12-19-2003

Development of Discontinuous Galerkin Method for 1-D Inviscid Burgers Equation

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DEVELOPMENT OF DISCONTINUOUS GALERKIN METHOD FOR 1-D INVISCID BURGERS EQUATION

A Thesis

Submitted to the Graduate Faculty of the
University of New Orleans
in partial fulfillment of the
requirements for the degree of

Master of Science
in
The Department of Mechanical Engineering

by

Kiran Voonna

B.E.,Mechanical Production and Industrial Engineering, Andhra University, India, 1999.

December 2003

ACKNOWLEDGEMENT

I wish to express my sincere gratitude to my advisor Professor Martin J. Guillot for his invaluable guidance, and support throughout this work. He has taught me many things and this work would not have been possible without his patience and encouragement.

I would like to special thank and respect Professor Kazim M. Akyuzlu who is one of my beloved professors. He gave his advices and directions for perfect work in studies during my course work. I have taken three courses with him. All of his courses have given me in-depth knowledge in subject. I would also like to thank Professors Dr. Paul D. Herrington and Dr. Carsie A. Hall for serving on my committee and for reviewing the work and the dissertation.

I would also like to thank my family especially my beloved brother Mr.Santhi Swaroop Voonna for his support and encouragement through out my studies.

Finally I would like to express my respect and thanks to Professor P.K.Sarma who has taught me courses in Fluids and Thermal Sciences during my four year undergrad in India.

DEDICATION

I dedicate this work to my mother Mrs.Vardhani Voonna, for whom I have tremendous respect, my father Mr.Satyanarayana Voonna, who is my inspiration and my cousin Mr.Venkata Ramana Rao Adapa, who is my friend, philosopher and guide.

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NOMENCLATURE

Symbols:

u : Velocity

C : Propagation wave speed

f : flux function

w : weight function

f^R : Riemann's flux

C : Courant number

υ : Viscosity

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ABSTRACT

The main objective of this research work is to apply the discontinuous Galerkin method to a classical partial differential equation to investigate the properties of the numerical solution and compare the numerical solution to the analytical solution by using discontinuous Galerkin method. This scheme is applied to 1-D non-linear conservation equation (Burgers equation) in which the governing differential equation is simplified model of the inviscid Navier-stokes equations. In this work three cases are studied. They are sinusoidal wave profile, initial shock discontinuity and initial linear distribution. A grid and time step refinement is performed. Riemann fluxes at each element interfaces are calculated. This scheme is applied to forward differentiation method (Euler's method) and to second order Runge-kutta method of this work.

CHAPTER 1

1. INTRODUCTION:

Burgers equation is a non-linear conservation equation of second order that contains a flux term. The numerical solution of the unsteady Navier –Stokes (NS) equation is challenging problem for computational fluid dynamics that requires careful mathematical and numerical formulation. As a simplified model of the Navier-Stokes equation, the one-dimensional viscous Burgers equation represents many of the properties of NS equations, such as non-linear convection and viscous diffusion, leading to shock waves and boundary layers. Burgers equation is used in computational fluid dynamics as a simplified model for turbulence, boundary layer behavior, shock wave formation, and mass transport.

In this work one-dimensional inviscid Burgers equation is studied. Many of the research developments for numerical solutions have been developed within the context of one-dimensional equations before attempting to apply the methods to high dimensional problems. Numerical methods can be applied to a two-dimensional equation by using operator-splitting method in which the two-dimensional problem is divided into a sequence of two separate one-dimensional problems. This approach uses upwind differentiating in inertia term and convective term of the inviscid Burger's equation, which requires the methods such as flux vector splitting or Godunov's discretization method.

This thesis work presents solutions to the inviscid Burgers equation using discontinuous Galerkin method (DGM). We have developed numerical solutions for initial conditions including sinusoidal wave profile, initial shock discontinuity and initial linear distribution and have compared them to analytical solutions. Forward differentiation and second order Runge-kutta methods are applied to develop the numerical solutions for inviscid Burgers equation.

There are several methods to develop the numerical solutions for inviscid Burgers equations. They are the finite element method, the finite difference method and the finite volume method. In this present work discontinuous Galerkin finite element method is used for 1-D inviscid Burgers equation. The Riemann problem is solved at the interface of each element. The Riemann problem is defined as a hyperbolic equation with discontinuous initial conditions.

This research work deals with solving the one-dimensional non-linear conservation equation using an explicit discontinuous Galerkin finite element method based on a Godunov's scheme. The flux at the interface of each element is solved by the Riemann problem. Boundary conditions are applied to the solution of the one-dimensional inviscid Burgers equation. Investigated properties of numerical solutions and comparison of these solutions with analytical solutions are documented.

CHAPTER 2

2. LITERATURE REVIEW:

The technical value of the computational fluid dynamics has become undisputed as it has been providing wide applications in evaluating numerical solutions for fluid engineering problems. A capability has been established to compute flows that can be investigated experimentally only at reduced Reynolds number, or at greater cost, or not at all, such as the flow around a space vehicle at re-entry, or a loss-of-coolant accident in nuclear reactor. Furthermore, modern computational fluid dynamics has become indispensable for design optimization, because many different configurations can be investigated at acceptable cost and in short time.

Fluid dynamics is a classic discipline. The physical principles governing the flow of simple fluids and gases such as water and air have been understood since the times of Newton. However numerical simulations have become essential in developing engineering analysis and design of the hydraulic systems to overcome the limitations of some flow fields such as, modeling of tunnels and big hydraulic flow fields.

Burgers equation is similar to transport equation except that the convective term is now nonlinear. Burgers equation was used as simple model of turbulence in an extensive study by Burgers (1974). Cole (1951) showed that the Burgers equation could be transformed into the diffusion equation. Kazuo Ito (1994) developed a globally existing solution to the inviscid Burgers equation with a nonlocal term. It is shown that the inviscid Burgers equation, with a nonlocal nonlinear term admits smooth global solutions for certain initial data which is smooth and nondecreasing. Burgers equation is a very suitable model for testing computational algorithms for flows where severe gradients or shocks are anticipated. This computational algorithm, facilitated by Cole-Hopf transformation, which allows exact solution of Burgers equation to be obtained for many combinations of initial and boundary conditions. Fletcher (1983a) provided many examples of the use of Burger's equation. Explicit schemes, implicit schemes have been developed to the propagation of shock wave governed by the viscous Burgers equation. Fletcher (1984) obtained a solution for the one-dimensional Burgers equation with a shock-like internal layer by using Galerkin spectral methods that shows global oscillations for a nine-term Legendre polynomial solution. Basdevant (1986) obtained solutions for 1-D Burgers equation for spectral tau and collocation methods based on Chebyshev polynomials. Burger's equation has been used to demonstrate various computational algorithms for convection-dominated flows.

Explicit MacCormack scheme has been applied to two-dimensional Burgers equation. Through the Runge-Kutta marching scheme it is possible to obtain stable solutions with even higher values of Courant number than MacCormack scheme. Courant and Friedrichs (1948) developed numerical methods for inviscid Burgers equation. The exact solution of the nonlinear inviscid Burgers equation is found using the Cole-Hopf transformation. Whitham (1974) developed numerical schemes for quasilinear form of conservation law equations.

The Finite element method is a numerical analysis technique for obtaining an approximate solution for a wide range of engineering problems. The finite element method first appeared in the 1950's as a technique for handling the problems in solid mechanics. Later on it was developed for fluid dynamics problems. Turner, Martin, and Topp (1956) had applied this method to aircraft structural analysis. Finite element methods have been applied to fluid flow problems steady-state heat conduction, potential flow in ideal fluids, nuclear engineering and aeronautical engineering problems.

The appearance of discontinuities even when starting from smooth initial data is a generic situation for non-linear hyperbolic PDE's were shown. To define what is meant by a solution in such cases, the concept of weak solutions was introduced, which involved integrating the discontinuous solution over some domain. This suggests that it might be advantageous to construct a numerical method, which involves an integration step. A case also exists where a large class of PDE's of practical interest is derived from conservation laws in which a direct expression of the quantity being conserved might prove useful in numerical algorithm. To evaluate a conserved quantity an integration step is again required.

In recent years, several numerical schemes have been developed. These schemes are satisfying hyperbolic conservation laws, montonicity preserving, hydraulic phenomenon and shock capturing properties.

Reed and Hill (1973) initially introduced the discontinuous Galerkin method as a technique to solve neutron transport problems. However, the technique lay dormant for several years and has only become popular as a method for solving fluid dynamics or electromagnetic problems. The DGM is somewhere between a finite element and a finite volume method and has many good features of both the methods. The use of DGM is uncommon in applications, but they rest on a reasonable mathematical basis for low-order cases and have local approximation features that can be exploited to produce very efficient schemes especially in parallel multi processor environment. The result of typical mathematical formulations that has influenced the development of many numerical schemes suggests discontinuous Galerkin methods. The local nature of the DGM requires communications of the solution only along sub-domain boundaries at each time step. DG method is only applied to a convective term in fully hyperbolic system of equations.

One reason for renewed interest in DGM is the advent of parallel computation. The decomposition of large –scale problems into several computational components that can be handled simultaneously by multiple processors, makes possible significant improvements in the efficiency with which large hyperbolic systems can be resolved.

Cockburn, Karnidakis and Shu (2000) introduced higher order schemes for hyperbolic equations in one and two-dimensions. Cockburn and Shu (1998) introduced the system of ordinary differential equations resulting from discontinuous Galerkin spatial approximation is marched in time using a multi-stage Runge-Kutta scheme. They have applied DG method with explicit Runge-kutta methods for the time discritisation to conservative hyperbolic systems. They also introduce the concepts of characteristic decomposition and splitting of element boundary fluxes and local projection limiters for resolving sharp gradient, such as shocks, without oscillations.

In addition, many interesting wave phenomenon in fluids are considered using examples such as acoustic waves, the emission of air pollutants, magneto hydrodynamic waves in the solar corona, solar wind interaction with the planet Venus, and ion-acoustic solutions.

Hydraulic conservation laws play a central role in mathematical modeling in several distinct disciplines of science and technology. Application areas include compressible, single (and multiple) fluid dynamics, shock waves, meteorology, elasticity, magneto hydrodynamics, relativity, and many others. The successes in the design and application of new and improved numerical methods of the Godunov type for hyperbolic conservation laws in the last twenty years have made a dramatic impact in these application areas. This method a cover very wide range of topics, such as design and analysis of numerical schemes, applications to compressible and incompressible fluid dynamics, multi-phase flows, combustion problems, astrophysics, environmental fluid dynamics, and detonation waves.

The following description reveals different schemes and methods invented on numerical solutions of computational fluid dynamics.

The method of upwinding explicit depends on directional descretization of the flux derivatives. After several extensions of second –order accuracy techniques, implicit time integrations have been developed.

The method of Lax –wendroff is second-order accurate and has led to a family of variants when applied to non-linear systems, characterized by their common property of being space centered, reducing to three-point central schemes in one-dimension, explicit in time and derived form a combined space and time discretization.

Runge-Kutta methods were proposed by Shu and Osher (1988). Runge-Kutta methods are mainly appreciated for their high-order of accuracy. There are many Rung-Kutta schemes. Using Runge-Kutta time stepping in finite-element algorithm gives good results in developing numerical solutions. Runge-Kutta methods can be used to obtain time-marching scheme.

CHAPTER 3

3.1 PROPERTIES OF HYPERBOLIC EQUATION:

The term hyperbolic equation refers to members of a specific class of partial differential equations. The most representative examples of this class are the partial differential equations that describe wave phenomena. In essence, the study of hyperbolic equations and the mathematical investigation of wave phenomena can be thought of as one and the same thing. A wave is a disturbance that propagates. The wave equation is a differential equation, which describes a harmonic wave passing through a certain medium. The equation has different forms depending on how the wave is transmitted, and in what medium. A system of quasi-linear partial differential equations, which expresses the conservative form of the physical conservation laws, will be hyperbolic, if the homogenous parts of the quasi-linear partial differential equations admit wave like phenomena. Wave phenomena are that oscillation cannot be particularly large. If we write the mathematical formulae that govern the propagating wave phenomena and describe the properties of wave phenomena, then the phenomena that conform to partial differential equations are known as hyperbolic equations. If partial differential equations allow the solution for damped waves then the system is known as parabolic. If the system of partial differential equations allows diffusion phenomena then the system of equations is known as elliptic.

3.2 MATHEMATICAL FORMULATION:

The main objective of this study is to develop a numerical solution for a 1-D inviscid Burgers equation by using discontinuous Galerkin method.

Burgers equation is a non-linear conservation equation, that possess readily computable exact solutions for many combination of initial and boundary conditions. For this reason it is an appropriate model equation on which various computational techniques can be tested. This feature has been exploited for one-dimensional Burger's equation.

We consider a 1-D convection term in Burgers equation to develop numerical solution.

Following assumptions are made to derive the invisicid Burgers equation.

1. The flow is unsteady 1-D conservation equation.
2. Fluid is incompressible and inviscid.
3. Velocity distribution is in x-direction as 1-D equation.

Standard form of viscous Burgers equation is

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\frac{u^2}{2}=\upsilon\frac{\partial^2 u}{\partial x^2} \tag{1}$$

where υ is viscosity.

By considering the above assumptions, if the viscous term dropped from equation (1) the result is the inviscid Burgers' equation.

The inviscid Burgers equation is a scalar conservation law with quadratic flux function.The standard form of this equation is,

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\frac{u^2}{2} = 0 \tag{2}$$

In quasi-linear form of equation (2) is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{3}$$

The nonlinearity in equation (2) allows discontinuous solution to develop. In the literature of hyperbolic equations, the inviscid problem (2) has been widely used for developing both theory and numerical methods.

Equation (2) can be written as

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \tag{4}$$

Where the flux is arbitrary function of velocity u and is denoted as $f(u)$. Consider the flux function $f = \frac{u^2}{2}$ and consider $u = u(x,t)$ as the dependent variable.

3.3 PHYSICAL PROBLEM:

In this work we resolved velocity distribution for inviscid Burgers flow by using discontinuous Galerkin method. With this velocity distribution we developed numerical solution for sinusoidal wave profile, initial shock discontinuity and initial linear distribution. Those physical descriptions are mentioned below.

1. Sinusoidal wave profile:

The time evaluation by sinusoidal wave profile is the good test case for unsteady flows with shock information and propagation.

$$g(x) = \begin{cases} A\sin\dfrac{\pi x}{L} + u_o & 0 \le x \le L \\ u_0 & x < 0, x > L \end{cases}$$

The formation of shock is given by the following relations.

For $t >> t_s$ the shock moves at speed

Where t_s is time of shock formation.

