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Abstract. In this paper we prove the global solvability of a class of fourth-order nonlinear boundary value problems that govern the deformation of a Hollomon's power-law plastic beam subject to an axial compression and nonlinear lateral constrains. For certain ranges of the acting axial compression force, the solvability of the equations follows from the monotonicity of the fourth order nonlinear differential operator. Beyond these ranges the monotonicity of the operator is lost. It is shown that, in this case, the global solvability may be generated by the lower order nonlinear terms of the equations for a certain type of constrains.

1. INTRODUCTION

The Euler buckling load of simply supported straight elastic beam subject to an end axial compressive load P can be modeled by the equation:

$$EIv'''' + Pv'' = 0, \ 0 < x < L, \tag{1}$$

with boundary conditions:
$$v(0) = v(L) = v''(0) = v''(L) = 0, \tag{2}$$

where L is the length of the beam, E the Young's modulus, and I the area moment of inertia. Integrating (1) two twice gives:
$$EIv'' + Pv = c_1 + c_2 x.$$

Applying boundary conditions (2), we get: $c_1 = c_2 = 0$

The boundary value problem (1), (2), then reduces to:
$$EIv'' + Pv = 0, \tag{3}$$

with the boundary conditions:
$$v(0) = v(L) = 0. \tag{4}$$

The general solution of (3) is: $v(x) = A\cos\sqrt{\dfrac{P}{EI}}x + B\sin\sqrt{\dfrac{P}{EI}}x,$

where A and B are arbitrary constants to be determined so that the conditions (4) are satisfied. This gives the following system of two equations in A and B:

$$\begin{cases} 0 = A\cos\sqrt{\dfrac{P}{EI}}0 + B\sin\sqrt{\dfrac{P}{EI}}0, \\ 0 = A\cos\sqrt{\dfrac{P}{EI}}L + B\sin\sqrt{\dfrac{P}{EI}}L, \end{cases}$$

whose non-zero solutions are given by:

$$A = 0, B \neq 0, \quad \sqrt{\dfrac{P}{EI}}L = k\pi, k = 1,2,3... \ ,$$

corresponding to the following sequence of solutions of (1):

$$v_k(x) = B\sin\dfrac{k\pi x}{L}, \ P_k = EI\left(\dfrac{k\pi}{L}\right)^2, k = 1,2,3,...$$

Therefore, the boundary value problem (1), (2) can be thought of as the following eigenvalue problem:
$$\begin{cases} v'''' + \lambda v'' = 0, & 0 < x < L, \\ v(0) = v(L) = v''(0) = v''(L) = 0, \end{cases} \tag{5}$$

with eigenpairs given by:

$$v_k(x) = \sin\dfrac{k\pi x}{L}, \ \lambda_k = (k\pi)^2, k = 1,2,3,...$$

$v_1(x)$ is called the first eigenfunction or the first buckling mode and $P_{crit} = \dfrac{\lambda_1}{L^2}EI$

is the well-known Euler critical buckling load, sometimes also called the onset buckling load, see [11] for details.

The above classical Euler buckling load is derived based on the classical Hooke's law: $\sigma = E\varepsilon$ and the assumption that during the deformation, the cross-sections of the column remains perpendicular to the center line. Modern advances in manufacturing of metal made available in the markets the following high strength materials satisfying a more general constitutive equation:

$$\sigma = K \mid \varepsilon \mid^{n-1} \varepsilon,$$

which is known as the Hollomon's power-law. Here, σ stands for the true stress ε true strain, n the strain hardening index, and K the strength coefficient.

The differential equation for the Hollomon's power-law beam subjected to an end axial compressive load P can be written as:

$$(KI_n \mid v'' \mid^{n-1} v'')'' + Pv'' = 0, \ 0 < x < L, \tag{6}$$

where $I_n = \iint\limits_{A} y^{n+1} dy dz$ is the first moment of area inertia.

We consider here the solvability of (6) with one of the pin-pin (PP), the pin-slide (PS) or the slide-slide (SS) boundary conditions:

(1) $v(0) = v(1) = v''(0) = v''(1) = 0$ (PP Conditions),

(2) $v'(0) = v(1) = v'''(0) = v''(1) = 0$ (PS Conditions), $\tag{7}$

(3) $v(0) = v(1) = v'''(0) = v'''(1) = 0$ (SS Conditions),

If KI_n in (6) is constant, then equations (6), (7) can be written as the following eigenvalue problems:

$$\begin{cases} \quad -(\mid v'' \mid^{n-1} v'')'' = \lambda v'', \ 0 < x < 1, \\ (1) \quad v(0) = v(1) = v''(0) = v''(1) = 0 \ \text{(PP Conditions)}, \\ (2) \ v'(0) = v(1) = v'''(0) = v''(1) = 0 \ \text{(PS Conditions)}, \\ (3) \ v(0) = v(1) = v'''(0) = v'''(1) = 0 \ \text{(SS Conditions)}, \end{cases} \tag{8}$$

where: $\lambda = \dfrac{PL^{2n}}{KI_n}$.

