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THE COHOMOLOGY OF RIGHT ANGLED ARTIN GROUPS WITH GROUP RING COEFFICIENTS

C. JENSEN AND J. MEIER

Abstract

We give an explicit formula for the cohomology of a right angled Artin group with group ring coefficients in terms of the cohomology of its defining flag complex.

1. *Introduction*

Let Γ be a finite simplicial graph and let $\widehat{\Gamma}$ be the induced flag complex, i.e., the maximal simplicial complex whose 1-skeleton is Γ. The associated *right angled Artin group* A_Γ is the group presented by

$$A_\Gamma = \langle V(\Gamma) \mid vw = wv \text{ if } \{v,w\} \in E(\Gamma)\rangle \ .$$

Because one can import topological properties of the associated flag complex $\widehat{\Gamma}$ into the group A_Γ, these groups have provided important examples of exotic behavior. (See for example [**1**], [**4**] and [**12**].) Here we refine the understanding of the end topology of right angle Artin groups by giving an explicit formula for the cohomology of A_Γ with group ring coefficients in terms of the cohomology of $\widehat{\Gamma}$ and links of simplices in $\widehat{\Gamma}$.

Definition 1.1. If K is a simplicial complex let $\mathcal{S}(K)$ denote the set of closed simplices — including the empty simplex — in K. The dimension of a simplex is denoted $|\sigma|$; the link is denoted $\mathrm{Lk}(\sigma)$; the star of σ is $\mathrm{St}(\sigma)$. By definition $|\emptyset| = -1$ and $\mathrm{Lk}(\emptyset) = K$.

Main Theorem. *Let Γ be a finite simplicial graph, let $\widehat{\Gamma}$ be the associated flag complex and A_Γ the associated right angled Artin group. As long as $\widehat{\Gamma}$ is not a single simplex,*

$$H^*(A_\Gamma, \mathbb{Z}A_\Gamma) = \bigoplus_{\sigma \in \mathcal{S}(\widehat{\Gamma})} \left[\bigoplus_{i=1}^{\infty} \overline{H}^{*-|\sigma|-2}\left(Lk(\sigma)\right)\right] \ .$$

If $\widehat{\Gamma}$ is a single simplex then A_Γ is free abelian and $H^*(A_\Gamma, \mathbb{Z}A_\Gamma)$ is simply $\mathbb{Z}$ in top dimension.

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Example 1.2. Let $\widehat{\Gamma}$ be $\mathbb{R}P^2$. Then the reduced cohomology of $\mathrm{Lk}(\emptyset) = \mathbb{R}P^2$ is concentrated in dimension 2 where it is $\mathbb{Z}_2$. The link of any other simplex σ is a $(1-|\sigma|)$-sphere hence its reduced cohomology is concentrated in dimension $(1-|\sigma|)$, where it is $\mathbb{Z}$. Thus $H^*(A_\Gamma, \mathbb{Z}A_\Gamma)$ is trivial except in dimension 3 where it is the sum of a countably generated free abelian group and a countable sum of $\mathbb{Z}_2$'s.

There are at least two approaches to establishing the Main Theorem. One can modify the techniques of [**9**] that were developed for computing the cohomology of Coxeter groups with group ring coefficients — as well as the cohomology with compact supports of any locally finite building — to compute this cohomology for right angled Artin groups. In fact, the formula given in the Main Theorem is quite similar to the formulas for cohomology with compact supports of locally finite buildings (Theorem 5.8 in [**9**]). We take a more efficient route, and use the fact that right angled Artin groups are commensurable with certain right angled Coxeter groups [**8**], and appeal to the formula for the cohomology of a right angled Coxeter group with group ring coefficients ([**7**] or [**9**]).

In the last section we explain how the formula of the Main Theorem extends results of [**4**] on the end topology of right angled Artin groups.