Shock speed is given by,

$C = u_0 + 1/2\sqrt{B/t}$

and has an intensity of velocity

$[u] = \sqrt{B/t}$

Where $\dfrac{B}{2}$ is the area under the sinusoidal curve, which remains constant.

$$B = 2\int_0^L [g(x) - u_0]dx = \frac{4AL}{\pi}$$

The expansion part takes on a linear shape asymptotically,

$$u(x,t) \approx \frac{x}{t} \qquad u_0 t < x < u_0 t + \sqrt{Bt}$$

So that the amplitude decreases as $t \to \infty$, while the shock velocity decreases, both at a rate $\propto 1/\sqrt{t}$ if $u_0 = 0$. This solution is illustrated in fig.1.

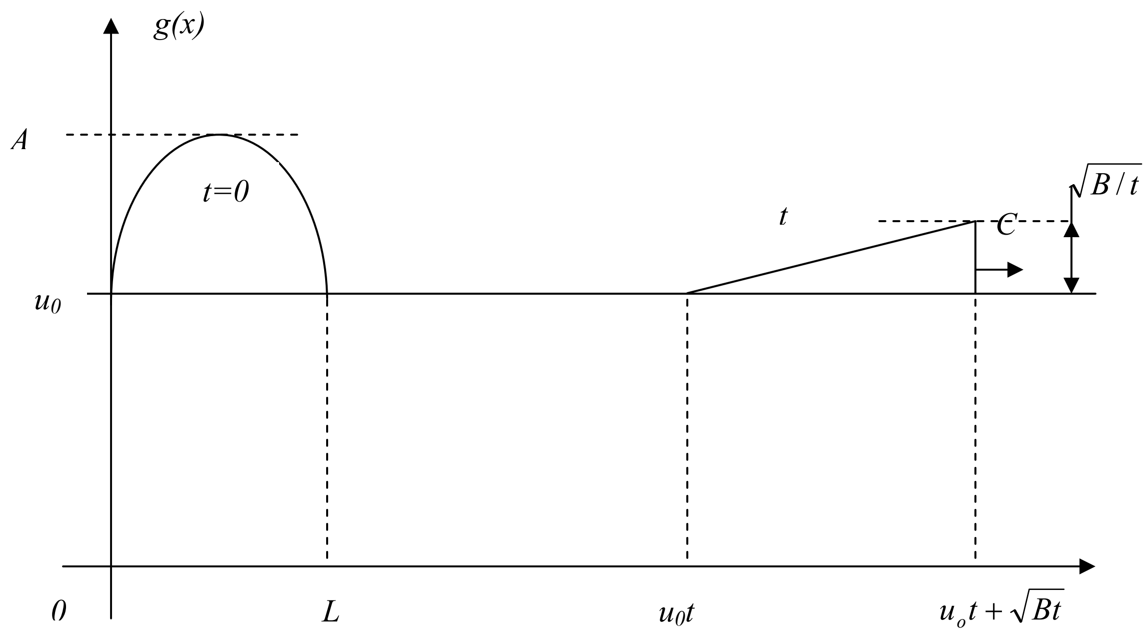


Fig.1 Solution for Burgers equation from an initial sinusoidal distribution.

2. Initial shock discontinuity:

This test case can be applied to steady and unsteady computations. This phenomena is provided by an initial discontinuous distribution

$$u(x,0) = g(x) = \begin{cases} u_L & x < 0, t = 0 \\ u_R & x > 0, t = 0 \end{cases}$$

Shock propagating speed $C = (u_L + u_R)/2$ with unmodified intensity $[u] = u_L - u_R$.

If $u_R = u_L$ the shock is stationary then this form is simple, although non-linear, test case for steady-state methods. This solution is illustrated in Fig.2

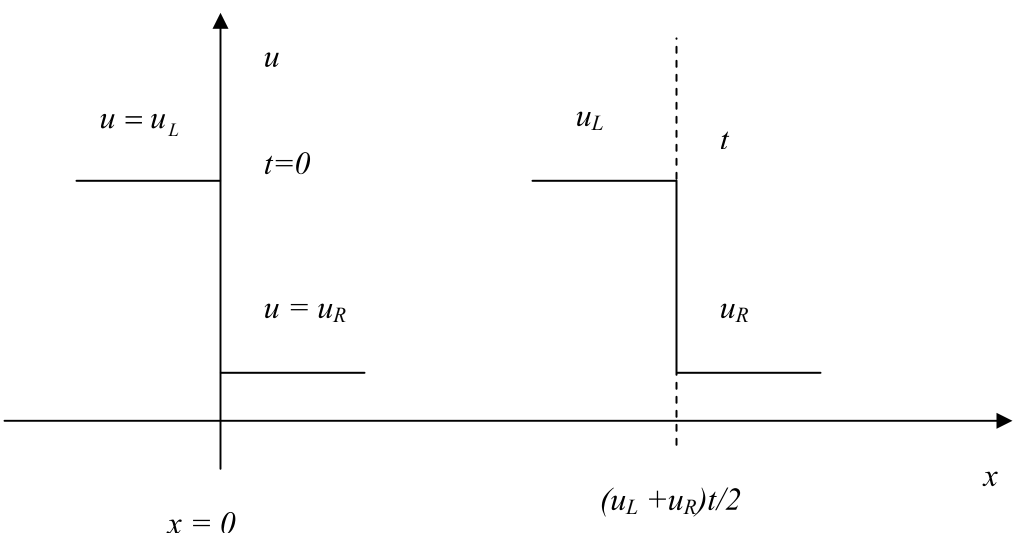


Fig.2 Burgers solution for a propagating discontinuity.

3. Initial linear distribution:

Any initial distribution with $g'(x) < 0$ between u_L and u_R will lead to the same shock structure for large times. For instance, a linear distribution is illustrated in fig.3.This formulation will lead to the solution shown in fig.3, where the characteristics are also indicated. The shock is formed at a time given t_s is shown below.

$$u(x,0) = g(x) = \begin{cases} u_L & x < 0 \\ u_L(1-\frac{x}{L}) + u_R\frac{x}{L} & 0 \le x \le L \\ u_R & x > L \end{cases}$$

$$t_s = \frac{L}{u_L - u_R}$$

and at the position $x_s = t_s.u_L = L + t_s.u_R$

The solution is therefore, for $t > t_s$

$$u(x,t) = \begin{cases} u_L & for \quad x < \frac{u_L + u_R}{2}t \\ \\ u_R & for \quad x > \frac{u_L + u_R}{2}t \end{cases}$$

This solution is illustrated in Fig.3

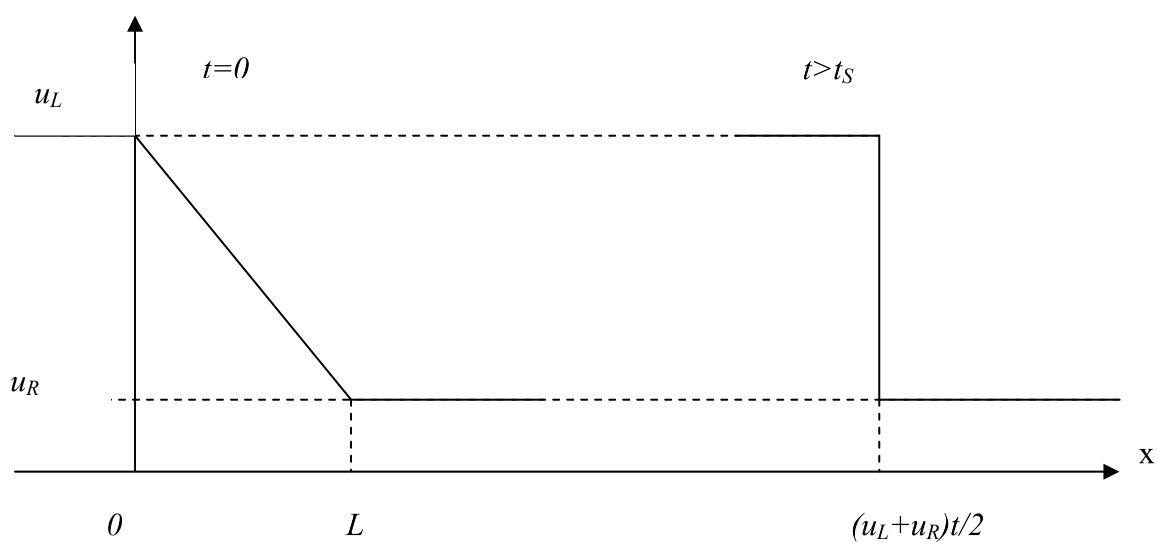


Fig.3 Shock formation from an initial linear distribution.

CHAPTER 4

4. NUMERICAL FORMULATION*:*

Historically, many of the fundamental ideas of numerical solutions were first developed for the special case of compressible gas dynamics (the Euler equations), for applications in aerodynamics, astrophysics, detonation waves, and related fields where shock waves arise. Burger's equation and the shallow water equations have played an important role in development of these methods.

In recent years a powerful class of numerical methods has been developed for approximating solutions, including both linear and non-linear conservation laws. These equations describe a wide range of wave propagation and transport phenomena arising in nearly every scientific and engineering discipline. Several applications are being made along with mathematical theory of hyperbolic equations. Recently developed high-resolution FEM (Finite Element Methods) techniques provide an excellent learning environment for understanding wave-propagation phenomena and finite element methods.

DGM (Discontinuous Galerkin Method) in FEM has wide applications even it is uncommon in use. The DG method has reasonable mathematical basis for low order cases and local approximation features produce very efficient schemes.

The mathematical theory of nonlinear hyperbolic problems is beautiful, and the development and analysis of finite element methods requires a good interplay between this mathematical theory, physical modeling, and numerical analysis, so that finite element methods can be developed for nonlinear conservation laws.

4.1 Finite Element method:

The basic concept of the finite element method is that a body or structure may be divided into smaller elements of finite dimensions called finite elements. It is important to insure that the numerical techniques conserve the physical quantities such as mass, momentum, charge and energy. Finite element methods have also been developed to solve integral equations. Consequently, it is often preferable to use a finite-element method rather than a finite difference method in which u_i^n is considered as an approximation to the average value of $u(x,t)$ over a element rather than a point wise value of u. This average value is the integral of u over the cell divided by its length. The spatial divergence (derivative) terms are expressed as a surface integral of fluxes that are approximated with the use of solutions at two adjacent finite-elements. The value $u(x,t)$ will be approximated by its average value over the j^{th} numerical element at the time $t_n = n\Delta t$,

$$\overline{u_j^n} \cong \frac{1}{\Delta x_j} \int_{x_{j,i}}^{x_{j,i+1}} u(x,t_n)dx$$

Where $\Delta x_j = x_{j,i+1} - x_{j,i}$, the index i and $i+1$ corresponds to left and right interfaces of the j^{th} element respectively.

4.2 Applying DG method to 1-D nonlinear Burgers flux equation:

Galerkin's method uses the set of governing equations in the development of an integral form. It is usually presented as one of the weighted residual methods. Discontinuous Galerkin method is well suited for large-scale time-dependent computations in which high accuracy is required. Its compact formulation can be applied near boundaries without special treatment, which greatly increases the robustness and accuracy of any boundary condition implementation. DG method can be combined with explicit time-marching methods, such as Runge-Kutta. One of the disadvantages of the method is its high storage and high computational requirements. In the Galerkin method, the weighting functions w_j are chosen from basis function.

Here DG method is defined by choosing a weight function w, this is denoted as shown below,

(5) $$w_j = w_{j,0} + (x - \bar{x})\frac{\partial w_j}{\partial x}$$

Where $\bar{x} = \frac{x_{j,i} + x_{j,i+1}}{2}$ is the average length of j^{th} element.

The weight functions are chosen as the approximating functions.

And velocity u function over the j^{th} element is

(6) $$u_j = u_{j,0} + (x - \bar{x})\frac{\partial u_j}{\partial x}$$

Where $u_{j,0}$ is the average velocity of j^{th} element and $\frac{\partial u_j}{\partial x}$ is linear slope of j^{th} element.

Multiplying the equation (4) with test or weight function on both sides and integrates over an j^{th} element. Then the equation (4) becomes,

(7) $$\int_{x_{j,i}}^{x_{j,i+1}} (\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x}) w dx = 0$$

4.3 1-D Discontinuous Galerkin finite element method:

In this research work discontinuous Galerkin finite element scheme is based on the discrete equations, which are constructed by expressing the integral conservation law on a discrete set of elements. Mathematical formulation for Burgers equation by using FEM based on integral form instead of differential equation. High resolution finite element methods that have provided to be very effective for computing discontinuous solution. A fundamental tool in the development of the discontinuous Galerkin finite element method is the Riemann problem, which is simply the hyperbolic equation together with discontinuous initial data. Since PDEs to continue away from discontinuities, one possible approach is to combine an FEM in smooth regions with some explicit procedure for tracking the locations of discontinuities. According to DG procedure techniques integrate the differential conservation law given in equation (7) over an jth element $x_{j,i} \le x \le x_{j,i+1}$. That can be shown below in equation (8). A finite element model of a problem gives piecewise approximation to the governing equations. The basic premise of finite element method is that a solution region can be analytically modeled or approximated by replacing it with an assemblage of discrete elements. 1-D mesh with nodes and cell nomenclature shown in fig.4.

(8) $$\int_{x_{j,i}}^{x_{j,i+1}} \frac{\partial u}{\partial t} w \ dx + \int_{x_{j,i}}^{x_{j,i+1}} \frac{\partial f}{\partial x} w \ dx = 0$$

Expansion of equation (8) becomes,

$$(9) \qquad \frac{d}{dt}\int_{x_{j,i}}^{x_{j,i+1}} u \;\; w \;\; dx + \left[f \;\; w \;\right]_{x_{j,i}}^{x_{j,i+1}} - \int_{x_{j,i}}^{x_{j,i+1}} f \;\; \frac{\partial w}{\partial x} dx = 0$$

We can use this equation (9) to develop an explicit time-marching algorithm.