Table 1.1below gives some typical values for the parameters K and n (see [7].)

Material	Strength Coefficient K(Mpa)	Hardening Index n
Aluminum 1100-O	180	0.20
Brass 70-30, annealed	895	0.49
Steel, low-carbon, annealed	530	0.26
Steel, 17-4 P-H annealed	1200	0.05
410 stainless steel, annealed	960	0.10

Table 1.1

In this case when KI_n is constant, Theorem 5.3 of [11] can be applied to imply that for each K and n, (8) has the following solutions corresponding to the three boundary conditions:

(1) For the PP boundary condition:

The first eigenfunction is $v_{1,n}(x) = \dfrac{1}{\pi_n^2} \sin_n(\dfrac{K\pi_n}{L}x)$, $x \in [0,1]$,

corresponding to the first eigenvalue $\lambda_1 = \dfrac{2n}{n+1}\pi_n^2$

and the critical buckling load $P_{cr}(n) = \left(\dfrac{2n}{n+1}\right)\dfrac{\pi_n^2 K I_n}{L^{2n}}$,

(2) For the PS boundary condition:

The first eigenfunction is $v_{1,n}(x) = -\dfrac{4\alpha}{\pi_n^2} \sin_n(\dfrac{\pi_n}{2}(1-x))$, $x \in [0,1]$,

corresponding to the critical buckling load $P_{cr}(n) = \dfrac{n\alpha^{n-1}K I_n}{2(n+1)L^{2n}}\pi_n^2$,

(3) For the SS boundary condition:

The first eigenfunction is $v_{1,n}(x) = \begin{cases} \dfrac{1}{\pi_n^2}(\dfrac{n+1}{2n})^{\frac{1}{n}}\left[2x-1-\sin_n(\pi_n(x+\dfrac{1}{2}))\right], & 0 \le x \le \dfrac{1}{2} \\[3mm] \dfrac{1}{\pi_n^2}(\dfrac{n+1}{2n})^{\frac{1}{n}}\left[2x-1-\sin_n(\pi_n(\dfrac{1}{2}-x))\right], & \dfrac{1}{2} < x \le 1 \end{cases}$,

corresponding to the critical load $P_{cr}(n) = \dfrac{K I_n}{L^{2n}}\left(\dfrac{2n}{n+1}\right)^{\frac{1}{n}}\pi_n^2$, $\alpha = v''(0)$.

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In the above, $\sin_n(t)$ is a generalized sine function and $\pi_n = B\left(\dfrac{n}{n+1},\dfrac{1}{2}\right)$ is a generalized Pi, where $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$. See, [3], [10], and [13] for definitions and properties of the generalized sine function. If $n=1$, $\sin_n x = \sin x$, and $\pi_n = \pi$, $P_{cr}(1) = \dfrac{\pi^2 EI}{L^2}$, and for $n \ne 1$ the corresponding onset buckling load is given by $P_{cr}(n)$.

This paper is concerned with the global existence and uniqueness of solutions to the following boundary value problems:

$$\begin{cases} (|v''|^{n-1}v'')'' + \lambda v'' + kv + G(x,v,v',v'') = f(x), \ 0 < x < 1 \\ \quad (1) \ v(0) = v(1) = v''(0) = v''(1) = 0 \ \text{ (PP Conditions)}, \\ \quad (2) \ v'(0) = v(1) = v''(0) = v''(1) = 0 \ \text{ (PS Conditions)}, \\ \quad (3) \ v(0) = v(1) = v'''(0) = v'''(1) = 0 \ \text{ (SS Conditions)}. \end{cases} \qquad (9)$$

These boundary value problems model the deflection of a power-law column subject to axial load and lateral force, with nonlinear foundation constrain. Our objectives are to extend the results established in [1] for the n=1 case. The main conclusion of our result here is to show that the linear lateral constrain in (9) cause a shift of the critical buckling load.