2. *Background and Definitions*

One of the classical approaches to the study of asymptotic properties of a group G is via its cohomology with $\mathbb{Z}G$-coefficients. For example, from Proposition 7.5 and Exercise 4 of [**5**], if G is a discrete group and X is a contractible G-complex with finite cell stabilizers and finite quotient, then

$$H^*(G, \mathbb{Z}G) \cong H^*_c(X; \mathbb{Z}),$$

where $H^*_c(X;\mathbb{Z})$ is the cohomology of X with compact supports. In particular, one can take as X either of the classifying spaces EG or $\underline{E}G$ provided they have finite quotients BG or $\underline{B}G$ (cf. [**11**]). Cohomology with group ring coefficients determines the cohomological dimension of G [**5**, VIII.6.7]: If G is of type FP then

$$\mathrm{cd}\ G = \ \max\{n : H^n(G, \mathbb{Z}G) \neq 0\}.$$

It is also closely related to connectivity at infinity and duality properties as is described at the end of the next section.

Definition 2.1. Right angled Artin groups admit CAT(0) $K(\pi, 1)$s formed as the union of tori. If Γ is a finite simplicial graph, let K_Γ be the complex formed by joining tori in the manner described by the flag complex $\widehat{\Gamma}$. That is, for each simplex $\sigma \subset \mathcal{S}(\widehat{\Gamma})$, let T_σ be the torus formed by identifying parallel faces of a unit $(|\sigma|+1)$-cube. (The torus $T_\emptyset$ is a single vertex.) The complex K_Γ is then the union of these tori, subject to $T_\sigma \cap T_{\sigma'} = T_{\sigma''}$ when $\sigma \cap \sigma' = \sigma''$ in $\widehat{\Gamma}$. For a proof that these K_Γ's are CAT(0) classifying spaces, see [**13**]. We denote the universal cover of K_Γ by $\widetilde{K}_\Gamma$.

The complex $\widetilde{K}_\Gamma$ is also the Davis complex for an appropriate right angled Coxeter group. Given a finite simplicial graph Γ the *right angled Coxeter group* C_Γ is the quotient of A_Γ formed by declaring that each generator is an involution

$$C_\Gamma = \langle V(\Gamma) \mid vw = wv \text{ if } \{v,w\} \in E(\Gamma) \text{ and } v^2 = 1 \text{ for all } v\rangle \ .$$

For a finite simplicial graph Γ let Γ' be the graph whose vertices are given by $V(\Gamma) \times \{-1, 1\}$ where

$$\{(v,\epsilon),(w,\epsilon)\} \in E(\Gamma') \Leftrightarrow \{v,w\} \in E(\Gamma)$$
$$\{(v,\epsilon),(w,-\epsilon)\} \in E(\Gamma') \Leftrightarrow v \neq w$$

for $\epsilon = 1$ or -1.

THEOREM 2.2 (Davis-Januszkiewicz [**8**]). *The Artin group A_Γ and the Coxeter group $C_{\Gamma'}$ are commensurable and in fact the complexes $\widetilde{K}_\Gamma$ and the Davis complex for $C_{\Gamma'}$ are identical.*

(Because $\widetilde{K}_\Gamma$ is the Davis complex for $C_{\Gamma'}$ we do not actually define the Davis complex for a Coxeter group; see [**8**] for a definition.)

One can now derive a formula for $H^*(A_\Gamma, \mathbb{Z}A_\Gamma)$ from known results in the literature. Namely, because

(1) cohomology with group ring coefficients can be expressed in terms of cohomology with compact supports of an $\underline{E}G$, and
(2) $\widetilde{K}_\Gamma$ is both an EA_Γ and an $\underline{E}C_{\Gamma'}$, and
(3) the cohomology of a Coxeter group with group ring coefficients has been computed, and can be expressed in terms of the cohomology of subcomplexes of links of vertices in the Davis complex ([**7**] or [**9**]),

we have the following formula for the cohomology of A_Γ with $\mathbb{Z}A_\Gamma$ coefficients.

COROLLARY 2.3. *Each $w \in C_{\Gamma'}$ has an associated simplex $\sigma(w) \in \mathcal{S}(\widehat{\Gamma}')$ such that*

$$H^*(A_\Gamma, \mathbb{Z}A_\Gamma) = H^*(C_{\Gamma'}, \mathbb{Z}C_{\Gamma'}) = H^*_c(\widetilde{K}_\Gamma; \mathbb{Z}) = \bigoplus_{w \in C_{\Gamma'}} \overline{H}^{*-1}\left(\widehat{\Gamma}' - \sigma(w)\right) \ .$$

Each simplex $\sigma \in \mathcal{S}(\widehat{\Gamma}') \setminus \{\emptyset\}$ occurs countably many times in this sum, while $\sigma = \emptyset$ occurs exactly once.

