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Single-layer antireflection coatings on absorbing substrates for the parallel and perpendicular polarizations at oblique incidence

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Explicit equations are derived that determine the refractive index of a single layer that suppresses the reflection of p- or s-polarized light from the planar interface between a transparent and an absorbing medium at any given angle of incidence. The required layer thickness and the system reflectance for the orthogonal unextinguished polarization also follow explicitly. This generalizes earlier work that was limited to normal incidence or to oblique incidence at dielectric–dielectric interfaces. Specific examples are given of p- and s-antireflection layers on Si and Al substrates at $\lambda = 6328$ Å at various angles of incidence.

I. Introduction

Single-layer antireflection coatings on dielectric substrates for normally incident monochromatic light are well known.[1] The layer refractive index N_1 must be chosen as[2]

$$N_1 = (N_0 N_2)^{1/2}, \tag{1}$$

i.e., equal to the geometric mean of the refractive indices N_0 of the ambient (incidence medium) and N_2 of the substrate.

At a general angle of (oblique) incidence ϕ, the condition of zero reflection by a transparent film on a transparent substrate, for the parallel p or perpendicular s polarization, has also been derived in explicit form,[3] yielding N_1 as a function of N_0, N_2, and ϕ.

When the substrate is absorbing, antireflection at normal incidence continues to be possible using a transparent film of refractive index[4]

$$N_1 = \left(N_0 n_2 + \frac{N_0^2 k_2^2}{N_0 n_2 - N_0^2}\right)^{1/2}, \tag{2}$$

where $N_2 = n_2 - jk_2$ is the substrate complex refractive index. Equation (2) reduces to Eq. (1) when $k_2 = 0$.

In this paper we further generalize these earlier results and derive explicit equations for the refractive index of a transparent film on an absorbing substrate necessary to suppress the reflection of p- or s-polarized light at any given angle of incidence. The thickness of the antireflection (polarizing) layer, and the associated unextinguished reflectance (for the orthogonal polarization) of the film–substrate system, are also determined. The results are illustrated by specific examples of antireflection layers on semiconducting (Si) and metallic (Al) substrates at one wavelength ($\lambda = 6328$ Å).

II. Basic Relations

In what follows, we will consider the antireflection condition for p- and s-polarized light separately. For either polarization, zero reflection by the ambient–film–substrate (0–1–2) system happens if the ambient–film and film–substrate interface reflectances are equal:

$$|r_{01\nu}| = |r_{12\nu}|, \nu = p,s. \tag{3}$$

Equation (3) is basic and has been recognized previously.[5–7] However, it appears that no attempt has been made to solve it for the film refractive index N_1 when the substrate is absorbing.[8] With N_2 complex $r_{12\nu}$ is also complex. In this case, manipulating Eq. (3) is simplified considerably by replacing it by the equivalent form

$$r_{01\nu}^2 = r_{12\nu} r_{12\nu}^*, \tag{4}$$

where * indicates the complex conjugate.

Once N_1 that satisfies Eq. (3) or (4) has been determined, the normalized polarizing film thickness is readily obtained as[6]

$$\zeta_\nu = -\arg(X_\nu)/2\pi, \qquad 0 \le \zeta_\nu < 1, \tag{5}$$

where

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$$X_\nu = -r_{01\nu}/r_{12\nu}. \quad (6)$$

The least film thickness is

$$d_\nu = \zeta_\nu D_\phi, \quad (7)$$

where

$$D_\phi = \frac{\lambda}{2}(N_1^2 - N_0^2 \sin^2\phi)^{-1/2} \quad (8)$$

is the film thickness period and λ is the free-space wavelength of light. Higher polarizing film thicknesses are obtained by adding integral multiples of D_ϕ to d_ν.

The complex-amplitude reflection coefficient of the coated substrate for the unextinguished orthogonal polarization ν' (if $\nu = p, \nu' = s$, and vice versa) is

$$R_{\nu'} = (r_{01\nu'} + r_{12\nu'}X_\nu)/(1 + r_{01\nu'}r_{12\nu'}X_\nu), \quad (9)$$

where X_ν is given by Eq. (6). The corresponding intensity reflectance is

$$\mathcal{R}_{\nu'} = |R_{\nu'}|^2. \quad (10)$$

III. Antireflection of the *s* Polarization

Fresnel's reflection coefficients of the ambient–film (01) and film–substrate (12) interfaces for the s polarization are given by[9]

$$r_{01s} = (S_0 - S_1)/(S_0 + S_1), \quad (11a)$$

$$r_{12s} = (S_1 - S_2)/(S_1 + S_2), \quad (11b)$$

where

$$S_i = (\epsilon_i - \epsilon_0 \sin^2\phi)^{1/2}, \quad (12)$$

$$\epsilon_i = N_i^2, i = 0,1,2. \quad (13)$$

ϵ_i is the dielectric constant of medium i. On substituting Eqs. (11) into Eq. (4), the antireflection condition for s-polarized light takes the form

