found that maps any three points \((\xi_1, \xi_2, \xi_3)\) in the complex \( \xi \) plane into the three points \((\eta_1, \eta_2, \eta_3)\) in the complex \( \eta \) plane and is given by
\[
\eta_n = \frac{(\eta - \eta_1)(\xi - \xi_3) - (\eta - \eta_3)(\xi - \xi_1)}{(\xi - \xi_2)(\eta - \eta_1) - (\xi - \xi_1)(\eta - \eta_2)} \times \frac{(\xi - \xi_2)(\eta - \eta_3) - (\xi - \xi_3)(\eta - \eta_2)}{(\eta - \eta_1)(\xi - \xi_2) - (\eta - \eta_2)(\xi - \xi_1)}.\]
dance that appears at one part of \( W \) when the other port is short- and open-circuited, respectively.

For each optical system there are two polarization states that pass through the system unchanged. These eigenpolarizations\(^3\) are obtained from Eq. (4) by setting \( \chi' = \chi \). This yields

\[
\chi_{\pm} = \frac{1}{2}(T_{\pm} + T_{\mp})
\]

(10)

Corresponding to the eigenpolarizations \( \chi_{\pm} \) and \( \chi_{\pm} \) of the optical system \( S \), the two-port network \( W \) has two iterative impedances \( Z_{\pm} \) and \( Z_{\pm} \) that when connected at one port appear unchanged at the other. By analogy, Eq. (7) gives

\[
Z_{\pm} = \frac{1}{2}(T_{\pm} + T_{\mp})
\]

(11)

Because Eqs. (10) and (11) involve ratios of the elements of the transfer matrices \( T \) and \( W \), the eigenpolarizations and iterative impedances are uniquely determined by the PTF and ITF, respectively.

The loci of polarization states that preserve either ellipticity or azimuth after propagation through an optical system\(^6\) are analogous to the loci of impedances that if connected at one port would appear at the other with either magnitude or angle unchanged, respectively. To determine the cartesian equation of these loci, both Eqs. (4) and (7) are put in the form

\[
\eta = (A\xi + B)/(C\xi + D),
\]

(12)

where \( \xi = \chi \) or \( Z \), \( \eta = \chi' \) or \( Z' \), and the coefficients \( A, B, C, \) and \( D \) correspond to the elements of either the \( T \) matrix of the optical system or the \( W \) matrix of the two-port network. The locus of polarizations that preserve their ellipticity or impedances that preserve their magnitude is determined by\(^12\)

\[
|\eta| = |\xi|,
\]

(13)

or

\[
|(A\xi + B)/(C\xi + D)| = |\xi|.
\]

(14)

Similarly the locus of polarizations that preserve their azimuth or impedances that preserve their angle is given by\(^12\)

\[
\arg(\eta) = \arg(\xi),
\]

(15)

or

\[
\arg((A\xi + B)/(C\xi + D)) = \frac{\arg(\xi)}{2},
\]

(16)

Expanding Eqs. (14) and (16) yields

\[
(x^2 + y^2)Q_1 - Q_2 = 0,
\]

(17)

and

\[
xQ_3 - yQ_4 = 0,
\]

(18)

respectively. In Eqs. (17) and (18) we have

\[
\xi = x + jy,
\]

(19)

and the \( Q \)'s are quadratics in \( x \) and \( y \) given by

\[
Q_1 = (C - D)(x^2 + y^2) + 2(C \times D)x + 2(C - D)x + 2(C \times D)y + (D - D),
\]

\[
Q_2 = (A - A)(x^2 + y^2) + 2(A \times B)x + 2(A \times B)y + (B - B),
\]

(20)

\[
Q_3 = (C - C)(x^2 + y^2) - (B \times C - D \times A)x - (B - C - D \times A)y + (D - B),
\]

\[
Q_4 = (C - A)(x^2 + y^2) - (B - C \times A)x - (B \times C - D \times A)y + (D - B).
\]

In Eq. (20) \( A, B, C, \) and \( D \) are the complex coefficients of the PTF or the ITF considered as vectors in the complex plane. The operations of the dot and cross products have their usual meanings except that for the cross product only the magnitude (with proper sign) is to be taken. Equations (17)–(20) can be easily used to determine these loci for any optical system or two-port network. The explicit form of the equations of the loci of invariant-ellipticity states and invariant-azimuth states of an optical system in the cartesian complex-plane representation can be found in Ref. 6.