Although the formula above is correct, it obfuscates the connection between $H^*(A_\Gamma, \mathbb{Z}A_\Gamma)$ and the cohomology of the flag complex $\widehat{\Gamma}$. As a first step toward expressing the right hand side in terms of the flag complex $\widehat{\Gamma}$, we give an alternate description of the flag complex $\widehat{\Gamma}'$.

For each $v \in V(\Gamma)$ let $\widehat{\Gamma}_v$ be the full subcomplex of $\widehat{\Gamma}$ induced by the vertices $V(\Gamma) \setminus \{v\}$. Thus $\widehat{\Gamma}_v$ is a deformation retract of $\widehat{\Gamma}$ with the vertex v removed.

Let $(W, V(\Gamma))$ be the Coxeter system where W is abelian and the generating set has been identified with the vertices of the graph Γ. Hence W is simply

$$W = \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{|V(\Gamma)| \text{ copies}} \ .$$

Let the $\widehat{\Gamma}_v$ be a set of mirrors related to this Coxeter system and form the associated W-complex in the following manner. For each $x \in \widehat{\Gamma}$ let W_x be the subgroup of W generated by the set of $v \in V(\Gamma)$ such that x belongs to $\widehat{\Gamma}_v$. In other words, W_x is generated by those v such that x is not in the open neighborhood of v in $\widehat{\Gamma}$. Define

$$L_\Gamma = W \times \widehat{\Gamma}/\sim$$

where $(w, x) \sim (v, y)$ if and only if $x = y$ and $w^{-1}v \in W_x$.

The complex $\widehat{\Gamma}'$ shows up in the formula of Corollary 2.3 because it is isomorphic to the link of any vertex in $\widetilde{K}_\Gamma$. One can find the following result in [**8**].

Lemma 2.4. *The complex $\widehat{\Gamma}'$ is isomorphic to L_Γ, and is isomorphic to the link of the vertex in K_Γ.*

If $\sigma \in \mathcal{S}(\widehat{\Gamma})$ one can form a subcomplex $L_\sigma \subset L_{\widehat{\Gamma}}$ by defining W_σ to be the subgroup of W generated by $\{v \in V(\Gamma) \mid v \notin \sigma\}$, and forming $W_\sigma \times \widehat{\Gamma}/\sim$ where as before $(w, x) \sim (v, y)$ if and only if $x = y$ and $w^{-1}v \in W_x$. In particular, if $\sigma = \emptyset$ (the empty simplex) then $L_\emptyset = L_{\widehat{\Gamma}}$.

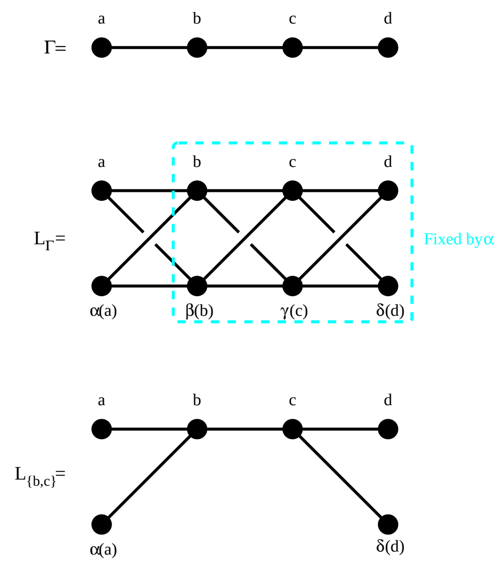

Figure 1. *A defining graph Γ, the associated complex L_Γ, and subcomplex $L_{\{b,c\}}$*

Example 2.5. Let $\Gamma = \widehat{\Gamma}$ be the simplicial arc indicated in Figure 1. The group W is then generated by four elements associated with the vertices. Switching to Greek letters we denote these generators as α, β, γ and δ, where the mirror associated to α is the subgraph induced by $\{b, c, d\}$, and similarly for the other three generators. The complex L_Γ is then as is indicated in Figure 1. The generator α acts on L_Γ by exchanging the vertices labeled a and $\alpha(a)$, and leaves all other vertices fixed. Similarly β exchanges b and $\beta(b)$, fixing all other vertices, and so on. Finally, if $\sigma = \{b, c\}$ then L_σ is the bottom complex in Figure 1.