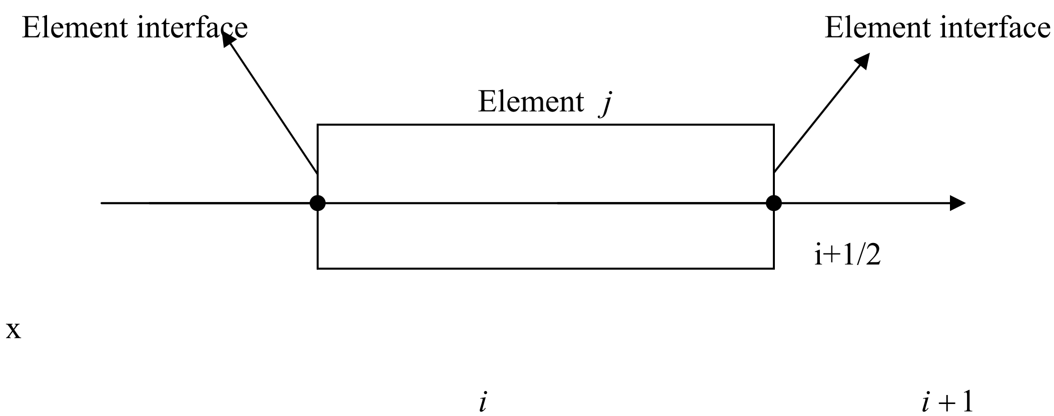


Fig.4. 1-Dimensional Finite element mesh.

Where i and $i+1$ are the j^{th} element left and right interfaces respectively.

Equation (9) can be written as,

$$(10) \qquad \frac{d}{dt}\int_{x_{j,i}}^{x_{j,i+1}} u \;\; w \;\; dx = -\left[f_i^R w_{j,i} + f_{i+1}^R w_{j,i+1} \right] + \int_{x_{j,i}}^{x_{j,i+1}} f \;\; \frac{\partial w}{\partial x} dx$$

Above equation (10) can be written in matrix form

Where

$$\int_{x_{j,i}}^{x_{j,i+1}} u \;\; w \;\; dx = \int_{x_{j,i}}^{x_{j,i+1}} \begin{bmatrix} w_{j,0} \\ (x - \bar{x}) \frac{\partial w_j}{\partial x} \end{bmatrix} \begin{bmatrix} u_{j,0} & (x - \bar{x}) \frac{\partial u_j}{\partial x} \end{bmatrix} dx$$

$$= \Delta x_j \begin{bmatrix} u_{j,0} & \dfrac{\partial u_j}{\partial x}(x - \overline{x}) \\ u_{j,0}(x - \overline{x}) & \dfrac{\partial u_j}{\partial x}(x - \overline{x})^2 \end{bmatrix}$$

$$= \quad \Delta x_j \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{\Delta x_j^2}{12} \end{bmatrix} \begin{bmatrix} u_{j,0} \\ \dfrac{\partial u_j}{\partial x} \end{bmatrix} \qquad (11)$$

Integrating the flux terms by parts,

$$\int_{x_{j,i}}^{x_{j,i+1}} f \quad \frac{\partial w}{\partial x} dx = \int_{x_{j,i}}^{x_{j,i+1}} \frac{\left[u_j\right]^2}{2} \frac{\partial w_j}{\partial x} dx$$

$$= \frac{1}{2} \int_{x_{j,i}}^{x_{j,i+1}} \left[\Delta x_j (u_{j,0})^2 + \frac{\Delta x_j^3}{12} (\frac{\partial u_j}{\partial x})^2 \right] \begin{bmatrix} 0 \\ \dfrac{\partial w_j}{\partial x} \end{bmatrix} dx$$

$$= \frac{\Delta x_j}{2} \begin{bmatrix} 0 & 0 \\ u_{j,0} & \dfrac{\Delta x_j^2}{12} \dfrac{\partial u_j}{\partial x} \end{bmatrix} \begin{bmatrix} u_{j,o} \\ \dfrac{\partial u_j}{\partial x} \end{bmatrix} \qquad (12)$$

and

$$\left[f_{i+1}^R w_{j,i+1} - f_i^R w_{j,i} \right]$$

$$= \left\{ \begin{bmatrix} w_{j,0} \\ (x_{j,i+1} - \overline{x}) \dfrac{\partial w_j}{\partial x} \end{bmatrix} f_{i+1}^R - \begin{bmatrix} w_{j,0} \\ (x_{j,i} - \overline{x}) \dfrac{\partial w_j}{\partial x} \end{bmatrix} f_i^R \right\}$$

$$= \begin{bmatrix} (f_{i+1}^R - f_i^R) \\ \dfrac{\Delta x_j}{2} (f_{i+1}^R + f_i^R) \end{bmatrix} \qquad (13)$$

Substitute all the above equations (11), (12), and (13) into equation (10)

Then the equation (10) becomes,

$$(14)\quad \frac{d}{dt}\Delta x_j \begin{bmatrix} 1 & 0 \\ 0 & \frac{\Delta x_j^2}{12} \end{bmatrix} \begin{bmatrix} u_{j,0} \\ \frac{\partial u_j}{\partial x} \end{bmatrix} = \frac{\Delta x_j}{2} \begin{bmatrix} 0 & 0 \\ u_{j,0} & \frac{\Delta x_j^2}{12}\frac{\partial u_j}{\partial x} \\ & \end{bmatrix} \begin{bmatrix} u_{j,o} \\ \frac{\partial u_j}{\partial x} \end{bmatrix} - \begin{bmatrix} (f_{i+1}^R - f_i^R) \\ \frac{\Delta x_j}{2}(f_{i+1}^R + f_i^R) \end{bmatrix}$$

Where Riemann's flux can be denoted as $f_i^R = f^R(u_{j-1,i}^n, u_{j,i}^n)$, $f_{i+1}^R = f^R(u_{j,i}^n, u_{j+1,i}^n)$. These two fluxes are evaluated at i and $i+1$ interface. This Riemann flux with nodal and cell nomenclature is shown in fig.5

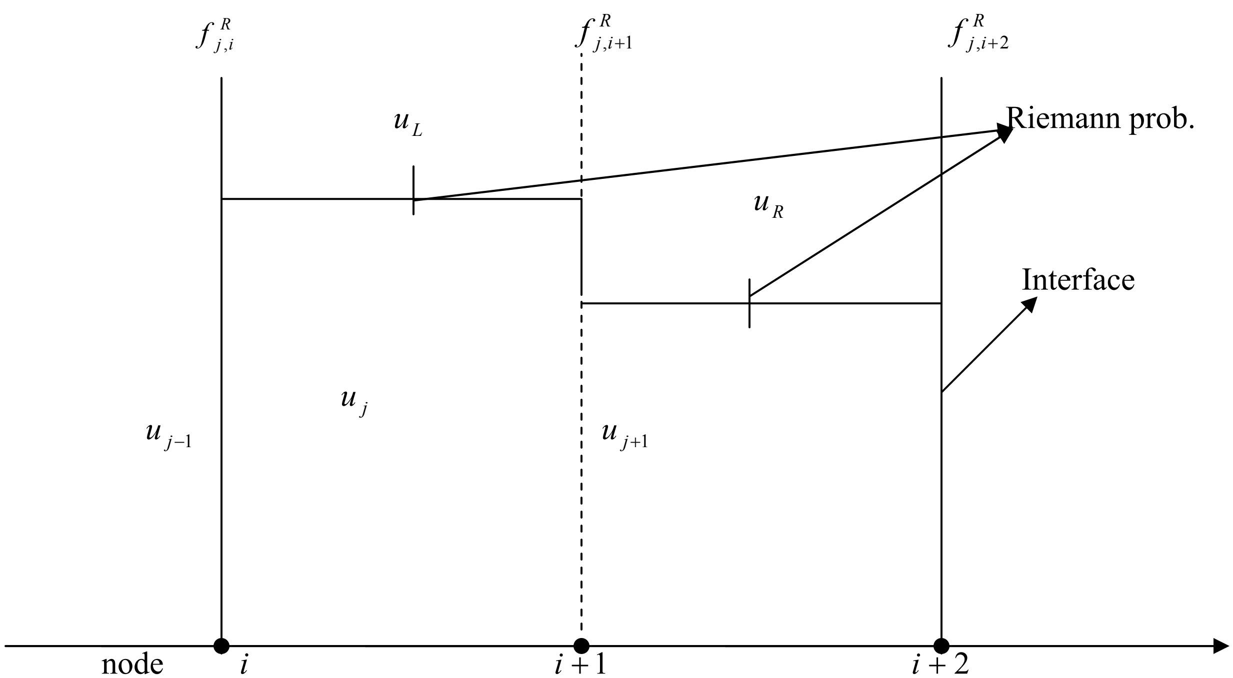


Fig.5 Schematic representation of Reimann flux.

4.4 Explicit forward differencing in time (Euler's method):

Here explicit forward differencing scheme in time is applied in equation (14). Velocity distribution and change of velocity with respect to distance at each interface of an j^{th} element can be approximated from these equations.

Discrete the system in space using the DG method with first order polynomial, and discretize in time using explicit forward differencing technique. Then the equation (14) becomes,

(15)

$$\Delta x_j \left\{ \frac{\begin{bmatrix} 1 & 0 \\ 0 & \frac{\Delta x_j^2}{12} \end{bmatrix} \begin{bmatrix} u_{j,0} \\ \frac{\partial u_j}{\partial x} \end{bmatrix}^{n+1} - \begin{bmatrix} 1 & 0 \\ 0 & \frac{\Delta x_j^2}{12} \end{bmatrix} \begin{bmatrix} u_{j,0} \\ \frac{\partial u_j}{\partial x} \end{bmatrix}^{n}}{\Delta t} \right\} = \frac{\Delta x_j}{2} \begin{bmatrix} 1 & 0 \\ 0 & \frac{\Delta x_j^2}{12} \frac{\partial u_j}{\partial x} \end{bmatrix} \begin{bmatrix} u_{j,0} \\ \frac{\partial u_j}{\partial x} \end{bmatrix}^{n} - \begin{bmatrix} (f_{i+1}^R - f_i^R) \\ \frac{\Delta x_j}{2} (f_{i+1}^R + f_i^R) \end{bmatrix}^{n}$$

Expand the equation (15) and divide with Δx_j on both sides.

Then the equation becomes,

$$(16) \begin{bmatrix} 1 & 0 \\ 0 & \frac{\Delta x_j^2}{12} \end{bmatrix} \begin{bmatrix} u_{j,0} \\ \frac{\partial u_j}{\partial x} \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\Delta x_j^2}{12} \end{bmatrix} \begin{bmatrix} u_{j,0} \\ \frac{\partial u_j}{\partial x} \end{bmatrix}^{n} + \frac{\Delta t}{2} \begin{bmatrix} 0 & 0 \\ u_{j,0} & \frac{\Delta x_j^2}{12} \frac{\partial u_j}{\partial x} \end{bmatrix}^{n} \begin{bmatrix} u_{j,0} \\ \frac{\partial u_j}{\partial x} \end{bmatrix}^{n} - \frac{\Delta t}{\Delta x_j} \begin{bmatrix} (f_{i+1}^{R^n} - f_i^{R^n}) \\ \frac{\Delta x_j}{2} (f_{i+1}^{R^n} - f_i^{R^n}) \end{bmatrix}$$

On expanding equation (16), average velocity and linear slope of j^{th} element at time level $(n+1)$ are derived as follows.

$$(17) \qquad u_{j,0}^{n+1} = u_{j,0}^{n} + (f_i^{R^n} - f_{i+1}^{R^n}) \frac{\Delta t}{\Delta x_j}$$

$$(18) \qquad \frac{\partial u_j^{n+1}}{\partial x} = \frac{\partial u_j^{n}}{\partial x} + \frac{\Delta t}{2} \left[\frac{12}{\Delta x_j^2} (u_{j,0}^{n})^2 + \left(\frac{\partial u_j^{n}}{\partial x}\right)^2 \right] - \frac{6 \Delta t}{\Delta x_j^2} (f_{i+1}^{R^n} + f_i^{R^n})$$

Where time averages of the flux at each interface can be given as $f_{i+1}^{R^n} = \frac{1}{\Delta t}\int_n^{n+1} f(u(x_{j,i+1},t))dt$ and $f_i^{R^n} = \frac{1}{\Delta t}\int_n^{n+1} f(u(x_{j,i},t))dt$.

For hyperbolic problem information propagates with finite speed, so it is reasonable to first suppose that we can obtain $f_i^{R^n}$ and $f_{i+1}^{R^n}$ based on the on the values of $f(u_{j-1,i}^n, u_{j,i}^n)$ and $f(u_{j,i}^n, u_{j,i+1}^n)$ respectively which are called as numerical flux functions. Numerical Riemann fluxes of Burgers equation at each interface are approximated by Godunov's scheme.

Piece wise constant approximation is shown in fig.6

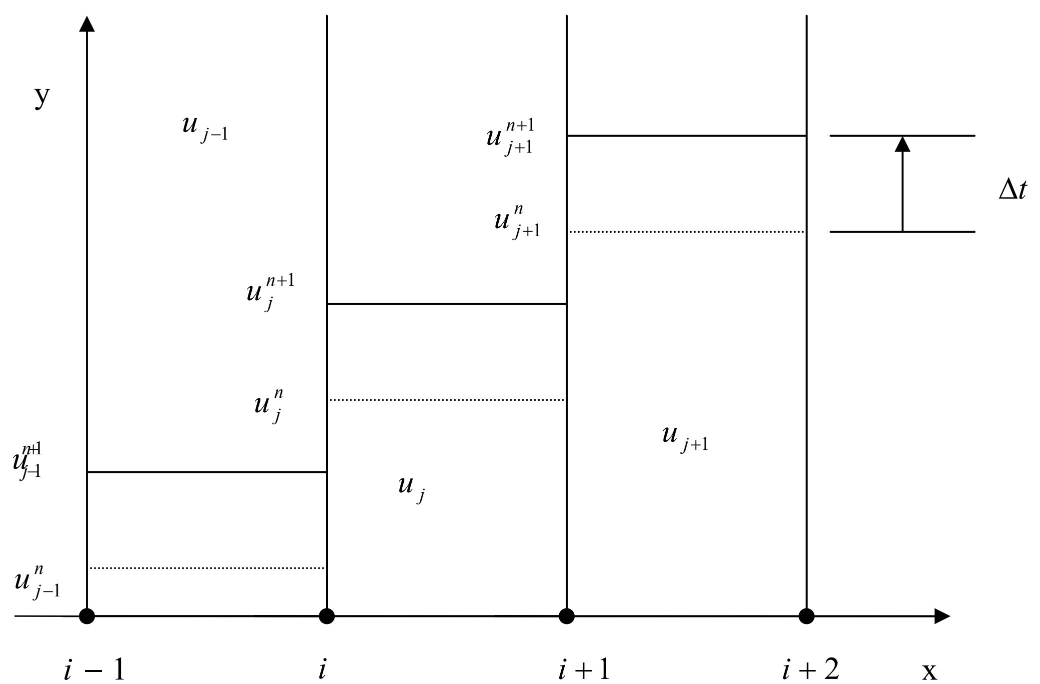


Fig.6. Piecewise constant distribution at t = (n+1) Δt

4.5 Runge-Kutta method:

Runge-Kutta methods are mainly appreciated for their high-order of accuracy. There are many Runge-kutta methods. Here we are approaching explicit scheme in second order RK method in time to obtain the solution. This method was proposed by Shu and Osher.