**IV. Simulation of Optical Systems by Equivalent Two-Port Networks and Vice Versa**

Because of the analogy between the terminal characteristics of optical systems and two-port networks, it is possible to simulate systems of one type by systems of the other. The equivalence is achieved when the PTF of the optical system [Eq. (4)] and the ITF (or the admittance transfer function) of the two-port network [Eq. (7) or (8)] are identical.

For a given optical system (two-port network) there is one and only one PTF (ITF). However, the opposite is not true, namely, that there can be many optical systems (two-port networks), all of which have the same PTF (ITF). In other words, there can be many possibilities for the internal structure of \( S \) or \( W \), all of which lead to the same terminal polarization- or impedance-mapping properties.
Consider the following problem. Given an optical system $S$, find a two-port network $W$ whose impedance- or admittance-transforming properties are the same as the polarization-mapping properties of $S$. The first step in the solution is to find the PTF of $S$. As is evident from Eq. (4), the Jones matrix $T$ has to be determined apart from a complex multiplying factor. Because $T$ depends on the internal structure of $S$ as well as the basis polarization states, the latter have to be chosen. Such a choice is determined by the system under consideration and in most cases it is between a pair of orthogonal linear polarizations, the right and left circular polarizations or a mixture of these. Once the basis states have been decided upon, the Jones matrix $T$ is obtained from

$$T = T_N \ldots T_2 T_1,$$

(21)

where $T_1, T_2 \ldots, T_N$ are the matrices of the individual devices encountered by the light beam, with $T_1$ referring to the first-to-be-encountered optical element. Substituting the elements of the matrix $T$ of Eq. (21) into Eq. (4) the PTF of the optical system is found. The ITF or the admittance transfer function of an equivalent two-port network $W$ is obtained from Eq. (4) simply by substituting $T$ or $T'$ for $x$ and $T$ or $T'$ for $x'$. This gives

$$Z' = (T_{21} T + T_{12})/(T_{21} T + T_{12}),$$

(22)

or

$$Y' = (T_{21} Y + T_{12})/(T_{21} Y + T_{12}).$$

(23)

Thus the equivalence is between the polarization-mapping properties of $S$ and either the impedance- or the admittance-mapping properties of $W$. To synthesize a two-port network that maps an impedance $Z$ connected at one port into an impedance $Z'$ at the other port in accordance with Eq. (22) we can rewrite the latter equation in the form

$$Z' = Z_1 + \left[Z_2 + (Z + Z_1)\right],$$

(24)

where

$$Z_1 = T_{11}/T_{12}, \quad Z_2 = (\det T)^{-1}/T_{12}, \quad Z_3 = T_{21}/T_{12},$$

(25)

and $\det T$ denotes the determinant of the bilinear transformation,

$$\det T = T_{11}T_{22} - T_{12}T_{21}. $$

(26)

Equation (24) leads to the network of Fig. 3(a), which shows two series impedances $Z_1$ and $Z_3$ with a $\lambda/4$ section of a transmission line of characteristic impedance $Z_2$ between them. A simpler (and more practical) synthesis is the $T$ network of Fig. 3(b) whose impedances $Z_1, Z_3,$ and $Z_2$ are given by

$$Z_1 = (T_{11}/T_{12}) - Z_{12}, \quad Z_2 = (\det T)^{-1}/T_{12}, \quad Z_3 = (T_{21}/T_{12}) - Z_{21}. $$

(27)

The corresponding alternative solutions based on the mapping of admittance instead of impedance between the two ports according to Eq. (23) are the duals of those shown in Fig. 3. Other network configurations with three independent impedances or admittances can be found that satisfy either Eq. (22) or Eq. (23).

The opposite problem is to find the optical analog $S$ of a given two-port network $W$. A direct procedure would be to search for an optical system $S$ with six real independent parameters (e.g., two dichroic retarders and a rotater) whose PTF is the same as the ITF or the admittance transfer function of $W$. In general the synthesis is not as straightforward as in the first case, when $S$ is given and $W$ is to be synthesized.

If $W_1, W_2, \ldots, W_n$ represent the two-port-network analogs of the optical systems $S_1, S_2, \ldots, S_n$, then the network formed by connecting $W_1, W_2, \ldots, W_n$ in succession will correspond to the cascade of the optical systems $S_1, S_2, \ldots, S_n$ placed one after the other along the direction of propagation and in the same order in which the networks are connected.