$$(S_0^2 + S_1^2)\,\mathrm{Re}(S_2) = S_0(S_1^2 + |S_2|^2), \quad (14)$$

where Re indicates the "real part of." To simplify reaching Eq. (14), we used the algebraic fact that if $(A - B)/(A + B) = (C - D)/(C + D)$, then $A/B = C/D$. From Eq. (14) we get

$$S_1^2 = S_0[|S_2|^2 - S_0\,\mathrm{Re}(S_2)]/[\mathrm{Re}(S_2) - S_0]. \quad (15)$$

If S_i from Eq. (12) are used in Eq. (15), we obtain

$$\epsilon_1 = \epsilon_0 \left\{ \sin^2\phi + \cos\phi \left[\frac{|\beta_2| - \cos\phi\,\mathrm{Re}(\beta_2{}^{1/2})}{\mathrm{Re}(\beta_2{}^{1/2}) - \cos\phi} \right] \right\}, \quad (16)$$

$$\beta_2 = (\epsilon_2/\epsilon_0) - \sin^2\phi. \quad (17)$$

Equation (16) gives the desired dielectric constant of the s-polarization antireflection layer in terms of the ambient dielectric constant ϵ_0, substrate complex dielectric constant $\epsilon_2 = (n_2 - jk_2)^2$, and angle of incidence ϕ. The corresponding refractive index is

$$N_1 = \epsilon_1^{1/2}. \quad (18)$$

Knittl's result[3] for a transparent film on a transparent substrate is obtained as a special case of Eq. (16) if the imaginary part of ϵ_2 is set equal to zero.

As a first example, consider the reflection of light ($\lambda = 6328$ Å) in air ($\epsilon_0 = 1$) by a Si substrate of complex refractive index[10] $N_2 = 3.85$–$j0.02$. The refractive index N_1 of the s-polarization antireflection layer was computed from Eqs. (16)–(18) and is plotted in Fig. 1 as a function of ϕ from normal ($\phi = 0$) to grazing ($\phi = 90°$) incidence. For $0 \le \phi \le 80°$, we have $1.9622 \ge N_1 \ge 1.2713$, which correspond to several exisitng thin-film coating materials.[11,12] For $\phi > 80°$, N_1 is too close to 1 to be realizable by a thin solid film.

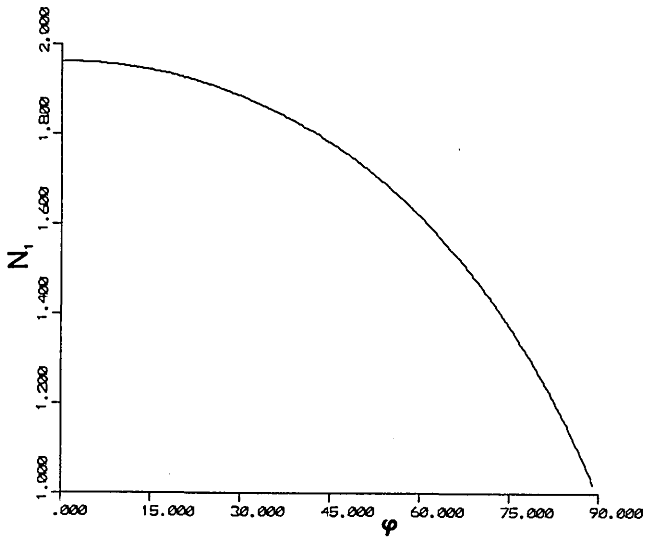

Fig. 1. Refractive index N_1 of s-polarization antireflection layer on Si ($N_2 = 3.85$–$j0.02$) as a function of angle of incidence ϕ (degrees). Light ($\lambda = 6328$ Å) is assumed to be incident from air ($N_0 = 1$).