Two steps are involved in applying second order Runge-Kutta discontinuous Galerkin method (RKDGmethod)

1. Discretize in space using the DG method with first order polynomial.

2. Discretize in time using a second order Runge-Kutta time scheme.

In RK method Riemann problem needs piecewise constant variation at time levels in a time-marching solution. It is shown fig.7.

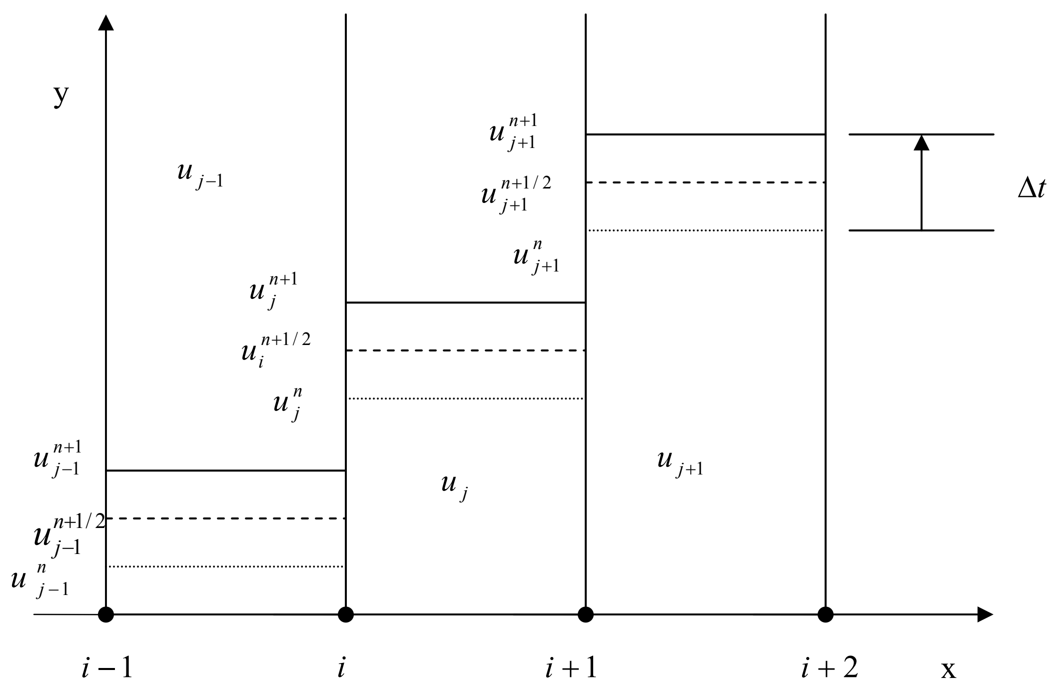


Fig.7. 2^{ND} order Runge-Kutta Piecewise constant distribution

4.5.1 First step of Runge-Kutta method:

First step Runeg-Kutta method is applied at intermediate time level $n+1/2$ into the equation (14). Thus the velocity distribution and velocity variation with respect to distance equations at time level n+1/2 can be derived. In first step Runge-Kutta method, equation (14) will be written as shown below.

(20)

$$\Delta x_j \left\{ \frac{ \begin{bmatrix} 1 & 0 \\ 0 & \frac{\Delta x_j^2}{12} \end{bmatrix} \begin{bmatrix} u_{j,0} \\ \frac{\partial u_j}{\partial x} \end{bmatrix}^{n+1/2} - \begin{bmatrix} 1 & 0 \\ 0 & \frac{\Delta x_j^2}{12} \end{bmatrix} \begin{bmatrix} u_{j,0} \\ \frac{\partial u_j}{\partial x} \end{bmatrix}^{n} }{\Delta t} \right\} = \frac{\Delta x_j}{4} \begin{bmatrix} 1 & 0 \\ 0 & \frac{\Delta x_j^2}{12}\frac{\partial u_j}{\partial x} \end{bmatrix} \begin{bmatrix} u_{j,0} \\ \frac{\partial u_j}{\partial x} \end{bmatrix}^{n} - \frac{1}{2} \begin{bmatrix} (f_{i+1}^R - f_i^R) \\ \frac{\Delta x_j}{2}(f_{i+1}^R + f_i^R) \end{bmatrix}^{n}$$

Expand the equation (20) and divide the equation with Δx_j on both sides.

Then the equation becomes,

(21)

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{\Delta x_j^2}{12} \end{bmatrix} \begin{bmatrix} u_{j,0} \\ \frac{\partial u_j}{\partial x} \end{bmatrix}^{n+1/2} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\Delta x_j^2}{12} \end{bmatrix} \begin{bmatrix} u_{j,0} \\ \frac{\partial u_j}{\partial x} \end{bmatrix}^{n} + \frac{\Delta t}{4} \begin{bmatrix} 0 & 0 \\ u_{j,0} & \frac{\Delta x_j^2}{12}\frac{\partial u_j}{\partial x} \end{bmatrix}^{n} \begin{bmatrix} u_{j,0} \\ \frac{\partial u_j}{\partial x} \end{bmatrix}^{n} - \frac{\Delta t}{2\Delta x_j} \begin{bmatrix} (f_{i+1}^{R^n} - f_i^{R^n}) \\ \frac{\Delta x_j}{2}(f_{i+1}^{R^n} + f_i^{R^n}) \end{bmatrix}$$

On expanding the equation (21), average velocity and linear slope of j^{th} element at intermediate time level ($n+1/2$) are derived as follows.

Those are,

(22) $$u_{j,0}^{n+1/2} = u_{j,0}^{n} + (f_i^{R^n} - f_{i+1}^{R^n})\frac{\Delta t}{2\Delta x_j}$$

(23) $$\frac{\partial u_j^{n+1/2}}{\partial x} = \frac{\partial u_j^{n}}{\partial x} + \frac{\Delta t}{4}\left[\frac{12}{\Delta x_j^2}(u_{j,0}^{n})^2 + \left(\frac{\partial u_j^{n}}{\partial x}\right)^2\right] - \frac{3\Delta t}{\Delta x_j^2}(f_{i+1}^{R^n} + f_i^{R^n})$$

4.5.2 Second step of Runge-Kutta method:

Second step Runeg-Kutta method is applied at time level $n+1$ into the equation (14). In second step Runge-Kutta method discretisation of system in space, and discretisation in time can be shown as below.

(25)

$$\Delta x_j \left\{ \frac{\begin{bmatrix} 1 & 0 \\ 0 & \frac{\Delta x_j^2}{12} \end{bmatrix} \begin{bmatrix} u_{j,0} \\ \frac{\partial u_j}{\partial x} \end{bmatrix}^{n+1} - \begin{bmatrix} 1 & 0 \\ 0 & \frac{\Delta x_j^2}{12} \end{bmatrix} \begin{bmatrix} u_{j,0} \\ \frac{\partial u_j}{\partial x} \end{bmatrix}^{n+1/2}}{\Delta t} \right\} = \frac{\Delta x_j}{2} \begin{bmatrix} 1 & 0 \\ 0 & \frac{\Delta x_j^2}{12}\frac{\partial u_j}{\partial x} \end{bmatrix} \begin{bmatrix} u_{j,0} \\ \frac{\partial u_j}{\partial x} \end{bmatrix}^{n+1/2} - \begin{bmatrix} (f_{i+1}^R - f_i^R) \\ \frac{\Delta x_j}{2}(f_{i+1}^R + f_i^R) \end{bmatrix}^{n+1/2}$$

Expand the equation (25), and divide the equation by Δx_j on both sides.

Then the equation becomes,

(26)

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{\Delta x_j^2}{12} \end{bmatrix} \begin{bmatrix} u_{j,0} \\ \frac{\partial u_j}{\partial x} \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\Delta x_j^2}{12} \end{bmatrix} \begin{bmatrix} u_{j,0} \\ \frac{\partial u_j}{\partial x} \end{bmatrix}^{n+1/2} + \frac{\Delta t}{2} \begin{bmatrix} 0 & 0 \\ u_{j,0} & \frac{\Delta x_j^2}{12}\frac{\partial u_j}{\partial x} \end{bmatrix}^{n+1/2} \begin{bmatrix} u_{j,0} \\ \frac{\partial u_j}{\partial x} \end{bmatrix}^{n+1/2} - \frac{\Delta t}{\Delta x_j} \begin{bmatrix} (f_{i+1}^{R^{n+1/2}} - f_i^{R^{n+1/2}}) \\ \frac{\Delta x_j}{2}(f_{i+1}^{R^{n+1/2}} + f_i^{R^{n+1/2}}) \end{bmatrix}$$

On expanding equation (26), average velocity and linear slope of j^{th} element at time level $(n+1)$ are derived as follows.

(27) $$u_{j,0}^{n+1} = u_{j,0}^{n} + (f_i^{R^{n+1/2}} - f_{i+1}^{R^{n+1/2}})\frac{\Delta t}{\Delta x_j}$$

(28) $$\frac{\partial u_j^{n+1}}{\partial x} = \frac{\partial u_j^{n}}{\partial x} + \frac{\Delta t}{2}\left[\frac{12}{\Delta x_j^2}(u_{j,0}^{n+1/2})^2 + \left(\frac{\partial u_j^{n+1/2}}{\partial x}\right)^2\right] - \frac{6\Delta t}{\Delta x_j^2}(f_{i+1}^{R^{n+1/2}} + f_i^{R^{n+1/2}})$$

Numerical fluxes of Burger's equation at each interface are approximated by Godunov's scheme.

4.6 Godunov's method:

Godunov's method is only first order accurate. Higher order schemes like Lax-Wendroff type schemes, such as MacCormack' schemes are second order and show spurious wiggles near discontinuities. Here we focus on first order accurate methods for non-linear equations, in particular the upwind method for advection and for hyperbolic systems. This is the non-linear version of Godunov's method, which is the fundamental starting point for methods for nonlinear conservation laws. Finite element methods are derived on the basis of the integral form of the conservation law, a starting point that turns out to have many advantages.

Godunov introduced a method to resolve numerical solutions of hyperbolic equations where the cell averages of the solution of the system of equations at time level t_n are to be assumed to be piecewise constant. So that we can approximate the cell averages at next time level t_{n+1} with the known value of cell average at time level t_n. This time step of length can be denoted as $\Delta t = t_{n+1} - t_n$. Thus we can obtain the cell average at time level t_{n+1} from equation (10).

Godunov's method indicates the Riemann problem's existence in the solution. The Riemann problem can be solved at each interface by the technique of piecewise constant solution like one time level to next time level. So that local Riemann problem at each element interface can be solved analytically. Therefore, the local solutions at each element interface are the exact solutions to the conservation laws of hyperbolic equation subject to the given discontinuous initial conditions.

However we cannot evaluate the time integrals on the right-hand side of the equations (9) and (10) exactly, since $u(x_i, t)$ and $u(x_{i+1}, t)$ varies with time along each edge of the element, and we don't have the exact solution to work with.

4.7 Riemann Problem:

This method is named after German mathematician G.F. Bernard Riemann who first attempted its solution in 1858. The Riemann problem was initiated and solved for one-dimensional Euler equations of isentropic flow in gas dynamics by Riemann in 1860. Riemann's solutions reveal the elementary waves of isentropic flow shock waves and rarefaction waves. The Riemann problems lend themselves to a direct analytic solution of the unsteady, one dimensional Euler equations. The Riemann problems are Cauchy problems for non-linear conservation laws with scale invariant initial data. They describe the phenomena of wave-front interaction. The corresponding Riemann solutions are important both theoretically and computationally as building blocks of general solutions.

Here we used exact solutions of the Riemann problem with initial conditions which are derived for Godunov's scheme.

The numerical Riemann flux of inviscid Burgers equation for a Godunov's scheme is

$$f_i = \begin{cases} u_i^2/2 & \text{if } u_i \text{ and } u_{i-1} \text{ are both negative} \\ u_{i-1}^2/2 & \text{if } u_i \text{ and } u_{i-1} \text{ are both positive} \end{cases}$$

When u_i and u_{i-1} have opposite signs, one has

$$f_i = \begin{cases} 0 & \text{if} \quad u_i < 0 < u_{i-1} \quad \text{(expansion fan)} \\ u_i^2/2 & \text{if} \quad u_i > 0 > u_{i-1} \text{ and } c_i > 0 \\ u_{i-1}^2/2 & \text{if} \quad u_i > 0 > u_{i-1} \text{ and } c_i < 0 \end{cases}$$

Where c_i is propagating speed of wave.

4.8 Courant number:

Stability of numerical solution for each studied method was determined by Courant number. CFL condition, which is necessary condition that must be satisfied by the Godunov's method if we expect it to be stable and converge solution of the differential equation as the grid is refined. This is consequence of CFL condition, named after Courant, Friedrichs, and Lewy. It is very important to note that the CFL condition is only a necessary condition for stability (and hence convergence). DG method is required for lower courant number for stable numerical solution. Courant number is defined as

$$C = u\frac{\Delta t}{\Delta x} \qquad \text{where C is called Courant number.}$$

This equation is called the Courant-Friedrichs-Lewy condition. For all cases, the maximum courant number is remained constant for each of the methods. It is an important stability criterion for hyperbolic equations. For stable numerical solutions courant number must be less than or at most equal to unity. For stability, the numerical domain must include the entire analytical domain. Time period depends on courant number C where the stability criteria based on courant number for numerical solutions. Then numerical solution convergent to the analytical solution.

CHAPTER 5

5. BOUNDARY CONDITIONS:

Boundary conditions should be specified as part of the problem and are determined by the physical setup, generally not in terms of the characteristic variables. Since the hyperbolic problem is a time-dependent problem for which we need both initial data and boundary data. It is not always easy to see what the correct conditions are to impose on the mathematical equation. We may have several pieces of information about what is happening at the boundary. It often helps greatly to know that the characteristic structure is which reveals how many boundary conditions we need and allows us to check that we are imposing appropriate conditions for a well-posed problem.

In practice we must always compute on some finite set of grid cells covering a bounded domain, and in the first and last cells we will not have the required neighboring information. The solution of the problem is determined by specifying the boundary and initial conditions. For numerical solutions we need additional boundary conditions (numerical boundary conditions) in addition to analytical boundary conditions which are also known as physical boundary conditions. These analytical boundary conditions are defined as the free number of primitive variable that can be imposed at the boundary. This number depends on the signs of the eigen values, which indicates the propagation of information along the characteristics. The variables applied at boundaries of the domain are transported along characteristics that travel from boundary towards the interior domain. These variables are remaining the same during the computation but they change the state of flow by feeding the data from exterior domain. When the computation started along the domain the information propagates from interior domain into the other boundary, the situation along this boundary will be influenced and modified by the computed flow so the variables from one boundary to another through interior domain cannot be arbitrarily specified.