For any $\sigma \in \mathcal{S}(\widehat{\Gamma}) \setminus \{\emptyset\}$ let

$$\widehat{\Gamma}_{b(\sigma)} = \bigcup_{v \in \sigma^{(0)}} \widehat{\Gamma}_v \ ,$$

so that $\widehat{\Gamma}_{b(\sigma)}$ is a deformation retract of $\widehat{\Gamma}$ with the barycenter of σ removed.

Lemma 2.6. *The cohomology groups of L_σ are given by*

$$\overline{H}^*(L_\sigma) = \bigoplus_{\tau \in \mathcal{S}(\widehat{\Gamma} - \sigma)} \overline{H}^*\left(\widehat{\Gamma}, \widehat{\Gamma}_{b(\tau)}\right) \ ,$$

where in a small abuse of notation we let $\mathcal{S}(\widehat{\Gamma} - \sigma)$ denote all closed simplices of $\widehat{\Gamma}$ except those with non-empty intersection with σ.

Proof. In [**6**] Mike Davis gives a formula for the homology of a complex on which a Coxeter group acts. One can switch this to a formula for cohomology using universal coefficients, or via a minor rewriting of Davis's original argument. In our case the formula is rather simple. Since W_σ is abelian, each $w \in W_\sigma$ is determined by the set of generators $S(w)$ that are necessary to express w. Temporarily following Davis's notation, define

$$\widehat{\Gamma}^{S(w)} = \bigcup_{v \in S(w)} \widehat{\Gamma}_v \ .$$

(If $w = 1$ then $S(w) = \emptyset$ and so $\widehat{\Gamma}^{S(w)}$ is empty as well.) Davis's formula then gives

$$\overline{H}^*(L_\sigma) \simeq \bigoplus_{w \in W_\sigma} \overline{H}^*\left(\widehat{\Gamma}, \widehat{\Gamma}^{S(w)}\right) \ .$$

This can be simplified. If $S(w)$ is not the vertex set of a simplex in $\widehat{\Gamma}$, then $\widehat{\Gamma}^{S(w)} = \widehat{\Gamma}$; if $S(w) = \sigma^{(0)}$ for some $\sigma \in \mathcal{S}(\widehat{\Gamma})$, then $\widehat{\Gamma}^{S(w)} = \widehat{\Gamma}_{b(\sigma)}$. Thus the formula above can be rewritten as

$$\overline{H}^*(L_\sigma) \simeq \bigoplus_{\tau \in \mathcal{S}(\widehat{\Gamma} - \sigma)} \overline{H}^*\left(\widehat{\Gamma}, \widehat{\Gamma}_{b(\tau)}\right) \ .$$

□

3. *Proof of the Main Theorem*

From Corollary 2.3 we know that

$$H^*(A_\Gamma, \mathbb{Z}A_\Gamma) = \overline{H}^{*-1}\left(\widehat{\Gamma}'\right) \bigoplus_{\sigma \in \mathcal{S}(\widehat{\Gamma}') \setminus \{\emptyset\}} \left[\bigoplus_{i=1}^{\infty} \overline{H}^{*-1}\left(\widehat{\Gamma}' - \sigma\right)\right]$$

where we know there are infinitely many copies of $\overline{H}^{*-1}\left(\widehat{\Gamma}' - \sigma\right)$ since by its construction there are no non-trivial finite conjugacy classes in $C_{\Gamma'}$. To arrive at our Main Theorem we need a formula for $\overline{H}^*(\widehat{\Gamma}' - \sigma)$ where σ is any simplex in $\mathcal{S}(\widehat{\Gamma}')$. Thus our key lemma is:

Lemma 3.1. *Let $\sigma \in \mathcal{S}(\widehat{\Gamma}')$. Then $\widehat{\Gamma}' - \sigma$ is homotopy equivalent to L_σ and*

$$\overline{H}^*(L_\sigma) = \bigoplus_{\tau \in \mathcal{S}(\widehat{\Gamma}_\sigma)} \overline{H}^{*-|\tau|-1}(Lk(\tau)) \ .$$

Proof. The complex $\widehat{\Gamma}$ embeds in $\widehat{\Gamma}'$ in a number of ways. Let the *standard embedding* $\widehat{\Gamma} \hookrightarrow \widehat{\Gamma}'$ have image the subcomplex induced by $\{(v,1) \mid v \in V(\Gamma)\}$. Define $\widehat{\Gamma}^{\rm op}$ to be the subcomplex induced by $\{(v,-1) \mid v \in V(\Gamma)\}$. If σ is a simplex in $\mathcal{S}(\widehat{\Gamma}')$ then σ is defined by a set of vertices in Γ along with choices of ± 1. If

$$\sigma \sim \{(a,1),(b,1),\ldots,(c,1),(x,-1),\ldots,(y,-1),(z,-1)\}$$

then the automorphism $\alpha\beta\cdots\gamma$ takes σ to the simplex

$$\sigma' \sim \{(a,-1),(b,-1),\ldots,(c,-1),(x,-1),\ldots,(y,-1),(z,-1)\}.$$

(Here we have used the same convention on naming generators of W as in Example 2.5.) Thus in discussing the topology of $\widehat{\Gamma}' - \sigma$ for $\sigma \in \mathcal{S}(\widehat{\Gamma}')$, we may without loss of generality assume $\sigma \subset \widehat{\Gamma}^{\rm op} \subset \widehat{\Gamma}'$. But the space formed by removing the closed simplex $\sigma \subset \widehat{\Gamma}^{\rm op}$ from $\widehat{\Gamma}'$ deformation retracts onto the subcomplex formed by making all possible reflections of $\widehat{\Gamma}$ that do not involve the generators of W that correspond to vertices of σ. In other words, $\widehat{\Gamma}' - \sigma$ deformation retracts onto L_σ, which implies our first claim.

From Lemma 2.6 we know $\overline{H}^*(L_\sigma) = \bigoplus_{\tau \in \mathcal{S}(\widehat{\Gamma}-\sigma)} \overline{H}^*\left(\widehat{\Gamma}, \widehat{\Gamma}_{b(\tau)}\right)$, thus it suffices to establish

$$\overline{H}^*\left(\widehat{\Gamma}, \widehat{\Gamma}_{b(\tau)}\right) \simeq \overline{H}^{*-|\tau|-1}(\mathrm{Lk}(\tau)) \ .$$

First, if $\tau = \emptyset$, $\mathrm{Lk}(\tau) = \widehat{\Gamma}$ and $|\tau| = -1$, so we get $\overline{H}^{*-|\tau|-1}(\mathrm{Lk}(\tau)) = \overline{H}^*(\widehat{\Gamma})$. If $\tau \neq \emptyset$ then by excision, $\overline{H}^*(\widehat{\Gamma}, \widehat{\Gamma}_{b(\tau)}) \simeq \overline{H}^*(\mathrm{St}(\tau), S^{|\tau|}[\mathrm{Lk}(\tau)])$ where $\mathrm{St}(\tau)$ is the closed star of τ and $S^i[\cdot]$ denotes the i^{th} suspension. Because the star $\mathrm{St}(\tau)$ is contractible, the long exact sequence in cohomology shows

$$\overline{H}^*(\mathrm{St}(\tau), S^{|\tau|}[\mathrm{Lk}(\tau)]) = \overline{H}^{*-1}(S^{|\tau|}[\mathrm{Lk}(\tau)]) \ .$$