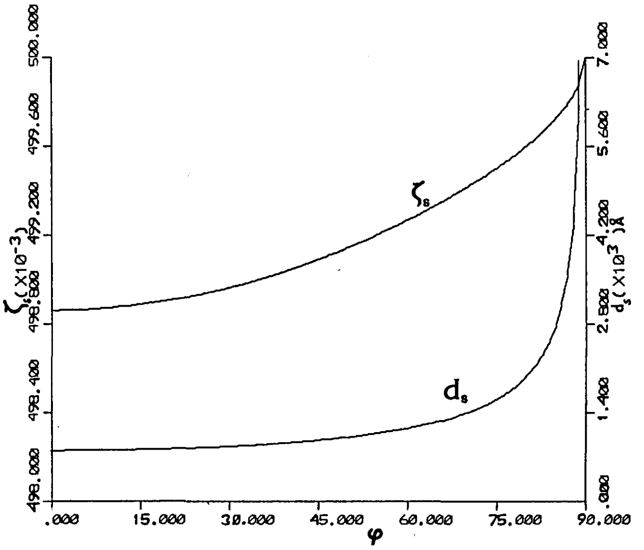

Fig. 2. Normalized ζ_s and actual d_s (angstroms) thicknesses of the s-polarization antireflection layer on Si at $\lambda = 6328$ Å vs angle of incidence ϕ (degrees).

Figure 2 shows the normalized and actual film thicknesses ζ_s and d_s, respectively, of the antireflection layer calculated from Eqs. (5)–(7). Because of the small but nonzero extinction coefficient of Si (0.02), ζ_s is slightly less than one-half, and the layer is strictly not of quarterwave thickness. ($\zeta_s = 1/2$ holds exactly when the substrate is transparent and $k_2 = 0$.)

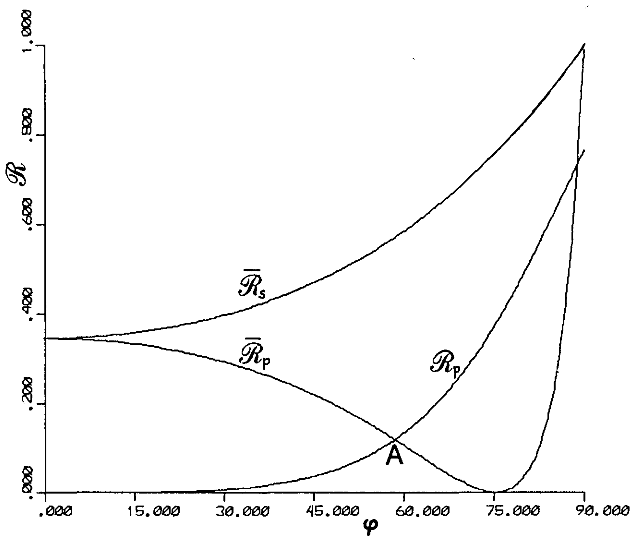

Fig. 3. Unextinguished p reflectance $\mathcal{R}_p$ vs angle of incidence ϕ (degrees) for Si, which is antireflection-coated for the s polarization. $\overline{\mathcal{R}}_p$ and $\overline{\mathcal{R}}_s$ are the p and s reflectances of the bare Si substrate.

Table I. s-Polarization Antireflection Layers on Si at Five Angles of Incidence [a]

ϕ	N_1	ζ_s	d_s	$\mathcal{R}_p$	$\overline{\mathcal{R}}_p$	$\overline{\mathcal{R}}_s$
0	1.96218	0.49886	804.4	0	0.3453	0.3453
30	1.88576	0.49896	868.3	0.0063	0.2933	0.3971
45	1.78218	0.49909	965.3	0.0361	0.2204	0.4694
60	1.62041	0.49927	1153.4	0.1333	0.1075	0.5849
75	1.37754	0.49950	1609.1	0.3710	0.0002	0.7571

[a] ϕ is the angle of incidence in degrees. N_1 is the layer required refractive index for s-polarization antireflection, and ζ_s is its normalized thickness (as a fraction of the thickness period). d_s is the corresponding actual (least) thickness in angstroms. $\mathcal{R}_p$ is the reflectance of the coated Si for the unextinguished p polarization. $\overline{\mathcal{R}}_p$ and $\overline{\mathcal{R}}_s$ are the reflectances of the film-free air–Si interface for the p and s polarizations, respectively. The complex refractive index of Si is taken as $N_2 = 3.85\text{–}j0.02$ at $\lambda = 6328$ Å.

Figure 3 presents the unextinguished p reflectance $\mathcal{R}_p$ of the coated Si, computed from Eqs. (6), (9) and (10), plotted vs ϕ. We have superimposed on Fig. 3 the reflectances $\overline{\mathcal{R}}_p$ and $\overline{\mathcal{R}}_s$ of the film-free air–Si interface. At point $A(\phi_A$ is between 58 and 59°) Abelès's condition[13] of equal p reflectances of the coated and uncoated substrate is satisfied. For $0 \leq \phi < \phi_A$, the coating that reduces the s reflectance to zero also diminishes the p reflectance below its bare-substrate value. At $\phi = 0$, the p and s polarizations are indistinguishable, and total antireflection of light occurs. For $0 < \phi \leq 45°$, suppression of the s polarization is accompanied by significant reduction of the p reflectance, so that excellent (but incomplete) overall antireflection is achieved by one layer.