One approach is to develop special formulas for use near the boundaries, which will depend both on what type of boundary conditions specified and on what sort of method we are trying to match. However, in general it is much easier to think of extending the computational domain to include a few additional cells on either end, whose values are set at beginning of each time step in some manner that depends on the boundary conditions and perhaps the interior solution. Those values at each boundary provide the neighboring-cell values needed in updating the cells near the physical domain. Generally three types of boundary conditions are implemented to solutions.

- Fixed boundary
- Periodic boundary
- Absorbing boundary

Fixed boundary:

A fixed boundary condition is easiest to implement. For the classical wave equation the right (left) traveling wave is set equal to the negative of the left (right) traveling wave at the left (right)boundary. For all other equations, we set the function to zero at the boundaries.

Periodic boundary:

Periodic boundaries are only slightly more difficult to implement. We require that the value of the function be calculated as if the ends were tied together. The wave at one end point is calculated as if it had the other endpoint as its nearest neighbor. This boundary condition is coded by using two extra grid points that have x-coordinates just outside the medium. These extra grid points are updated to the values at the opposite boundary after every time step but they are not displayed on the screen.

Absorbing boundary:

An absorbing boundary is the most complicated. Since the fixed boundaries for the diffusion equation already absorb or remove mass from the system, absorbing boundaries are redundant and have not been implemented for the diffusion equation.

For the classical wave equation, we set the first grid point of the right-traveling wave or last grid point of the left-traveling wave equal to zero and let the propagation algorithm carry these zeros into the medium.

Boundary conditions summary:

For all three cases:

At left boundary,

$f(1) = 0.0$

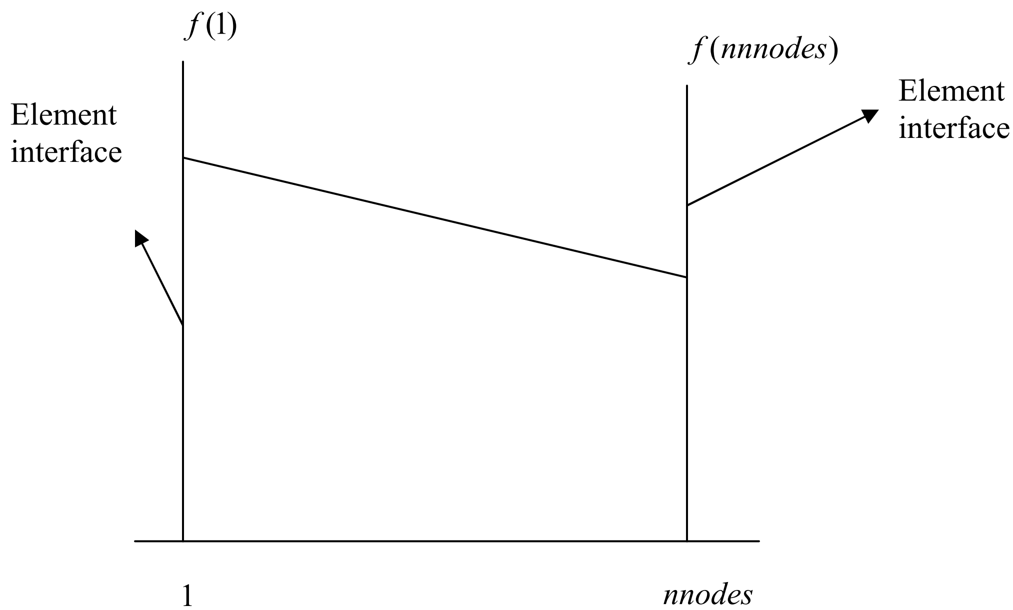
$unew(1) = 1.0$

$dudxnew(1) = 0.0$

At right boundary,

$$f(nnodes) = \frac{ue(2, nelements)^2}{2}$$

Nomenclature of these global boundary conditions is shown in below.



Where $f(1)$ is the flux at 1 interface of j^{th} element.

$f(nnodes)$ is the flux at $nnodes$ interface of j^{th} element.

$u(1)$ is the average velocity of j^{th} element.

$dudxnew(nnodes)$ is the slope of the j^{th} element.

CHAPTER 6

6. RESULTS AND DISCUSSIONS:

This chapter describes the results of numerical solution, analytical solution and comparison between them. Numerical solution results are obtained for all three cases by applying DG method to 1-D inviscid Burgers equation. A set of additional boundary conditions are added to the numerical solution besides physical boundary conditions or analytical boundary conditions.

The effects of different parameters on numerical solution are studied. Here results of numerical solutions and analytical solutions are obtained by applying different grid elements, different time steps and different total times. All results are divided into 4 sets.

Set 1 includes the grid independence study. figures8, and 10 the grid is refined by dividing the domain equally into 200, 400 and 800 elements for first and last cases (sinusoidal wave profile and initial linear distribution) at t=300seconds and t=200seconds respectively. In figure 9 the domain of the grid is divided into 80, 160 and 320 elements for second case (initial shock discontinuity) at t=20seconds for time interval of 0.01seconds. Figures 14 and 17 give the comparison between the analytical and RKDG numerical solutions for case1 and case3 at 400 elements, and time interval of 0.01 seconds. Figure 16 give the comparison between the analytical and RKDG numerical solutions for case2 at 200 elements, and time interval of 0.01 seconds. Grid refinement and courant numbers are listed in table 1.

In set 2 figures 15 shows the results obtained at different times for case1 study (sinusoidal wave profile) to prove that the solution is asymptotic. This is done at t=150, 300, and 600 seconds for the grid of 400 elements at time interval 0.01seconds. The description is shown in table 2.

Set 3 includes time convergence study for all three cases at different time intervals. Figure 11 shows time step convergence study for case1 which is obtained at t=300seconds for the grid of 400 elements at different time intervals of 0.01, 0.005, and 0.001 seconds. Figure 12 shows times step convergence study for case 2 which is studied at t=20 seconds for the grid of 160 elements at different time intervals of 0.01, 0.005, and 0.001 seconds. Figures13 shows time step convergence study for case 3 which is studied at t=200seconds for the grid of 400 elements at different time intervals of 0.01, 0.005, and 0.001 seconds. The time step convergence and courant numbers are shown in table3.

Set 4 includes the comparison between RKDG numerical solutions and Euler's numerical solutions. Figure 18, 19 and 20 shows comparison between RKDG and Euler's method for each case at the highest time interval of 0.01 seconds.

All plots are plotted to length of the domain vs. velocity. These results of numerical solutions obtained under valid initial and boundary conditions from RKDG method and Euler's method. Numerical solutions are stable up to a maximum courant number of 0.08 for case1 and case 3 and 0.32 for case 2.

TABLE 1

Grid refinement study for case1, case2, and case3

Number of elements	Time t sec	Δt sec	Courant number C
Case1 200	300	0.01	0.02
400	300	0.01	0.04
800	300	0.01	0.08
Case2 80	20	0.01	0.08
160	20	0.01	0.16
320	20	0.01	0.32
Case3 200	200	0.01	0.02
400	200	0.01	0.04
800	200	0.01	0.08

TABLE 2

Asymptotic study for case1

Time t sec	Δt sec	Number of elements	Courant number C
150	0.01	400	0.04
300	0.01	400	0.04
600	0.01	400	0.04

TABLE 3

Time step convergence study for case1, case2, and case3.

Δt sec	Number of elements	Time t sec	Courant number C
Case1 0.01	400	300	0.04
0.005	400	300	0.02
0.001	400	300	0.004
Case2 0.01	160	20	0.16
0.005	160	20	0.08
0.001	160	20	0.016
Case3 0.01	400	200	0.04
0.005	400	200	0.02
0.001	400	200	0.004

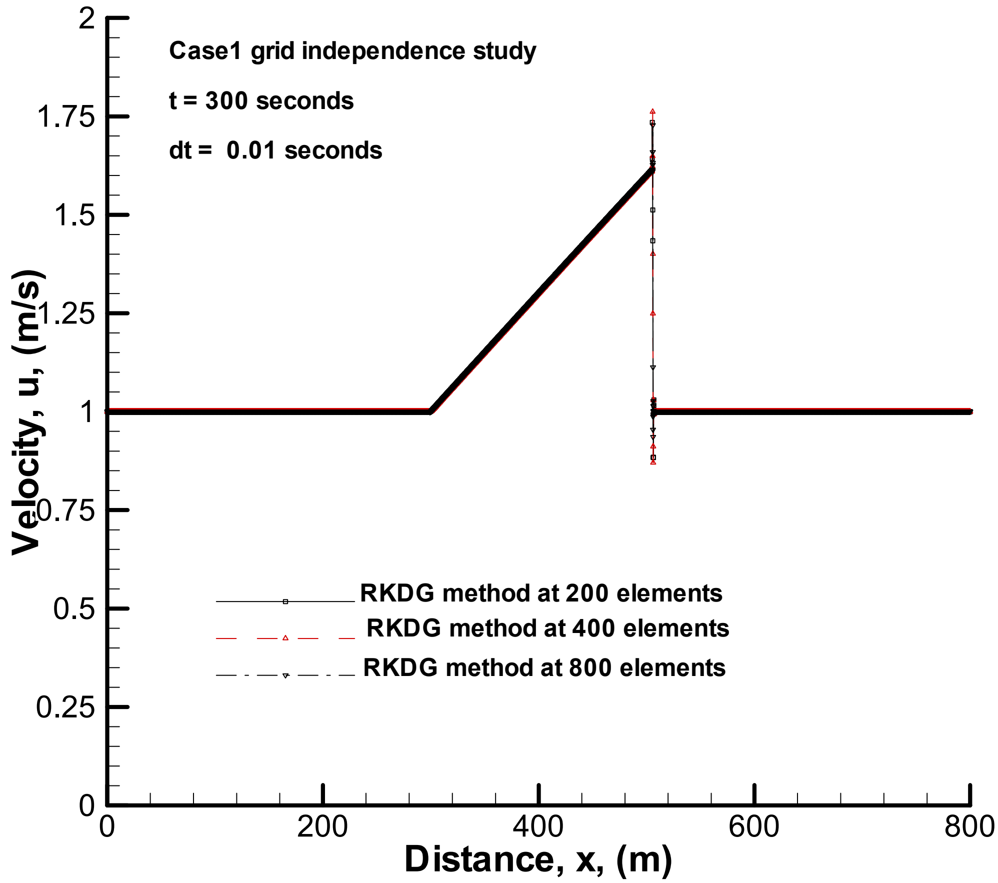


Figure 8: Case1 (sinusoidal wave profile) problem, grid independence study.

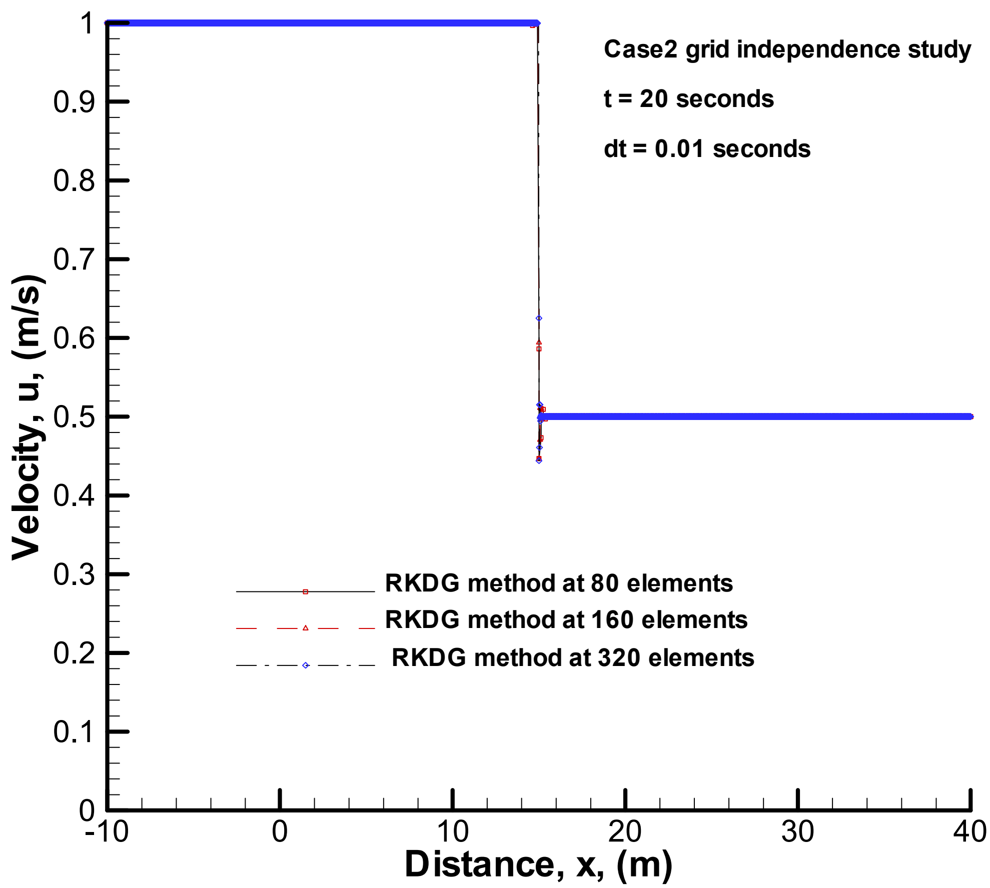


Figure9: Case2 (initial shock discontinuity) problem, grid independence study.