In the following sections of this paper, we only consider the global existence and uniqueness of solution for the differential equations in (9) for the "PP Condition." Similar results can be established for the other boundary conditions however, for simplicity and brevity, we omit the proofs.

2. ASSUMPTIONS AND PRELIMINARY RESULTS

Throughout the rest of this paper we will use the following set notation: $\mathrm{H} = \left\{ v \in W_0^{2,n+1}(0,1) \mid v''(0) = v''(1) = 0 \right\}$.

It can be easily seen that H is a Banach space.

We will make the following assumptions:

$(H_1) \ f \in L^{n+1}(0,1)$

$(H_2) \ G(x,v,v',v'') = g(v) + h(x,v,v',v'')$,

where $g(v)$ and $h(x,v,v',v'')$ are continous, and $h(x,v,v',v'') = H(x,v)$

defines a map $H : \ (0,1) \times \mathbf{H} \to L^{n+1}(0,1)$ which is continuous.

$(H_3) \ \int_0^1 [G(x,v,v',v'') - G(x,u,u',u'')](v-u)dx \geq 0, \ \text{for all } v,u \in \mathbf{H}.. \ \ . \ \ . \ \ .$

(H_4) Additionally, we assume that :

A. There exists $p > 1$ such that $g(rx) = r^p g(x)$ for $r,x \in R,$ with $r > 0.$

B. For any $v \in \mathbf{H}$ and $\int_0^1 g(v)vdx \geq 0,$ and $\int_0^1 g(v)vdx = 0$ *iff* $\mathrm{v} = 0;$

C. $\int_0^1 h(x,v,v',v'')vdx \geq 0, \forall v \in \mathbf{H}.$

The proof of our main result of the next section consists of verifying the conditions of a corollary of Leray-Schauder Fixed Point Theorem which we state here as the following lemma.

Lemma2.1. Let B be a Banach space and $K : B \to B$ be a compact operator. Suppose that there exists a priori bound $M > 0$ such that every solution of $v - tKv = 0$, for $t \in [0,1]$, satisfies $\| v \| \leq M$. Then K has a fixed point in B.

Let us define the operator:

$$A : W^{2,n+1}(0,1) \to W^{-2,n+1/n}(0,1) \qquad (10)$$

by defining, for $v \in W_0^{2,n+1}(0,1)$, $A(v) \in W_0^{-2,n+1/n}(0,1)$ by:

$$< A(v), u > = \int_0^1 | v'' |^{n-1} v'' u'' dx, \qquad \forall u \in W_0^{2,n+1}(0,1). \qquad (11)$$

It is known that $A(v) : W^{2,n+1}(0,1) \to R$ is a monotone operator satisfying the estimate (see, e.g. [8] and [9] for details):

$$\| v - w \|_{2,n+1}^2 \leq \frac{< A(v) - A(w), v - w >}{n} (\| v \|_{2,n+1}^2 + \| w \|_{2,n+1}^2)^{1-n}, \forall v, w \in W_0^{2,n+1}(0,1). \quad (12)$$

Therefore, for every $h \in W^{-2,n+1/n}(0,1)$ there exists a unique solution $u \in W_0^{2,n+1}(0,1)$ to the operator equation $Au = h$.

Let $A^{-1} : W^{-2,n+1/n}(0,1) \to W^{2,n+1}(0,1)$ denote the inverse of A.

Now, we formulate the boundary value problem (9) as an operator equation on $B \equiv W_0^{2,n+1}(0,1)$ by defining:

$$h = Kv \equiv A^{-1}[-\lambda v'' - kv - G(x,v,v',v'') + f], \text{ for } v \in W^{2,n+1}(0,1),$$

Provided that: $-\lambda v'' - kv - G(x,v,v',v'') + f \in W^{-2,n+1/n}(0,1)$ for $v \in W^{2,n+1}(0,1)$,

which is valid since the domain of A^{-1} is $W^{-2,n+1/n}(0,1)$.