Table I lists data on s-polarization antireflection layers on Si (at $\lambda = 6328$ Å) at five angles of incidence 0, 30, 45, 60, and 75°. Silicon nitride is a particularly suitable antireflection coating material for any ϕ from 0 to 45°. If prepared by sputtering, its refractive index can be finely tuned within the desired range $1.78 \leq N_1 \leq 1.96$ depending on its stoichiometry.[14]

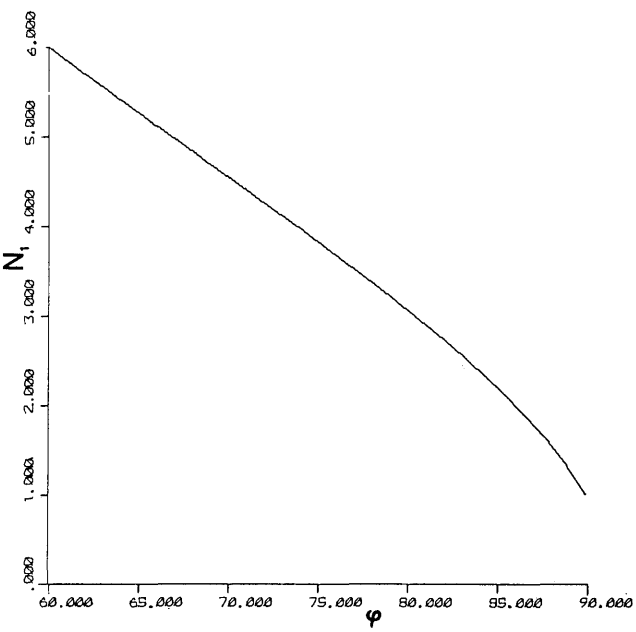

Fig. 4. Refractive index N_1 of s-polarization antireflection layer on Al ($N_2 = 1.212\text{–}j6.924$) as a function of angle of incidence ϕ (degrees). The range $\phi < 60°$ is excluded to avoid far from realistic refractive indices. Light ($\lambda = 6328$ Å) is assumed to be incident from air ($N_0 = 1$).

For $\phi = 45°$, the antireflection layer reduces the s reflectance of the Si surface from 46.94% to 0 and the p reflectance from 22.04 to 3.61%. Thus the residual reflectance of the coated Si for unpolarized incident light is only 1.8%. At $\phi = 75°$, $\mathcal{R}_p = 37.10\%$ is not sufficiently high to make the film–substrate system an efficient polarizer. (Reflection from bare Si at the pseudo-Brewster angle,[15] $\phi = 75.44°$, makes a simpler more efficient polarizer.)

As a second example, we consider s-polarization antireflection layers on an Al substrate with complex refractive index[16] $N_2 = 1.212\text{–}j6.924$ at $\lambda = 6328$ Å in air ($\epsilon_0 = 1$). As ϕ is increased from 0 to 90°, N_1, computed from Eqs. (16)–(18), decreases from 15.07821 to 1 monotonically. To exclude far from realistic values of the film refractive index, $N_1(\phi)$ is plotted in Fig. 4 vs ϕ over the restricted range $60° \leq \phi < 90°$. The corresponding required normalized ζ_s and actual d_s film thicknesses are plotted in Fig. 5. Notice that ζ_s differs appreciably from ½, so that the layer thickness is no longer close to quarterwave. The unextinguished reflectance $\mathcal{R}_p$ of the coated surface and the reflectances $\overline{\mathcal{R}}_p$ and $\overline{\mathcal{R}}_s$ of the bare Al substrate are shown in Fig. 6 as functions of ϕ. At A, Abelès condition is satisfied, as before.

To cite a specific numerical result, we give the characteristics of a thin film on Al that suppresses the reflection of the s polarization at 85°. The required refractive index of the film is $N_1 = 2.220$ (rounded to three decimal places) corresponding to ZnS, for example.[12] The normalized and actual film thicknesses are $\zeta_s =$

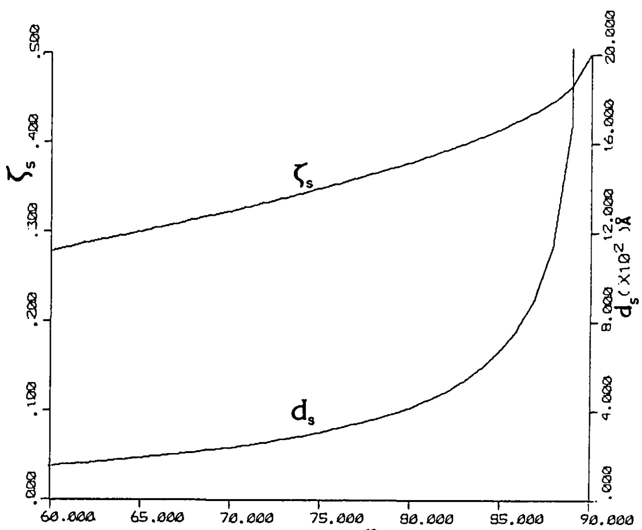

Fig. 5. Normalized ζ_s and actual d_s (angstroms) thicknesses of the s-polarization antireflection layer on Al at $\lambda = 6328$ Å vs angle of incidence ϕ (degrees).