But the cohomology of a suspension is just a shifted copy of the cohomology of the original complex

$$\overline{H}^{*-1}(S^{|\tau|}[\mathrm{Lk}(\tau)]) = \overline{H}^{*-|\tau|-1}(\mathrm{Lk}(\tau))$$

and the result follows. □

Example 3.2. In Example 2.5 we considered $\Gamma = \widehat{\Gamma} =$ a simplicial arc, and two associated complexes, L_Γ and $L_{\{b,c\}}$. (The first claim of Lemma 3.1 states that $L_{\{b,c\}}$ is homotopy equivalent to L_Γ with the closed edge $\{\beta(b), \gamma(c)\}$ removed.) The formula of Lemma 3.1 says, for example, that

$$\overline{H}^1(L_\Gamma) = \bigoplus_{\sigma \in \mathcal{S}(\widehat{\Gamma})} \overline{H}^{1-|\sigma|-1}(\mathrm{Lk}(\sigma)) = \bigoplus_{\sigma \in \mathcal{S}(\widehat{\Gamma})} \overline{H}^{-|\sigma|}(\mathrm{Lk}(\sigma)) \ .$$

This then becomes

$$\begin{aligned} H^1(L_\Gamma) = {} & \overline{H}^1(\mathrm{Lk}(\emptyset)) \oplus \overline{H}^0(\mathrm{Lk}(a)) \oplus \overline{H}^0(\mathrm{Lk}(b)) \oplus \overline{H}^0(\mathrm{Lk}(c)) \oplus \overline{H}^0(\mathrm{Lk}(d)) \\ & \oplus \overline{H}^{-1}(\mathrm{Lk}(\{a,b\}) \oplus \overline{H}^{-1}(\mathrm{Lk}(\{b,c\}) \oplus \overline{H}^{-1}(\mathrm{Lk}(\{c,d\}) \end{aligned}$$

$$H^*(A_\Gamma, \mathbb{Z}A_\Gamma)$$
$$= \overline{H}^1(\widehat{\Gamma}) \oplus \left(\overline{H}^0(\bullet)\right)^2 \oplus \left(\overline{H}^0(\bullet\quad\bullet)\right)^2 \oplus \left(\overline{H}^{-1}(\emptyset)\right)^3 = \mathbb{Z}^5 \ ,$$

using the convention that $\overline{H}^{-1}(\emptyset) = \mathbb{Z}$.

In the case of $L_{\{b,c\}}$ one drops all the terms involving b or c, which are precisely the non-trivial terms above, hence $H^1(L_{\{b,c\}}) = 0$.

We can now prove our Main Theorem.

THEOREM 3.3. *Let Γ be a finite simplicial graph, let $\widehat{\Gamma}$ be the associated flag complex and A_Γ the associated right angled Artin group. As long as $\widehat{\Gamma}$ is not a single simplex,*

$$H^*(A_\Gamma, \mathbb{Z}A_\Gamma) = \bigoplus_{\sigma\in\mathcal{S}(\widehat{\Gamma})} \left[\bigoplus_{i=1}^{\infty} \overline{H}^{*-|\sigma|-2}\left(Lk(\sigma)\right)\right] \ .$$

If $\widehat{\Gamma}$ is a single simplex then A_Γ is free abelian and $H^*(A_\Gamma, \mathbb{Z}A_\Gamma)$ is simply $\mathbb{Z}$ in top dimension.

Proof. From Corollary 2.3 we have

$$H^*(A_\Gamma, \mathbb{Z}A_\Gamma) = \bigoplus_{w\in C_{\Gamma'}} \overline{H}^{*-1}\left(\widehat{\Gamma}' - \sigma(w)\right) \ .$$

By Lemma 3.1 this gives

$$H^*(A_\Gamma, \mathbb{Z}A_\Gamma) = \bigoplus_{w\in C_{\Gamma'}} \left[\bigoplus_{\tau\in\mathcal{S}(\widehat{\Gamma}-\sigma(w))} \overline{H}^{*-|\tau|-2}\left(\mathrm{Lk}(\tau)\right)\right] \ .$$

If $\widehat{\Gamma}$ is not a single simplex, then each $\tau \in \mathcal{S}(\widehat{\Gamma})$ will show up in the product inside the square brackets for infinitely many $w \in C_{\Gamma'}$, and the formula in the theorem follows.