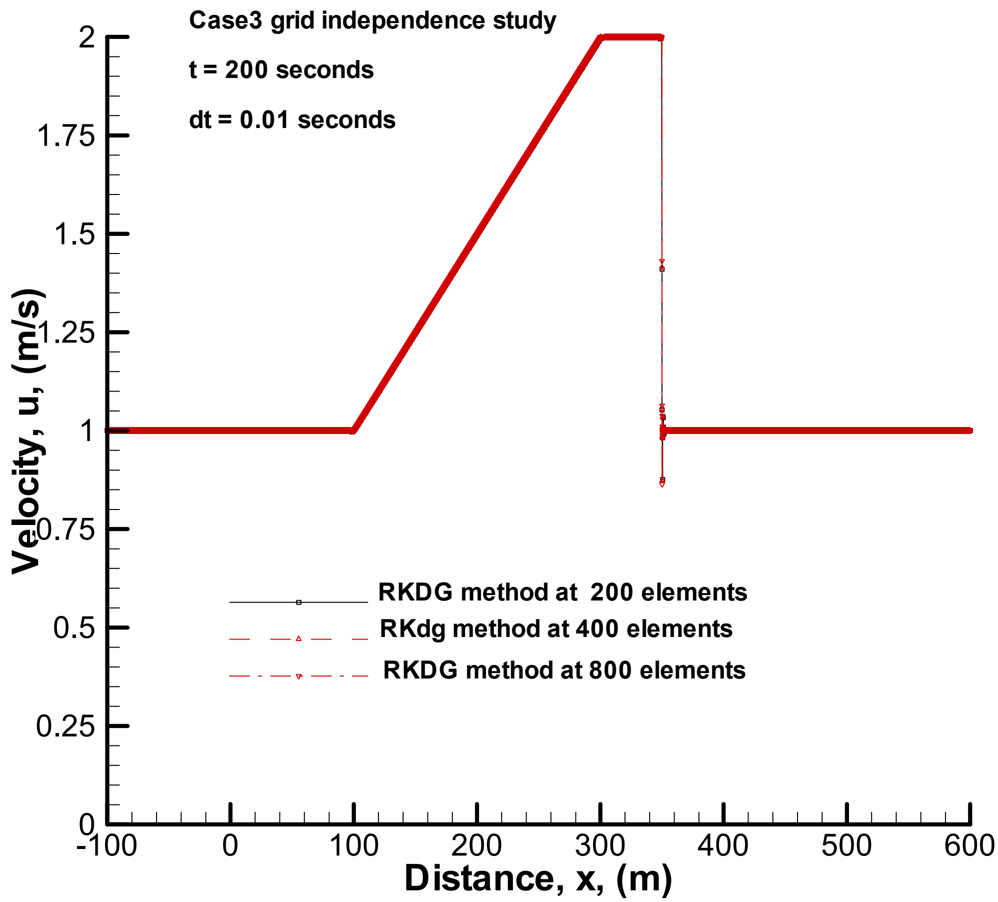


Figure10: Case3 (initial linear distribution) problem, grid independence study.

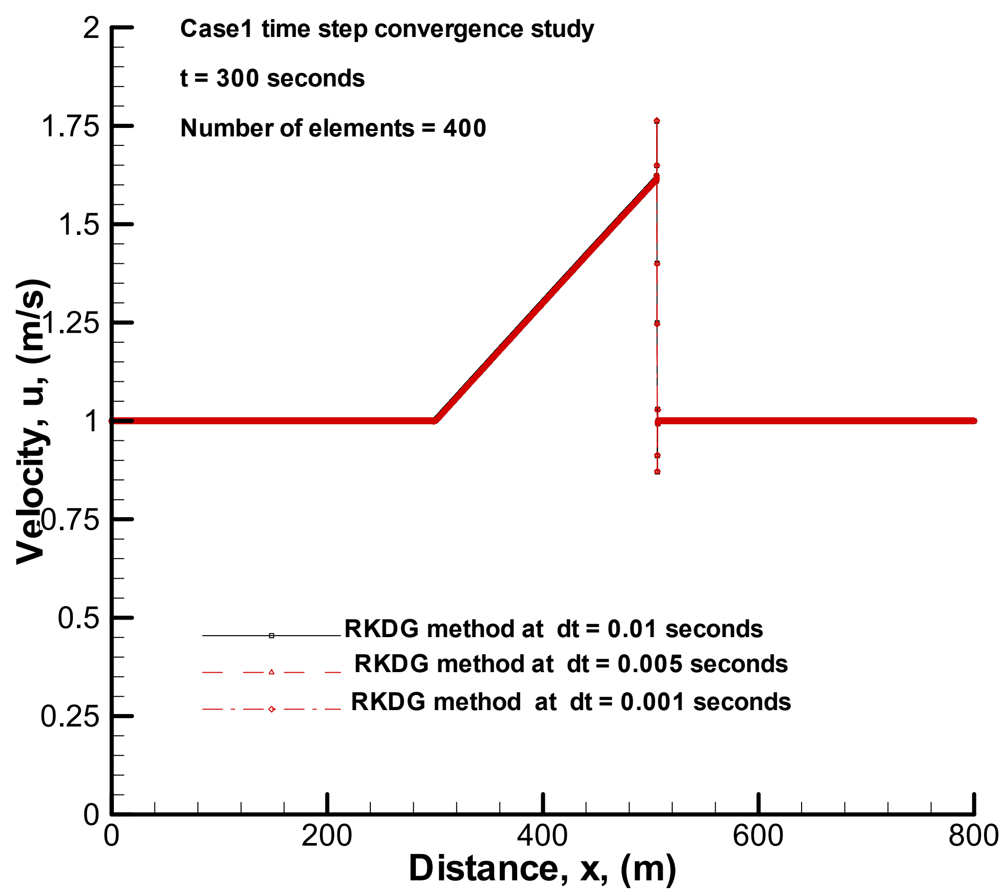


Figure 11: Case1 (sinusoidal wave profile) problem, time step convergence study.

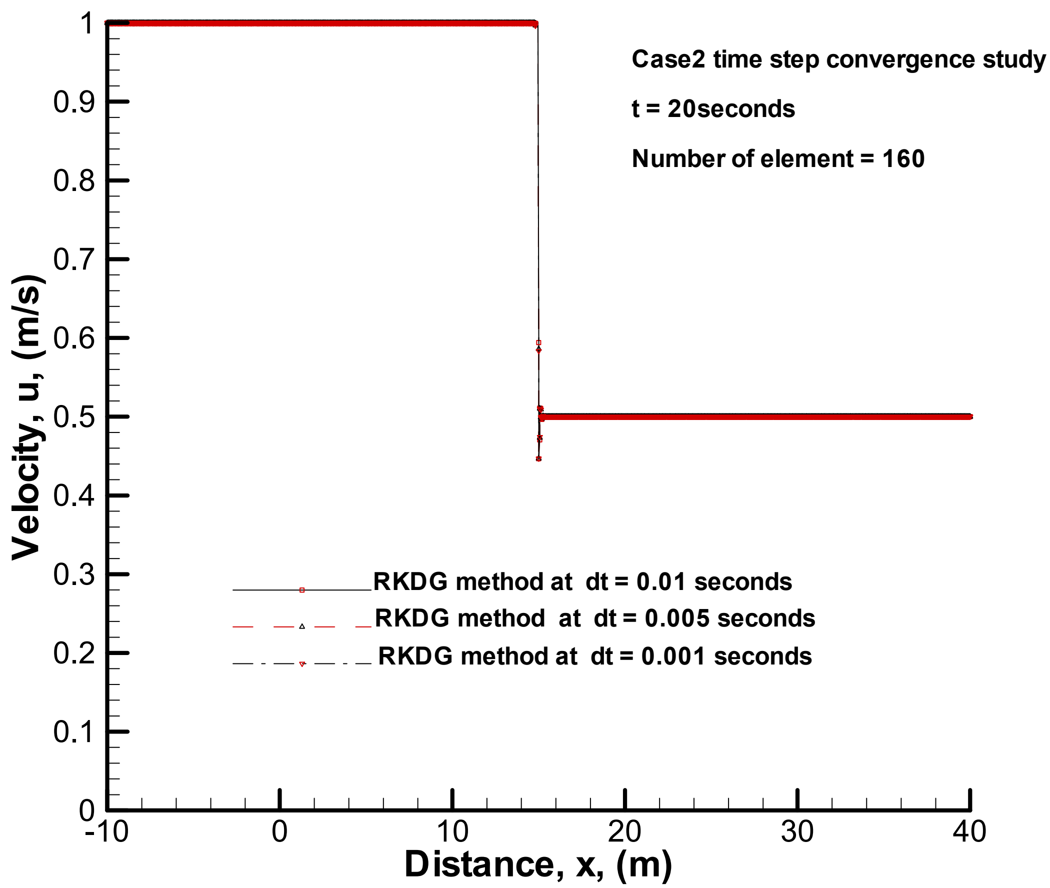


Figure 12: Case2 (initial shock discontinuity) problem, time step convergence study.

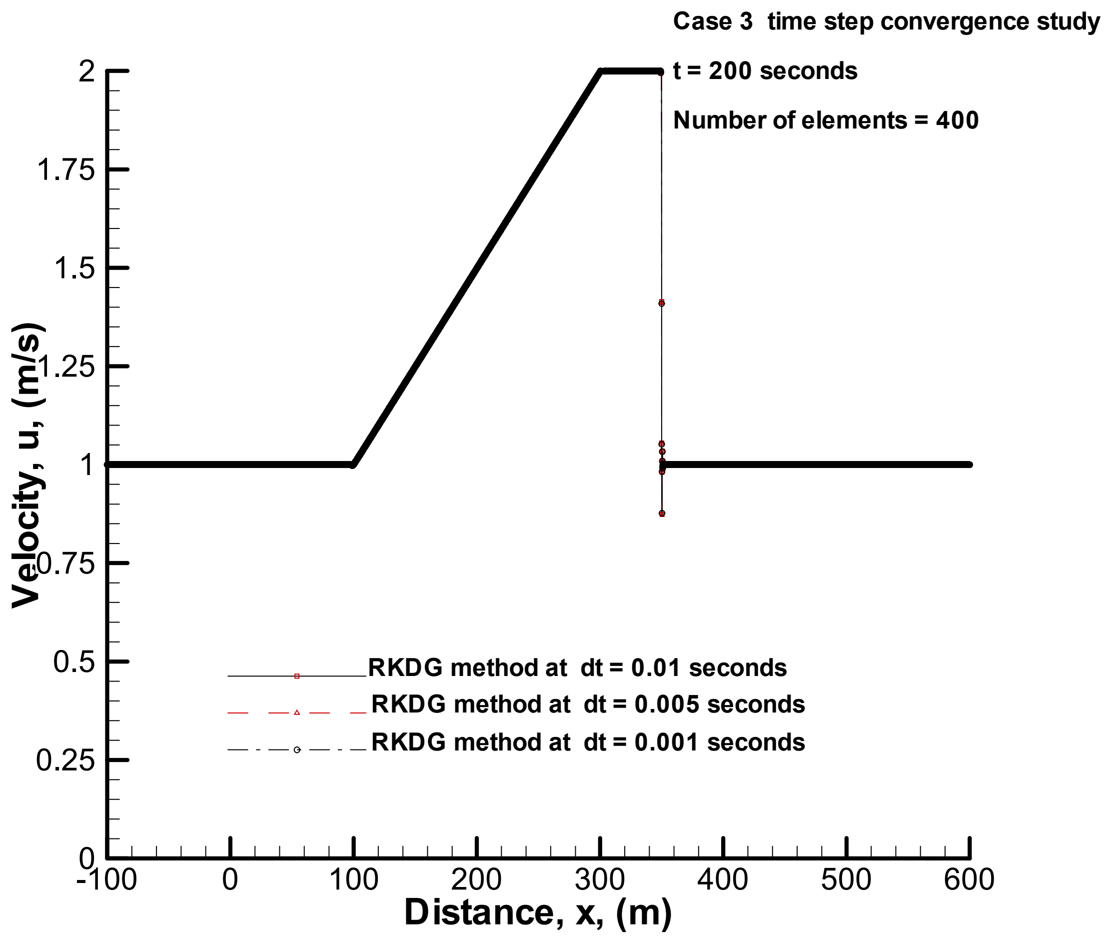


Figure 13: Case3 (initial linear distribution) problem, time step convergence study.

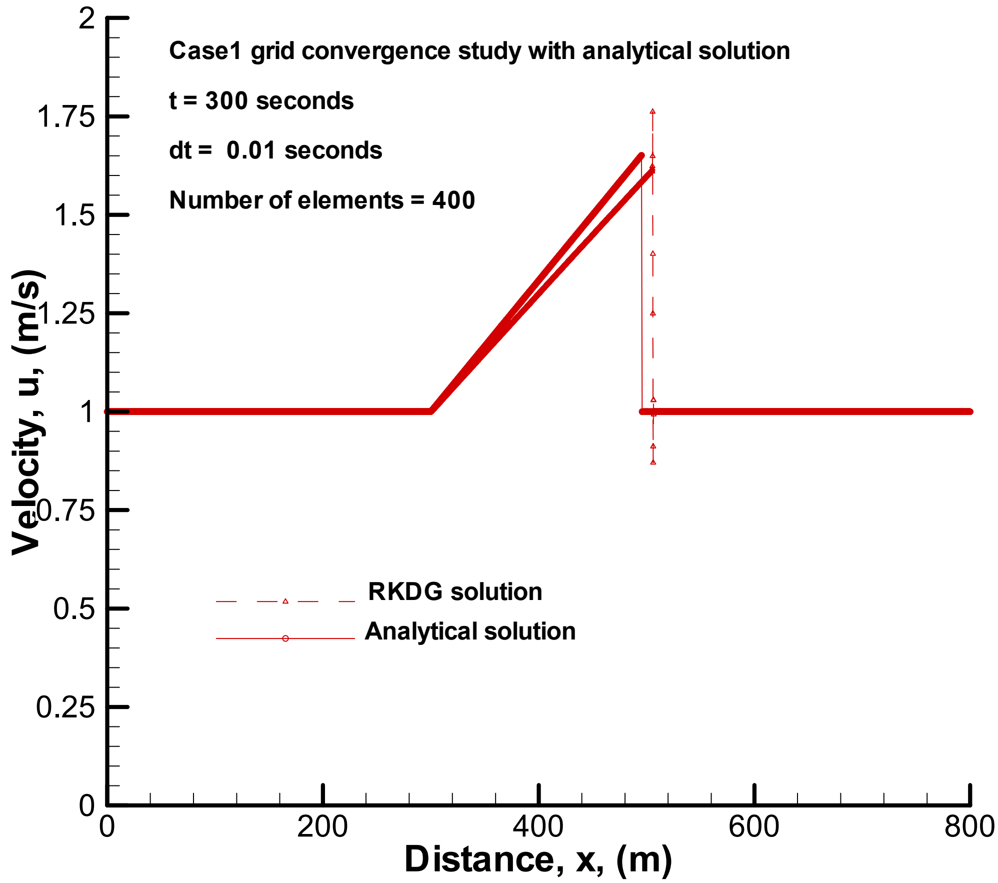


Figure 14: Case1 (sinusoidal wave profile) problem, grid convergence study comparison with analytical solution.