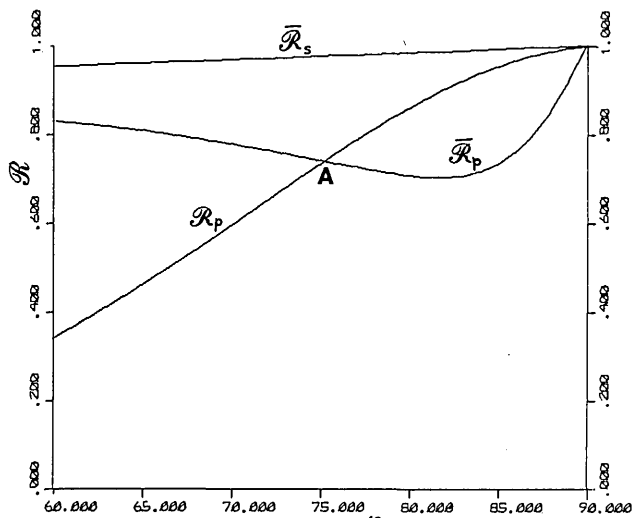

Fig. 6. Unextinguished p reflectance $\mathcal{R}_p$ vs angle of incidence ϕ (degrees) for Al, which is antireflection-coated for the s polarization. $\bar{\mathcal{R}}_p$ and $\bar{\mathcal{R}}_s$ are the p and s reflectances of the bare Al substrate.

0.41425 and $d_s = 661$ Å. Such a layer reduces the reflectance of Al for the s polarization from the bare-surface value of 99.17% to zero, while enhancing its p reflectance from 73.03 to 95.26%. Thus this film–substrate system functions as an excellent reflection polarizer.

IV. Antireflection of the p Polarization

Fresnel's reflection coefficients of the ambient–film and film–substrate interfaces for the p polarization are given by[9]

$$r_{01p} = (\epsilon_1 S_0 - \epsilon_0 S_1)/(\epsilon_1 S_0 + \epsilon_0 S_1), \tag{19a}$$

$$r_{12p} = (\epsilon_2 S_1 - \epsilon_1 S_2)/(\epsilon_2 S_1 + \epsilon_1 S_2), \tag{19b}$$

where ϵ_i and S_i are the same as previously defined in Eqs. (12) and (13). Substitution of Eqs. (19) into Eq. (4) puts the antireflection condition for the p polarization in the form

$$(\epsilon_1^2 S_0^2 + \epsilon_0^2 S_1^2)\,\mathrm{Re}(\epsilon_2^* S_2) = S_0(|\epsilon_2|^2 S_1^2 + \epsilon_1^2 |S_2|^2). \tag{20}$$

By replacing S_i from Eq. (12) into Eq. (20) and rearranging, a quadratic equation

$$A\bar{\epsilon}_1^2 + B\bar{\epsilon}_1 + C = 0 \tag{21}$$

is obtained, where

$$\begin{aligned} A &= \cos^2\phi\,\mathrm{Re}[\bar{\epsilon}_2^*(\bar{\epsilon}_2 - \sin^2\phi)^{1/2}] - \cos\phi|\bar{\epsilon}_2 - \sin^2\phi|, \\ B &= \mathrm{Re}[\bar{\epsilon}_2^*(\bar{\epsilon}_2 - \sin^2\phi)^{1/2}] - \cos\phi|\bar{\epsilon}_2|^2, \\ C &= -(\sin^2\phi)B; \end{aligned} \tag{22}$$

$$\bar{\epsilon}_1 = \epsilon_1/\epsilon_0,\ \bar{\epsilon}_2 = \epsilon_2/\epsilon_0 \tag{23}$$

are normalized film and substrate dielectric constants, respectively. Solving Eq. (21) gives

$$\epsilon_1 = \epsilon_0[-B \pm (B^2 - 4AC)^{1/2}]/2A, \tag{24}$$

from which

$$N_1 = \epsilon_1^{1/2}. \tag{25}$$

Equation (18) has been repeated as Eq. (25) for ease of reference.