Let $w \in W_0^{2,n+1}(0,1)$, then by Holder's inequality and the competence of the imbedding $W^{2,n+1}(0,1) \subseteq L^{\frac{n+1}{n}}(0,1)$, we have:

$$|< Kv, w > = \int_0^1 (-\lambda v''(x) - kv(x) - G(x,v(x),v'(x),v''(x)) + f(x))w(x)dx |$$

$$\leq \| f(x) - \lambda v''(x) - kv(x) - G(x,v(x),v'(x),v''(x)) \|_{L^{n+1}} \| w \|_{L^{\frac{n+1}{n}}}$$

$$\leq C[\| f \|_{L^{n+1}} + \lambda \| v'' \|_{L^{n+1}} + \| g(v) \|_{L^{n+1}} + \| h(x,v,v',v'') \|_{L^{n+1}}] \| w \|_{W^{2,n+1}} .$$

The term $\left\| g(v) \right\|_{L^{\frac{n+1}{n}}}$ is bounded since g is continuous, $v \in C[0,1]$, $\max_{0 \leq x \leq 1} | v(x) | \leq C \| v'' \|_{L^{n+1}}$ (by the Sobolev imbedding theorem) and h is continuous. It follows that:

$$|< Kv, w > | \leq C \| w \|_{W^{2,n+1}} , \text{ where the constant depends on the norm of } v \in W^{2,n+1}(0,1).$$

We now show that $K : B \to B$ is a compact operator. Let $h \in B$, and $v = A^{-1}h$, than $(Av, v) = (h, v)$ and $\phi(v'')'' = h$, where $\phi(v) = | v |^{n-1} v$.

Lemma 2.2 The operator $K : B \to B$ is a compact operator.

Proof: Suppose   that $\{v_k \mid k = 1, 2,...\} \subset B$ and $\max_{1 \leq k \leq \infty} \|v_k\| \leq M$,   where

$\| v_k \| = \left(\sum_{i=0}^{2} \int_0^1 | v_k^{(i)} |^{n+1} \right)^{1/n+1}$.   Let $h_k = Kv_k$, then We will show that the sequence

$\{h_k \mid k = 1, 2,...\}$ is bounded and equicontinuous. The compactness of K then follows from the Arzelà–Ascoli theorem. $Ah_k = -\lambda v_k'' - kv_k - G(x, v_k, v_k', v_k'') + f$.

$\phi(h_k'')'' = w_k$, where $w_k = -\lambda v_k'' - kv_k - G(x, v_k, v_k', v_k'') + f \in L^{n+1}$.   So

$h_k''(x) = \phi^{-1}(w_k(x))$, and $h_k'''(x) = | w_k(x) |^{\frac{1}{n}-1} w_k'(x)$

$z_k(x) \equiv \phi(h_k''(x)) = \int_0^x \int_0^t w_k(\tau) d\tau dt - \int_0^1 \int_0^t w_k(\tau) d\tau dt$ and

$| z_k(x) | \leq 2 \int_0^1 | w_k(x) | dx \leq 2 \| w_k \|_{L^{n+1}}$

$z_k'(x) = \int_0^x w_k(t) dt$ and $| z_k'(x) | \leq \int_0^1 | w_k(t) | dt \leq \| w_k \|_{L^{n+1}}$

So $h_k''(x) = \phi^{-1}(z_k(x))$, $h_k'''(x) = | z_k(x) |^{\frac{1}{n}-1} z_k'(x)$,   and   we   have

$| h_k'''(x) | \leq (2 \| w_k \|_{L^{n+1}})^{\frac{1}{n}-1} \| w_k \|_{L^{n+1}} \leq C$

Therefore, the Arzelà–Ascoli theorem applies and the operator K is compact.

With the notation:

$L(v) \equiv \phi(v''')''$, with $\phi(v'') \equiv | v'' |^{n-1} v''$ and $D(L) = \{v \in H \mid \phi(v'') \in W_0^{2,n+1}(0,1)\}$,   (13)

we note that:

$< A(v), u > = \int_0^1 L(v)u \, dx$, $\forall u \in W_0^{2,n+1}(0,1)$.   (14)

By Lemma 2.1, 2.2, and the result in [13], we have just completed the following proposition about the operator *L*:

Proposition 2.1

   (A) L, as an operator on $L^{n+1}(0,1)$, is densely defined;

   (B) $\| v \|_{L^{n+1}}^2 \leq \| v' \|_{L^2}^2 \leq \left(\dfrac{n+1}{2n\pi_n^2} \right) \| v'' \|_{L^{n+1}}^{n+1}$ for $v \in D(L)$;

   (C) For $y \in D(L)$, $L(y) = 0$ iff $y = 0$;

   (D) there exists a unique bounded operator $\psi : L^{\frac{n+1}{n}}(0,1) \to D(L)$ such that

        $L(\psi(h)) = h$, for any $h \in L^{\frac{n+1}{n}}(0,1)$;

   (E) $K = i \circ \psi$, $i : D(L) \to W^{2,n+1}(0,1)$ is the identity map, is a compact operator.