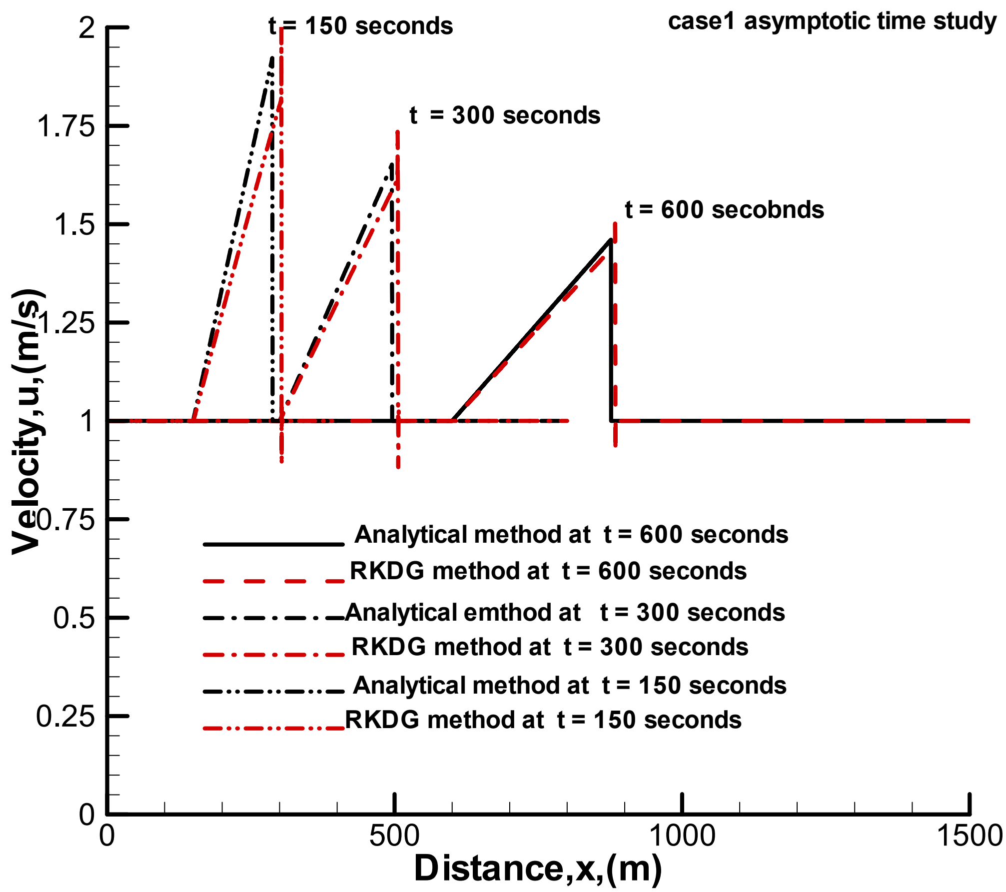


Figure 15: Case1 (sinusoidal wave profile) problem, time study, t =150, 300 and 600seconds, domain of 400elements, Δt =0.01seconds.

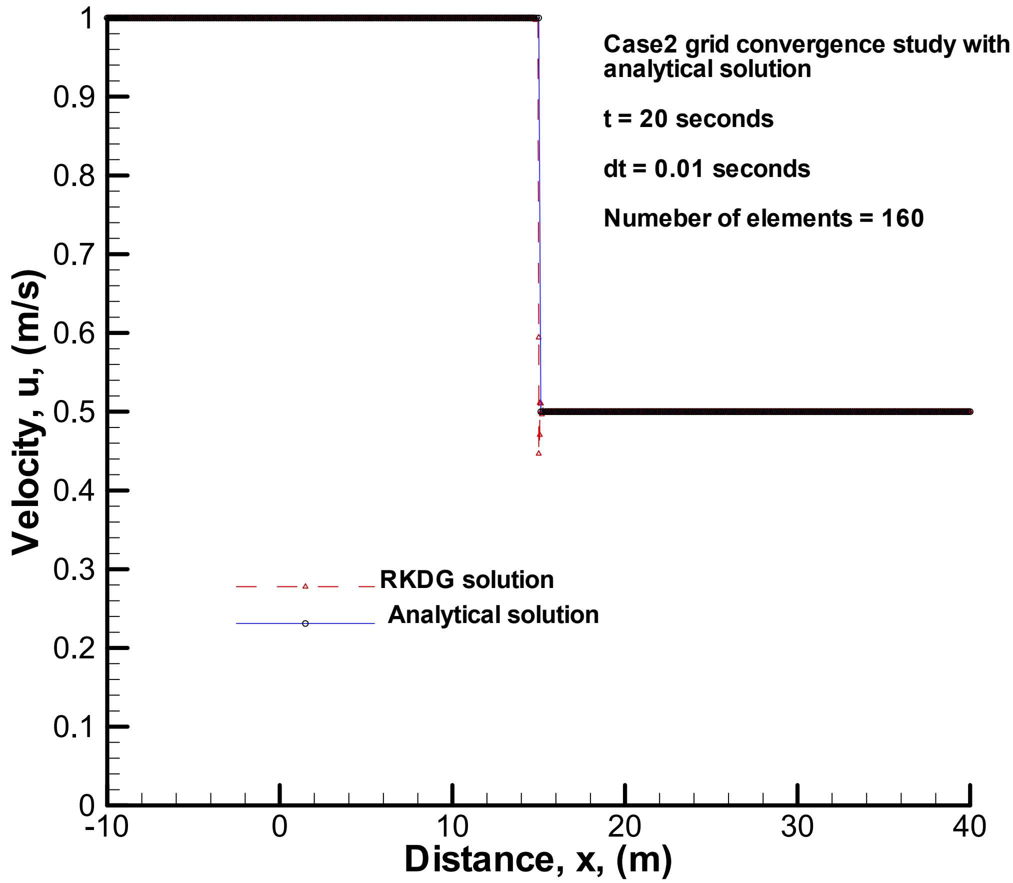


Figure16: Case2(initial shock discontinuity)problem, grid convergence study comparison with analytical solution.

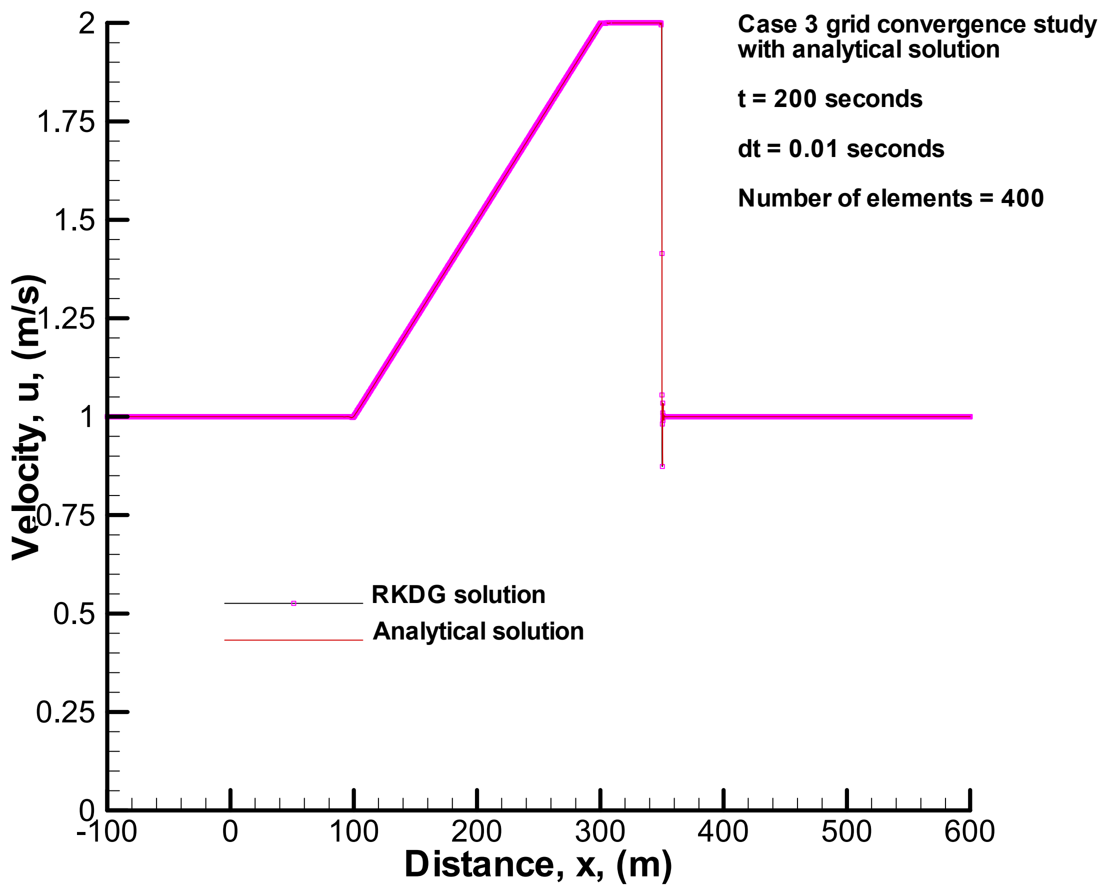


Figure17: Case3(initial linear distribution)problem, grid independence study comparison with analytical solution.

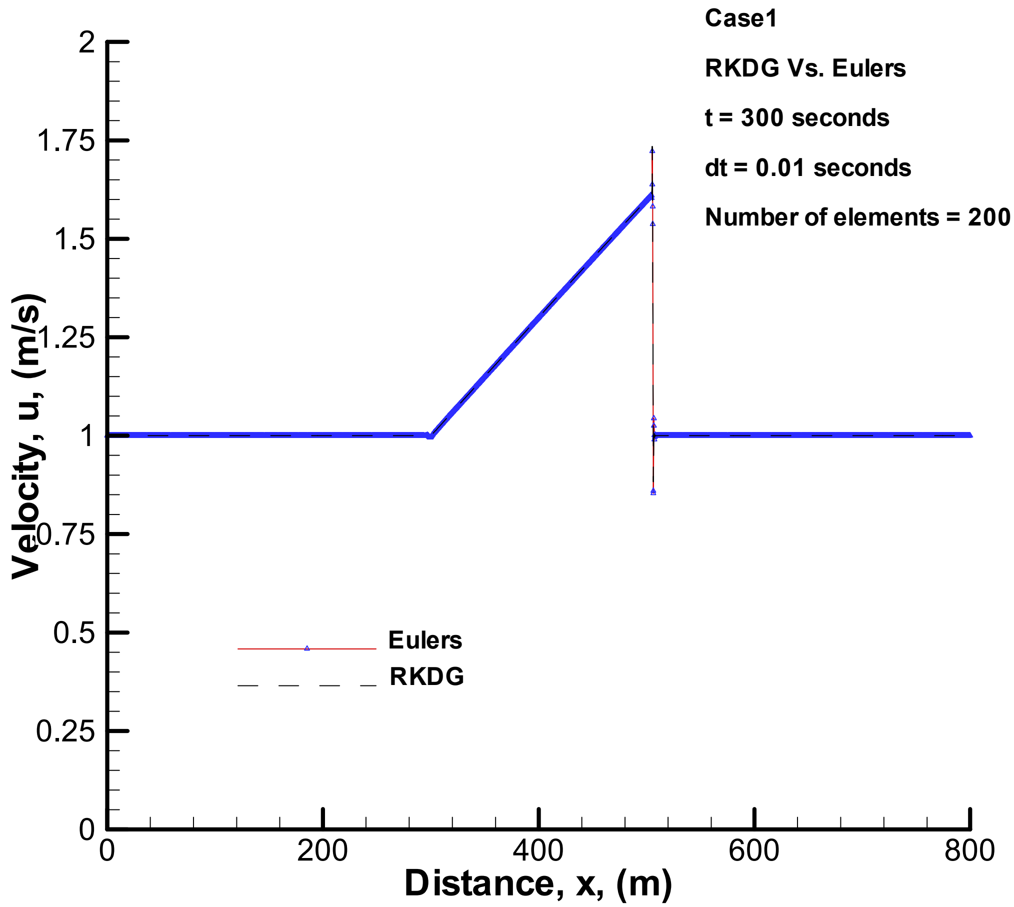


Figure 18: Case1 (sinusoidal wave profile) problem, comparison between RKDG and Euler's solutions.

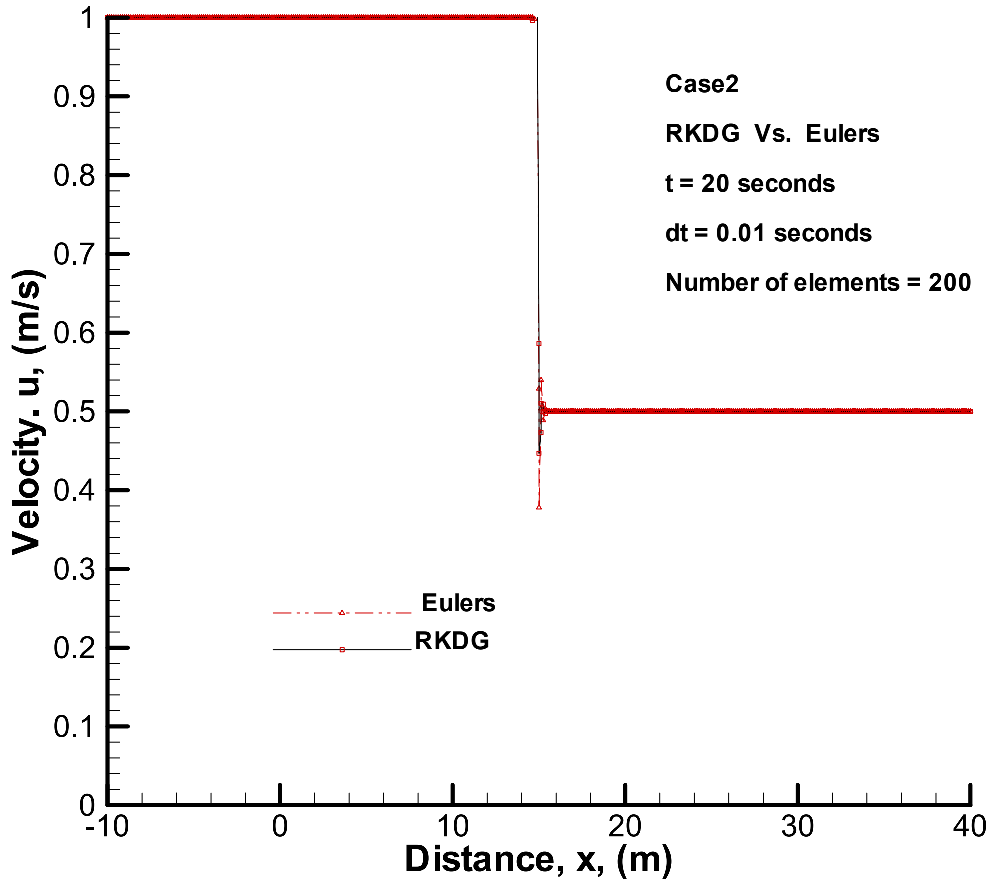


Figure 19: Case2 (initial shock discontinuity)problem, comparison between RKDG and Euler's solutions.