Equations (21)–(25) give the desired refractive index of the p-polarization single-layer antireflection coating in terms of the ambient dielectric constant ϵ_0, substrate complex dielectric constant $\epsilon_2 = (n_2 - jk_2)^2$, and angle of incidence ϕ. In the special case of a transparent substrate, ϵ_2 is real, and the result reduces to that given by Knittl.[3] Two values of N_1 are possible that correspond to the two roots of the quadratic equation. They will be denoted by the additional subscripts + and −, according to the + and − signs that appear in Eq. (24). Because N_1 must be real and positive, the following conditions must be satisfied:

$$B^2 - 4AC \geq 0, \tag{26a}$$

$$\epsilon_1 > 0. \tag{26b}$$

We now consider examples of p-polarization antireflection layers on the same semiconducting (Si) and metallic (Al) substrates (at $\lambda = 6328$ Å and in air) as in Sec. III.

Figure 7 shows N_{1+} and N_{1-} plotted vs ϕ between $\phi = 0$ and $\phi = 90°$ for Si. Notice the crossover between the two solutions that takes place as ϕ passes through the pseudo-Brewster angle,[17] $\phi_{pB} = 75.44°$, of Si. If the two solutions of N_1 are ordered according to their magnitudes as low and high, N_{1l} and N_{1h}, we find that $N_{1l} < 1$ for $\phi < \phi_{pB}$ and that N_{1l} is very slightly >1 for $\phi > \phi_{pB} + 0.1°$. This low-index branch does not correspond to practical thin-film coating materials and will not be pursued further.

Figure 8 shows the normalized and actual film thicknesses ζ_p ($= \zeta_{p+} = \zeta_{p-}$) and d_p, respectively, of the p-polarization antireflection layer on Si; d_p is associated with and calculated from N_{1h}, the higher of the two refractive indices N_{1+} and N_{1-}. Significant deviation of the thickness from a quarterwave occurs in the vicinity of the pseudo-Brewster angle.[17]

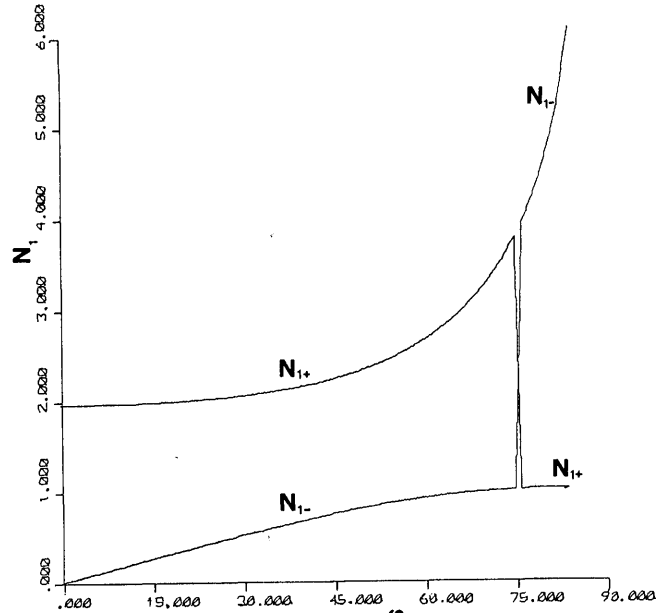

Fig. 7. Refractive indices N_{1+} and N_{1-} of p-polarization antireflection layers on Si ($N_2 = 3.85-j0.02$) as functions of angle of incidence ϕ (degrees). Light ($\lambda = 6328$ Å) is assumed to be incident from air ($N_0 = 1$).

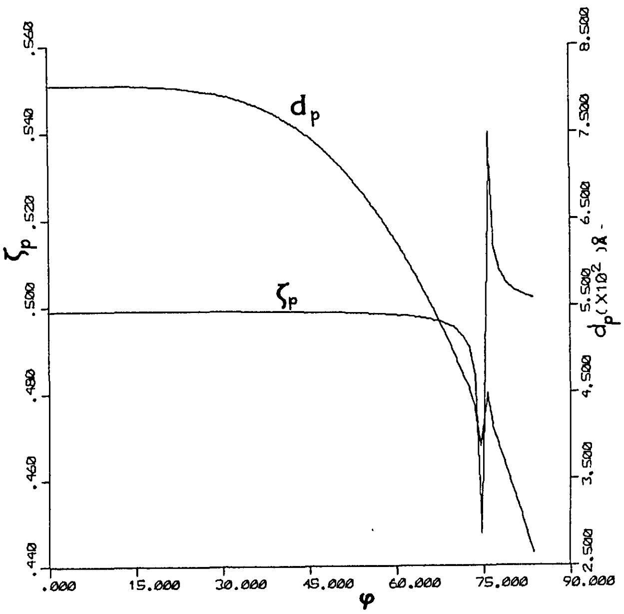

Fig. 8. Normalized ($\zeta_{p+} = \zeta_{p-} = \zeta_p$) and actual ($d_p$, angstroms) thicknesses of the p-polarization antireflection layer on Si at $\lambda = 6328$ Å vs angle of incidence ϕ (degrees). d_p corresponds to the higher of the two refractive indices N_{1+} and N_{1-} of Fig. 7.