The existence of solution proofs of section 3 are based Proposition 2.1.

3. GLOBAL EXISTENCE OF SOLUTIONS

In this section we consider the global solvability of the boundary value problem (9) (with the (PP) boundary condition) in the following two theorem.

Theorem 3.1

Under the assumptions H (1)-H (3), the boundary value problem (9) (with the (PP) boundary condition) has at least one solution for each $k \geq 0$ and each $\lambda \geq 0$.

PROOF

The boundary value problem (9) (with the (PP) boundary condition) can be written in the form of an operator equation as:

$$v = Tv = -K[\lambda v + kv + G(x, v, v', v'') - f(x)], \tag{15}$$

where K is the operator defined in Proposition 2.1.

We will prove the existence of a solution of (15) by verifying the conditions of Lemma 2.1.

Assume that the solutions of $v - tKv = 0$ are not uniformly bounded with respect to $t \in [0,1]$. Then there are sequences $\{t_m\} \subset (0,1)$ and $\{v_m\} \subset W^{2,n+1}(0,1)$ such that:

$$v_m = t_m K v_m, \ m = 1, 2, ..., \ \text{and} \ \| v_m \|_{L^{n+1}}^2 \to \infty \ \text{as} \ m \to \infty.$$

It follows that:

$$(|v_m''|^{n-1} v_m'')'' + t_m^n [\lambda v_m'' + g(v_m'') + h(x, v_m, v_m', v_m'')] = t_m^n f(x),$$

which in turn implies (upon multiplying by v_m, integrating and using the boundary conditions):

$$\| v_m \|_{L^{n+1}}^2 + \lambda t_m \int_0^1 v_m'' v_m \, dx + t_m k \| v_m \|_{L^{n+1}}^2 + t_m \int_0^1 G(x, v_m, v_m', v_m'') v_m \, dx = \int_0^1 t_m f v_m \, dx. \tag{16}$$

Set $w_m = \dfrac{v_m}{\| v_m \|_{L^{n+1}}}$, then the sequence $\{w_m\} \subset W^{2,n+1}(0,1)$ is bounded, and hence weakly relatively compact. Hence $\{w_m\}$ has a weakly convergent subsequence in $W^{2,n+1}(0,1)$, which we will still call $\{w_m\}$. Since the embedding $i : W^{2,n+1}(0,1) \to C^1(0,1)$ is compact, the sequence $\{w_m\}$ has a subsequence, which we will still call $\{w_m\}$, which converges in $C^1(0,1)$, say to w_0.

From (16), and using assumption H (2) we get:

$$t_m \int_0^1 g(v_m)v_m dx = - \| v_m'' \|_{L^{n+1}}^2 - t_m \lambda \int_0^1 v_m'' v_m dx - t_m k \| v_m \|_{L^{n+1}}^2 - t_m \int_0^1 h(x, v_m, v_m', v_m'') v_m dx + t_m \int_0^1 f v_m dx$$

$$\leq t_m \lambda \| v_m \|_{L^{n+1}} \| v_m'' \|_{L^{n+1}} + t_m \int_0^1 f v_m dx$$

$$\leq t_m C \lambda \| v_m'' \|_{L^{n+1}}^{\frac{n+3}{2}} + t_m \| f \|_{L^{n+1}} \| y_m \|_{L^{n+1}}, \tag{17}$$

where: $C = \sqrt{\dfrac{n+1}{2n\pi_n^2}}.$

Using (17) and assumption H(2) we get:

$$0 \leq \int_0^1 g(w_m)w_m dx \leq \frac{\lambda \| v'' \|_{L^{n+1}}^{\frac{n+3}{2}}}{C \| v_m \|_{L^{n+1}}^{p+1}} + \frac{\| f \|_{L_2} \| v_m \|_{L^{n+1}}}{\| v_m \|_{L^{n+1}}^{p+1}} \to 0 \text{ as m} \to \infty \tag{18}$$