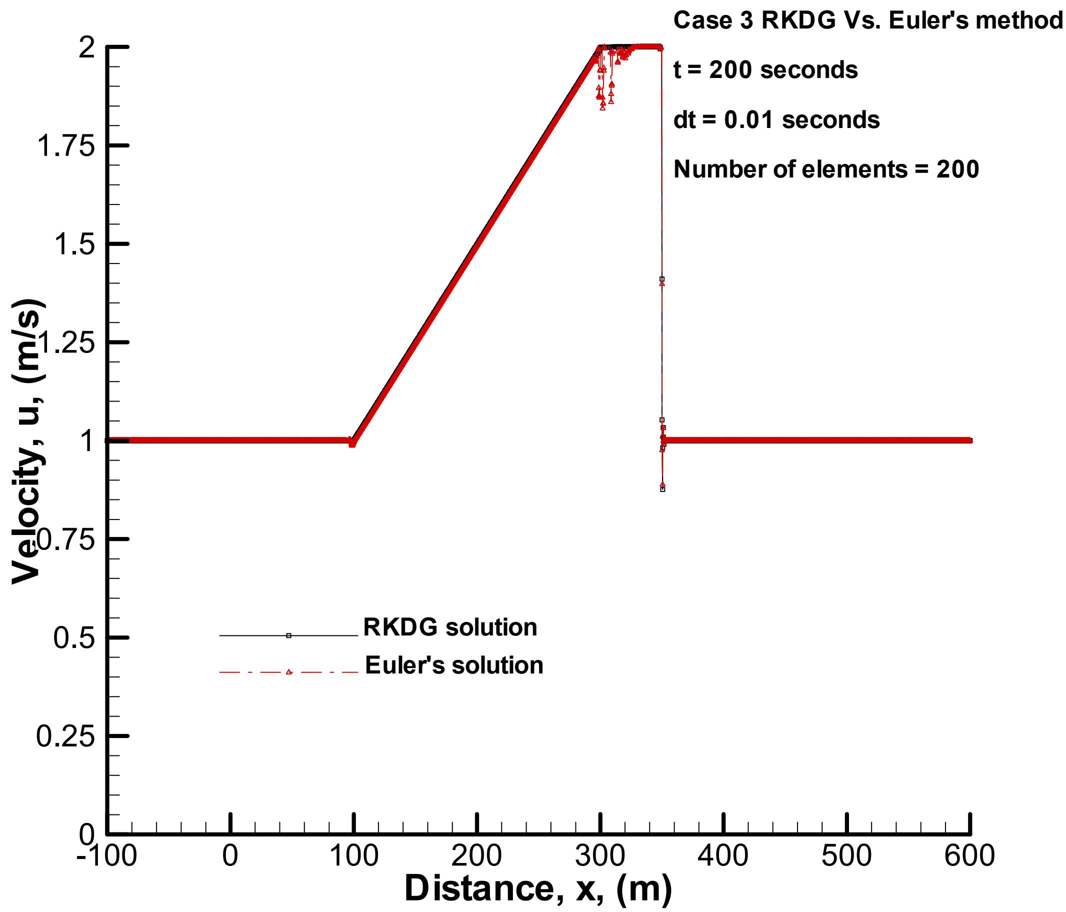


Figure 20: Case3 (initial linear distribution) problem, comparison between RKDG and Euler's solutions.

CHAPTER 7

7. CONCLUSIONS AND DISCUSSIONS:

The Discontinuous Galerkin method is applied to 1-Dimensional inviscid Burgers equation for 3 test cases. For all 3 cases the numerical and analytical solutions are obtained by applying initial and boundary conditions.

In grid convergence study numerical solutions and comparison of numerical solution with analytical solution are presented. Figures 8, 9, 10, 14, 16, and 17 shows that the results obtained from RKDG method and analytical solution are independent of grid elements. There is no significant change in solutions when the number of elements is changed from coarser grid to refined grid. Results obtained from refined grid are more accurate than coarser grid. Numerical solutions converge into exact solution. But for case1 figure 14 shows that the numerical solution converges at infinite. This phenomena is show in figure 15. Therefore the solution is independent of grid.

Fig .15 shows the asymptotic solution for case 1 (sinusoidal wave profile). The plot is drawn for different times at t=150, 300, and 600 seconds to show that the sinusoidal wave profile is the simple asymptotic situation as the analytical solution approaches the numerical solution at $t \to \infty$. This plot shows clear view of approach of analytical solution towards numerical solution at t=150, 300 and 600 seconds. As the time increases the analytical solution moving close towards numerical solution. Thus the fig.15 can be proved that as the time increases the numerical solution converges to analytical solution.

In time convergence study figures 11, 12, and13 shows that the convergence of solution under time step refinement. Transient solutions can be obtained under refined time step. The stability criterion is studied by courant number. For higher values for courant number the solution may blow up. The resulting plots show transient solutions and the given boundary conditions are satisfied.

Figures 18, 19, and 20 shows the comparison between RKDG solutions and Euler's solutions where transient solutions are shown from RKDG method than Euler' method even at higher time intervals. Figure 19, 20 clearly show the oscillations of the solutions at shock formation for Euler's method. Finally these results conclude that the RKDG method is more stable than Euler's method.

Developed DG method has been implemented into code. Plots obtained from code are compared with analytical solution of the problem. The solutions obtained for numerical and analytical methods are almost same and the written algorithm into the code for 1-D inviscid Burgers equation is valid. Under grid and time step study all numerical solutions converges. So that the implementation of developed DG method into the code has been verified and correctly implemented.

- Grid independence study results show that the solutions are independent of grid as there is no significant change in solutions at different grid elements and the numerical solutions converge into the analytical solution. But transient results can be obtained from refined grid than coarser grid.

- Asymptotic study for case1 (sinusoidal wave profile) clearly shows that the analytical solution converges the numerical solution at $t \to \infty$ as the sinusoidal wave profile case study is simple asymptotic situation.
- Time step convergence shows that the solutions converge under time refinement. Stable solutions are obtained at appropriate courant numbers i.e. stability criteria of the solutions depend on courant numbers. If the time interval is increased to higher value then the solution may blow up. Therefore results concluded that maximum time interval is justified by courant number. Numerical solutions are convergent to analytical solutions.
- Comparison of RKDG solution and Euler's solution shows that the RKDG solutions are more stable and accurate than Euler's solutions even at allowable higher time intervals.

RECOMMENDATIONS FOR FUTURE WORK

- Future work can be developed for 2-D inviscid Burgers equation.
- It can also be studied for viscous Burgers equation to capture some key futures of gas dynamics, aerodynamics, astrophysics, detonation of waves and related fields where shock waves arise.
- It can be extended to systems of non-linear hyperbolic equations such as shallow water equations and other Euler equations.

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APPENDIX

Discontinuous Galrekin method developed for 1-D inviscid Burgers equation by Euler's method and RKDG method.

$$\int_{x_{j,i}}^{x_{j,i+1}} \frac{\partial u}{\partial t} w dx + \int_{x_{j,i}}^{x_{j,i+1}} \frac{\partial f}{\partial x} w dx = 0 \text{ ------------------A1}$$

$$\frac{d}{dt} \int_{x_{j,i}}^{x_{j,i+1}} uwdx + \left[fw \right]_{x_{j,i}}^{x_{j,i+1}} - \int_{x_{j,i}}^{x_{j,i+1}} f \frac{\partial w}{\partial x} dx = 0 \text{ --------------------A2}$$

Equation A2 can be simplified as,

$$\frac{d}{dt} \int_{x_{j,i}}^{x_{j,i+1}} u \ w \ dx = -\left[f_i^R w_{j,i} + f_{i+1}^R w_{j.i+1} \right] + \int_{x_{j,i}}^{x_{j,i+1}} f \ \frac{\partial w}{\partial x} dx \text{ ....A3}$$

Where Riemann flux $f_i^R = f^R(u_{j-1,i}^n, u_{j,i}^n)$ and $f_{i+1}^R = f^R(u_{j,i}^n, u_{j+1,i}^n)$ which are evaluated at $i, i+1$ interfaces.

By applying the linear test and velocity functions of the discontinuous Galerkin method produces integral form of equations are derived as shown in below.

Write equation A3 in matrix form,

Then each term can be written as,

$$\int_{x_{j,i}}^{x_{j,i+1}} u \ w \ dx = \int_{x_i}^{x_{i+1}} \begin{bmatrix} w_{j,0} \\ (x - \bar{x}) \dfrac{\partial w_j}{\partial x} \end{bmatrix} \begin{bmatrix} u_{j,0} & (x - \bar{x}) \dfrac{\partial u_j}{\partial x} \end{bmatrix} dx$$

Where $\bar{x} = \dfrac{x_{j,i} + x_{j,i+1}}{2}$

Let $\frac{\partial w_j}{\partial x} = 1$ and $w_j = 1$

On matrix multiplication and simplification, the above terms becomes,

$$= \Delta x_j \begin{bmatrix} u_{j,0} & \frac{\partial u_j}{\partial x}(x - \bar{x}) \\ u_{j,0}(x - \bar{x}) & \frac{\partial u_j}{\partial x}(x - \bar{x})^2 \end{bmatrix}$$

$$= \quad \Delta x_j \begin{bmatrix} 1 & 0 \\ 0 & \frac{\Delta x_j^2}{12} \end{bmatrix} \begin{bmatrix} u_{j,0} \\ \frac{\partial u_j}{\partial x} \end{bmatrix} \qquad \text{------------------A4}$$

$$\because \int_{x_i}^{x_{i+1}} (x - \bar{x}) dx = 0 \ , \ \int_{x_i}^{x_{i+1}} (x - \bar{x})^2 dx = \frac{\Delta x_i^3}{12}$$

$$x_{j,i+1} - \bar{x} = \frac{\Delta x_j}{2} \text{ and } x_{j,i} - \bar{x} = \frac{-\Delta x_j}{2}$$

Now flux terms are evaluated as,

$$\int_{x_{j,i}}^{x_{j,i+1}} f \ \frac{\partial w}{\partial x} dx = \int_{x_{j,i}}^{x_{j,i+1}} \frac{[u_j]^2}{2} \frac{\partial w_j}{\partial x} dx$$

$$= \frac{1}{2} \int_{x_{j,i}}^{x_{j,i+1}} \left[u_{j,0} + (x - \bar{x}) \frac{\partial u_j}{\partial x} \right]^2 \frac{\partial w_j}{\partial x} dx$$

$$= \frac{1}{2} \int_{x_{j,i}}^{x_{j,i+1}} \left[u_{j,0}^{n\ 2} + (x - \bar{x})^2 \left[\frac{\partial u_j}{\partial x} \right]^2 + 2(x - \bar{x}) \frac{\partial u_j}{\partial x} u_{j,0} \right] \frac{\partial w_j}{\partial x} dx$$

$$=\frac{1}{2}\int_{x_{j,i}}^{x_{j,i+1}}\left[\Delta x_j(u_{j,0})^2+\frac{\Delta x_j^3}{12}(\frac{\partial u_j}{\partial x})^2\right]\begin{bmatrix}0\\ \frac{\partial w_j}{\partial x}\end{bmatrix}dx$$

$$=\frac{1}{2}\int_{x_{j,i}}^{x_{j,i+1}}\begin{bmatrix}0\\1\end{bmatrix}\left[\Delta x_{ji}(u_{j,0})^2+\frac{\Delta x_j^3}{12}(\frac{\partial u_j}{\partial x})^2\right]dx$$

$$=\frac{\Delta x_j}{2}\begin{bmatrix}0\\ (u_{j,0})^2+\frac{\Delta x_j^2}{12}\left(\frac{\partial u_j}{\partial x}\right)^2\end{bmatrix}$$

Separate the above equation into matrix from.

$$=\frac{\Delta x_j}{2}\begin{bmatrix}0 & 0\\ u_{j,0} & \frac{\Delta x_j^2}{12}\frac{\partial u_j}{\partial x}\end{bmatrix}\begin{bmatrix}u_{j,o}\\ \frac{\partial u_j}{\partial x}\end{bmatrix}\text{-----------------------------A5}$$

At each interface of the element flux terms (Riemann flux) can be simplified as,

$$\left[f_{i+1}^R w_{j,i+1}-f_i^R w_{j,i}\right]$$

$$=\left\{\begin{bmatrix}w_j\\ (x_{j,i+1}-\bar{x})\frac{\partial w_j}{\partial x}\end{bmatrix}f_{i+1}^R-\begin{bmatrix}w_j\\ (x_{j,i}-\bar{x})\frac{\partial w_j}{\partial x}\end{bmatrix}f_i^R\right\}$$

$$=\left\{\begin{bmatrix}1\\ (x_{j,i+1}-\bar{x})\end{bmatrix}f_{i+1}^R-\begin{bmatrix}1\\ (x_{j,i}-\bar{x})\end{bmatrix}f_i^R\right\}\qquad\because\frac{\partial w_j}{\partial x}=1\quad\text{and}\quad w_j=1$$

$$=\left\{\begin{bmatrix} 1 \\ \dfrac{\Delta x_j}{2} \end{bmatrix} f_{i+1}^R \quad - \begin{bmatrix} 1 \\ \dfrac{-\Delta x_j}{2} \end{bmatrix} f_i^R \quad \right\}$$

$$==\begin{bmatrix} (f_{i+1}^R - f_i^R) \\ \dfrac{\Delta x_j}{2}(f_{i+1}^R + f_i^R) \end{bmatrix} \text{---------------------------A6}$$

Substitute all equations A4, A5, and A6 in equation A3.

Then the discrete system in space will be,

$$\frac{d}{dt}\Delta x_j \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{\Delta x_j^2}{12} \end{bmatrix} \begin{bmatrix} u_{j,0} \\ \dfrac{\partial u_j}{\partial x} \end{bmatrix} = \frac{\Delta x_j}{2} \begin{bmatrix} 0 & 0 \\ u_{j,0} & \dfrac{\Delta x_j^2}{12}\dfrac{\partial u_j}{\partial x} \end{bmatrix} \begin{bmatrix} u_{j,o} \\ \dfrac{\partial u_j}{\partial x} \end{bmatrix} - \begin{bmatrix} (f_{i+1}^R - f_i^R) \\ \dfrac{\Delta x_j}{2}(f_{i+1}^R + f_i^R) \end{bmatrix}$$

VITA

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