Figure 9 shows the unextinguished reflectance $\mathcal{R}_s$ of Si coated with the p-antireflection layer of index N_{1h}. The bare −Si reflectances $\overline{\mathcal{R}}_p$ and $\overline{\mathcal{R}}_s$ for the p and s polarizations are also indicated as functions of ϕ.

Table II summarizes data on p-antireflection layers on Si at $\lambda = 6328$ Å for the same five angles of incidence $\phi = 0, 30, 45, 60$, and 75° considered in Table I. Only the high-index solution is included. The first lines of Tables I and II (at $\phi = 0$) are identical, as expected.

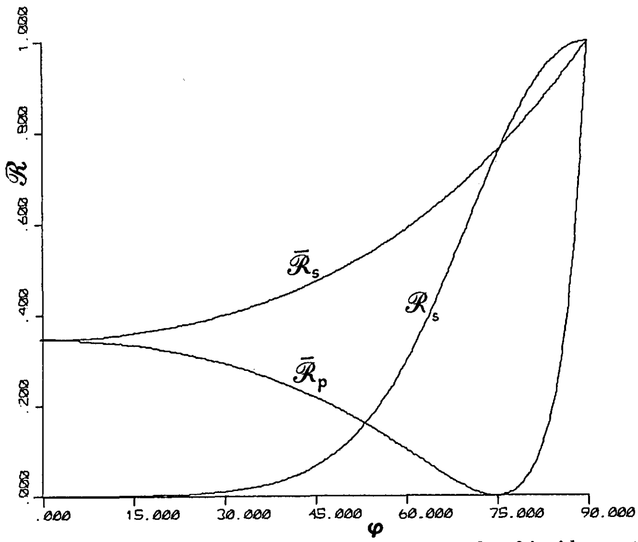

Fig. 9. Unextinguished s reflectance $\mathcal{R}_s$ vs angle of incidence ϕ (degrees) of Si, which is antireflection-coated for the p polarization. The higher of the two refractive indices N_{1+} and N_{1-} of Fig. 7 is assumed. $\overline{\mathcal{R}}_p$ and $\overline{\mathcal{R}}_s$ are the p and s reflectances of the bare Si substrate.

Table II. p-Polarization Antireflection Layers on Si at Five Angles of Incidence [a]

ϕ	N_1	ζ_p	d_p	$\mathcal{R}_s$
0	1.96218	0.49886	804.4	0
30	2.05377	0.49874	792.2	0.0083
45	2.23235	0.49850	744.9	0.0637
60	2.65778	0.49774	626.8	0.2937
75	3.79230	0.44725	385.9	0.7500

[a] ϕ is the angle of incidence in degrees. N_1 is the higher of the two possible refractive indices of the p-polarization antireflection layer. ζ_p and d_p are the normalized and actual (angstroms) layer thicknesses, respectively. $\mathcal{R}_s$ is the unextinguished s reflectance of the p-antireflection-coated Si. The complex refractive index of Si is taken as $N_2 = 3.85-j0.02$ at $\lambda = 6328$ Å. The reflectances $\overline{\mathcal{R}}_p$ and $\overline{\mathcal{R}}_s$ of the bare Si substrate are given in Table I.

For overall antireflection at 45°, s suppression is preferred over p suppression because of the lower associated residual unextinguished reflectance. (At $\phi = 45°$, $\mathcal{R}_p = 3.6\%$, while $\mathcal{R}_s = 6.4\%$ from Tables I and II.)

For the Al substrate, as ϕ is increased from 0 to 34°, Eqs. (21)–(25) yield N_{1+} that increases from 15.07822 to 77.51710 and N_{1-} that increases from $\simeq 0$ to 0.55922 monotonically. These refractive indices are obviously mathematically but not physically acceptable for a single thin film. Between $\phi \simeq 35°$ and $\phi = 88°$, no solutions exist because one or the other of Eqs. (26) is not satisfied. Solutions begin to reappear at $\phi \simeq 88.67°$, and realistic refractive indices are obtained only over the very narrow interval from 88.67 to 89°. N_{1+} and N_{1-} are plotted vs ϕ over this range in Fig. 10. The two solutions merge, $N_{1+} = N_{1-} \simeq \sqrt{2}$, at a point Q that corresponds to an angle ϕ_Q between 88.66 and 88.67°. (At ϕ_Q light is refracted in the film at 45°, and the s polarization is also suppressed at the same film index but at a different film thickness.[18])