Since g is continuous and p>1, it follows from (18) that

$$\int_0^1 g(w_0)w_0 dx = 0,$$

which, in view of assumption H (2) (b) implies that $w_0 = 0,$ and $w_m \to 0$ in $C^1(0,1)$.
On the other havd, from (16), we have:

$$\frac{\| v_m'' \|_{L^{n+1}}^2}{\| v_m \|_2^2} = -t_m \lambda \int_0^1 \frac{v_m' v_m}{\| v_m \|_2^2} dx - t_m k \frac{\| v_m \|_{L^{n+1}}^2}{\| v_m \|_2^2} - t_m \int_0^1 G(x, v_m, v_m', v_m'') \frac{v_m}{\| v_m'' \|_2^2} dx + \frac{t_m}{\| v_m \|_2^2} \int_0^1 f v_m dx$$

$$\leq -t_m \lambda [w_m'(1)w_m(1) - w_m'(0)w_m(0)] + t_m \lambda \int_0^1 (w_m')^2 dx + t_m \frac{\| v_m \|_{L^{n+1}} \| f \|_{L^{n+1}}}{\| v_m \|_2^2},$$

which implies (by the fact that $w_m \to 0$ in $C^1(0,1)$) that:

$$\frac{\| v_m'' \|_{L^{n+1}}^2}{\| v_m \|_2^2} \to 0. \tag{19}$$

However, (from Lemma (2.3) part (B) and the fact that $\| v_m' \|_{L^{n+1}} \leq \| v_m'' \|_{L^{n+1}}$ for $v \in D(L)$) we have:

$$\| v_m \|_2^2 = \| v_m \|_{L^{n+1}}^2 + \| v_m' \|_{L^{n+1}}^2 \leq (2 + C^2) \| v_m'' \|_{L^{n+1}}^2,$$

which contradicts (19). This completes the proof for the theorem.

4. UNIQUENESS

Assuming that $G(x, v_m, v'_m, v''_m)$ satisfies the condition

$$H(3) \int_0^1 [G(x, v_m, v'_m, v''_m) - G(x, u_m, u'_m, u''_m)](v-u)dx > 0, \text{ for all v, u } \in W^{2,n+1}(0,1),$$

we obtain the following result on the uniquesness of the solution.

THEOREM 4.1. Assume that H (3) holds, then the solution of the boundary value problem (9) with the (PP) boundary condition is unique, provided that

$$\lambda < \frac{n}{(\sqrt{2})^{1-n}} \alpha \pi_n^2 + \frac{k}{\pi^2}$$

where α is determined by representes the minimum of the functional:

$$f(v) = \int_0^1 |v|^{n+1} dx, \text{ over } D(f) = \{v \in L^2(0,1): \|v\|_{L^2} = 1\}.$$

Proof.

Let v, u be two solutions of the boundary value problem and set $w = v - u$.

Then we have:

$$[\phi(v'') - \phi(u'')]'' + \lambda w'' + kw + [G(x, v_m, v'_m, v''_m) - G(x, u_m, u'_m, u'')] = 0. \tag{20}$$

Multiplying both sides of (20) by $w = v - u$, intetgrating the first two terms

twice by parts using the boundary conditions and H (3) we get:

$$\langle \phi(v'') - \phi(u''), v'' - u'' \rangle + \lambda \langle w'', w \rangle + k \langle w, w \rangle \le 0$$

By inequality (12), we get

$$n \|v'' - w''\|^2_{L^{n+1}} (\| v'' \|^2_{L^{n+1}} + \| u'' \|^2_{L^{n+1}})^{1-n} + \lambda \langle w'', w \rangle + k \langle w, w \rangle \le 0. \tag{21}$$

We will prove that if λ satisfies (20), then (21) can hold if and only if $w = 0$.

This will then complete the proof of Theorem 4.1.

Using Holder's inequality three times, we can write (21) as:

$$\frac{n}{2^{\frac{1-n}{2}}} \|w''\|^{3-n}_{L^{n+1}} - \lambda \| w'' \|_{L^2} \| w \|_{L^2} + k \| w \|^2_{L^2} \le 0$$

Using the notations: $A = \|w''\|_{L^{n+1}}$, $D = \|w''\|_{L^2}$, $B = \|w\|_{L^2}$, we have:

$$\frac{n}{2^{\frac{1-n}{2}}} A^{3-n} - \lambda DB + kB^2 > 0 \text{ holds if and only if } \lambda \text{ satisfies:}$$

$$\lambda < \frac{n}{2^{\frac{1-n}{2}}} \frac{A^{2-n}}{D} \frac{A}{B} + k \frac{B}{D}. \tag{22}$$

However, if λ satisfies (20), it will satisfy (22).

5. CONCLUSIONS

The results of this paper generalize the previous results of the papers [1] and [2] for elastic materials for the elasto-plastic materials based on the Hollomon's power-law. Similar results for material following other power laws, for instant Ludwick and Ramberg and Osgood laws, as well as for other elasto-plastic basic structures such as rings, plates and arches are under preparation.

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