It is interesting to find the electrical analogs of some of the simple optical elements. A polarizer is by definition an optical device that transforms any of the various polarization forms of an incident light wave into a unique polarization state at its output. In this case the determinant $T_{21} T_{11} - T_{22} T_{12}$ of the bilinear transformation vanishes and Eq. (4) becomes

$$\chi' = \chi_0 = T_{21}/T_{12} = T_{11}/T_{12},$$

(28)

where $\chi_0$ represents the outgoing polarization state, which is generally elliptic. The equivalent circuit analog is shown in Fig. 4(a), where $Z_0 = \chi_0$. A polarizer is said to exhibit leakage if it has a small, nonzero transmission for the incident polarization state that is orthogonal with $\chi_0$. This condition is taken care of by replacing the short circuit of Fig. 4(a) by a small impedance $Z_1$ such that $|Z_1| \ll |Z_0|$. As another example, a linear dichroic plate shows nonequal but inphase transmittances along two orthogonal principal directions in its plane. In this case it can be easily shown that Eq. (4) degenerates to the simple linear transformation

$$\chi' = K \chi, $$

(29)

where $K$ is the ratio of the amplitude transmittances along the two principal axes of dichroism. Obviously the circuit analog is an ideal transformer [Fig. 4(c)] whose turns ratio is $K^4$. A rotater is a device that rotates the major axis of the polarization ellipse through a fixed angle in a fixed sense for all orientations and ellipticities of the incident polarization ellipse. The cartesian Jones matrix is
Fig. 4. The circuit analogs of some of the simple optical devices: (a) ideal polarizer, (b) imperfect polarizer with leakage $|Z_2| \ll |Z_1|$, (c) purely dichroic plate, and (d) optical rotator.

![Circuit Diagrams](Image)

The parameters of the equivalent T network are

$$Z_1 = -\tan(\alpha/2), \quad Z_2 = \frac{\text{cosec} \alpha}{\sin \alpha}, \quad Z_3 = -\tan(\alpha/2),$$

which follows from Eqs. (27) and (30), and leads to the resistive circuit of Fig. 4(d). Note that in this case the equivalent circuit analog is not as simple as those encountered in the above two examples.

The simplicity of producing and measuring light of known polarization coupled with the fact that the optical analogs of negative impedance networks are readily realizable may make the optical simulation of electrical two-port networks of considerable practical value. However, the opposite does not appear to be as simple, namely, to simulate optical systems by electrical two-port networks as can be seen from the example of the optical rotator.

Finally, it should be noted that linear mechanoacoustic systems are described by equations similar to Eqs. (1) and (5). In this case the pair of oscillating quantities are the mechanical force (or pressure) and velocity (or displacement). Therefore these systems have their optical analogs as well as their well-known electrical analogs.

V. Conclusions

Analogy between optical systems and two-port electrical networks is demonstrated. In each case we deal with the transformation of a pair of oscillating quantities between input and output. The mapping of polarization by an optical system and of impedance by a two-port network is described by a bilinear transformation. For each terminal characteristic of a system of one type there is a similar characteristic for the system of the other type. For example, the two eigenpolarizations of the optical system are the analogs of the two iterative impedances of the two-port network. The loci of polarization states that preserve ellipticity or azimuth and those of impedances (or admittance) that preserve magnitude or angle are described by the same equations in the complex plane. Two-port electrical networks are synthesized whose impedance- (or admittance-) mapping properties are the same as the polarization-mapping properties of a given optical system. The opposite problem is also considered. The analogy presented in this paper is useful because (1) it unifies the methods of treating both kinds of systems and (2) it leads to the reciprocal simulation of systems of one type by systems of the other, which is of practical value.

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References

1. When there is more than one incident beam or when a single incident beam generates a number of emergent beams (e.g., in the presence of a diffraction grating), the electrical analog of the optical system becomes a multiport network. There is no loss of generality, however, when a two-port network is used to represent the relation between one incident and one emergent beam.


7. Large-signal equivalent circuits are sometimes used to relate voltages and currents of the same frequency in the presence of nonlinear effects [see, e.g., M.A.H. El-Said, *IEEE Trans. Circuit Theory* 17, 8 (1970)]. Such circuits might simulate nonlinear optical systems if light waves of the same frequency are coherent at input and output.

8. Conventionally voltages are listed before currents in the input and output vectors.


12. When the orthogonal left and right circular polarizations are used as basis states, the azimuth $\theta$ and ellipticity $\epsilon$ of the ellipse of polarization are obtained from the complex polarization variable $\chi$ [Eq. (3)] by the relations $\theta = \frac{1}{2} \text{Arg} (\chi)$ and $\epsilon = (|\chi| - 1)/(|\chi| + 1)$. These lead to the equi-azimuth-equi-ellipticity chart of Fig. 2 (left).