The normalized film thicknesses ζ_{p+} and ζ_{p-}, associated with N_{1+} and N_{1-}, respectively, are equal (ζ_{p+}

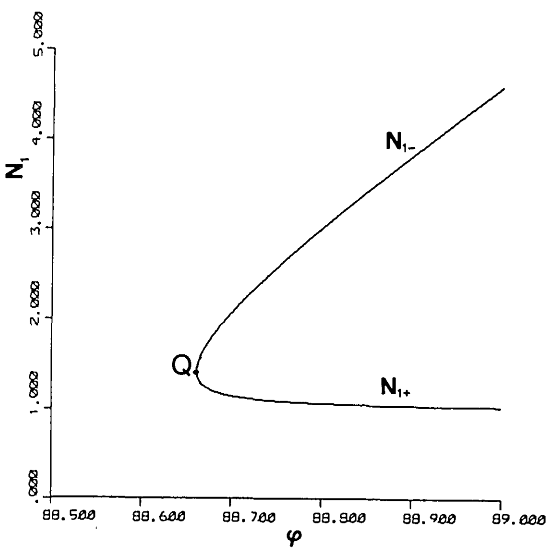

Fig. 10. Refractive indices N_{1+} and N_{1-} of p-polarization antireflection layers on Al ($N_2 = 1.212{-}j6.924$) as functions of the angle of incidence ϕ over the very narrow range $88.5° \le \phi \le 89°$. No solutions for N_1 exist, or far from practical values are obtained, outside this range of ϕ. The two solutions N_{1+} and N_{1-} merge at point Q. Light ($\lambda = 6328$ Å) is assumed to be incident from air ($N_0 = 1$).

$= \zeta_{p-} = \zeta_p$) and decrease monotonically from 0.908415 to 0.811834 as ϕ increases from 88.67 to 89°. The unextinguished reflectance $\mathcal{R}_{s+}$ decreases very slightly from 99.774 to 99.769%, while $\mathcal{R}_{s-}$ increases little from 99.784 to 99.880% monotonically over the same range of ϕ. Except that the angle of incidence is too close to 90°, this film–substrate system acts as a nearly ideal reflection polarizer.

V. Summary

For any given interface between (linear, nonmagnetic, and optically isotropic) transparent and absorbing media, the refractive index N_1 of an intermediate transparent layer can be found that allows suppression of the reflection of p- or s-polarized light at a specified angle of incidence ϕ. The explicit equations that determine N_1 are Eqs. (16)–(18) for the s polarization, and Eqs. (21)–(25) for the p polarization. N_1 must correspond to the refractive index of an existing thin-film coating material for the mathematical solution to be physically realizable by a single film. Thus p and s antireflection will often be possible only over limited ranges of ϕ, depending on the substrate and ambient optical constants. The thickness of the antireflection layer and the reflectance of the coated substrate for the unextinguished orthogonal polarization are calculated, also explicitly, from Eqs. (5)–(8) and Eqs. (9) and (10), respectively. The method is applied to Si and Al substrates at $\lambda = 6328$ Å, and the results appear graphically and in tables.

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8. For given optical constants of the ambient, film, and substrate, Eq. (3) becomes a transcendental equation in the angle of incidence ϕ that can be solved by iteration.[5–7] For high-reflectance (metal) substrates, explicit equations for the polarizing angles, that are approximate but accurate, can be obtained.
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17. Examining the behavior of the solutions in the immediate neighborhood (within ±0.01°) of the pseudo-Brewster angle ϕ_{pB} reveals some interesting detail. For example, there is a gap in this already very narrow range of ϕ where no solutions exist. The gap is approximately bounded by the angles $\phi_1 = 75.4397°$ and $\phi_2 = 75.4401°$ and includes ϕ_{pB}. The solutions N_{1+} and N_{1-} merge at the upper edge of the gap to form one continuous curve. Furthermore, the concident ζ_{p+} and ζ_{p-} tend to zero as the lower gap edge is approached from the left and tend to one as the upper gap edge is approached from the right.
18. Elsewhere we consider directly and in detail the conditions for the extinction of the p and s polarizations at the same angle of incidence by a transparent film on an absorbing substrate along with interesting applications; see R. M. A. Azzam, "Extinction of the p and s Polarizations of a Wave on Reflection at the Same Angle from a Transparent Film on an Absorbing Substrate: Applications to Parallel-Mirror Crossed Polarizers and a Novel Integrated Polarimeter," J. Opt. Soc. Am. A **2**, in press (1985).