On the other hand, if $\widehat{\Gamma}$ is a single simplex σ, then σ only occurs in the summand corresponding to $1 \in C_{\Gamma'}$. All other simplices occur infinitely often, but if $\tau \neq \sigma$ then $\mathrm{Lk}(\tau)$ is contractible, and $\overline{H}^*(\mathrm{Lk}(\tau))$ is zero. Thus $H^*(A_\Gamma, \mathbb{Z}A_\Gamma) = H^{*-|\sigma|-2}(\emptyset)$, consistent with the fact that $A_\Gamma = \mathbb{Z}^{|\sigma|+1}$. ☐

As was alluded to in the previous section, cohomology with group ring coefficients is closely related to asymptotic properties. A group G that admits a finite $K(G,1)$ is *n-acyclic at infinity* if roughly speaking, complements of compact sets in the universal cover have trivial homology through dimension n (see [**9**] for a precise definition.) It was from this perspective that Brady and Meier determined when a right angled Artin group was n-acyclic at infinity. Their approach was via a combinatorial Morse theory argument using the K_Γ complexes. However, there is an algebraic characterization that says a group G is n-acyclic at infinity if and only if $H^i(G, \mathbb{Z}G) = 0$ for $i \leq n+1$ and $H^{n+2}(G, \mathbb{Z}G)$ is torsion-free (see [**10**]). The group G is an *n-dimensional duality group* if there is a dualizing module D such that $H_i(G, M) \simeq H^{n-i}(G, M \otimes D)$ for all i and all G-modules M. This too can be recast in terms of cohomology with group ring coefficients: G is an n-dimensional duality group if its cohomology with group ring coefficients is torsion-free and concentrated in dimension n [**2**]. Thus our Main Theorem implies three results of [**4**]. It is important to remember that $\emptyset \in \mathcal{S}(\widehat{\Gamma})$, and the formal dimension of $\emptyset$ is -1.

Corollary 3.4 (Prop. 4.1 in [**4**]). *A right angled Artin group A_Γ is n-acyclic at infinity if and only if for all $\sigma \in \mathcal{S}(\widehat{\Gamma})$, $Lk(\sigma)$ is $(n - |\sigma| - 1)$-acyclic.*

Proof. Since $\mathrm{Lk}(\sigma)$ is $(n - |\sigma| - 1)$-acyclic it follows by universal coefficients that its cohomology is trivial up to dimension $n - |\sigma| - 1$ and that $\overline{H}^{n-|\sigma|}(\mathrm{Lk}(\sigma))$ is torsion-free. Thus the formula of the Main Theorem implies that $H^i(A_\Gamma, \mathbb{Z}A_\Gamma)$ is zero for $i \leq n + 1$ and $H^{n+2}(A_\Gamma, \mathbb{Z}A_\Gamma)$ is torsion-free. □

Corollary 3.5 (Theorem C in [**4**]). *A right angled Artin group A_Γ is a duality group if and only if $\widehat{\Gamma}$ is Cohen-Macaulay.*

Proof. A simplicial complex K is *Cohen-Macaulay* if for any simplex $\sigma \in \mathcal{S}(K)$, the cohomology of $\mathrm{Lk}(\sigma)$ is concentrated in top dimension (and is torsion free). It follows from the formula of the Main Theorem that $H^*(A_\Gamma, \mathbb{Z}A_\Gamma)$ is torsion free and concentrated in top dimension if and only if $\widehat{\Gamma}$ is Cohen-Macaulay. □

Recall that an n-dimensional duality group is called a *Poincaré* duality group if and only if $H^n(G, \mathbb{Z}G) = \mathbb{Z}$ [**3**]. After the statement of Theorem C in [**4**] it was remarked that a Theorem of Strebel combined with Theorem C implies that a right angled Artin group A_Γ is a Poincaré duality group if and only if A_Γ is free abelian. This characterization follows directly from the formula in our Main Theorem.

Corollary 3.6. *A right angled Artin group A_Γ is a Poincaré duality group if and only if it is free abelian.*

Proof. The Main Theorem implies that $H^n(A_\Gamma, \mathbb{Z}A_\Gamma)$ is not finitely generated — in particular it is not equal to $\mathbb{Z}$ — unless $\widehat{\Gamma}$ is a simplex and hence A_Γ is free abelian. □

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