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Principal angles and principal azimuths of frustrated total internal reflection and optical tunneling by an embedded low-index thin film

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The condition for obtaining a differential (or ellipsometric) quarter-wave retardation when p- and s-polarized light of wavelength \( \lambda \) experience frustrated total internal reflection (FTIR) and optical tunneling at angles of incidence \( \phi \), the critical angle by a transparent thin film (medium 1) of low refractive index \( n_1 \), and thickness \( d \), which is embedded in a transparent solid medium 0 of high refractive index \( n_0 \), takes the simple form: 

\[
- \tan^2 \varphi = \tan \delta_p \tan \delta_s , \quad \text{where} \quad x = 2 n_1 \sin \varphi = \sin \delta_p = \sin \delta_s \quad \left( n_0 / n_1 \right)^{1/2} , \quad N = n_0 / n_1 , \quad \text{and} \quad \delta_p , \delta_s \quad \text{are 01 interface Fresnel reflection phase shifts for the} \quad p \quad \text{and} \quad s \quad \text{polarizations.} 
\]

From this condition, the ranges of the principal angle and not-internal angle by a transparent thin film of low refractive index are obtained explicitly. At a given principal angle, the associated principal azimuths \( \psi_p , \psi_s \), in reflection and transmission are determined by 

\[
\tan^2 \psi_p = - \sin 2 \delta_p / \sin 2 \delta_s \quad \text{and} \quad \tan^2 \psi_s = - \tan \delta_p / \tan \delta_s , \quad \text{respectively.}
\]

At a unique principal angle \( \phi_p \) given by \( \sin \phi_p = 2 / (N^2 + 1) \), \( \varphi_p = 45^\circ \) and linear-to-circular polarization conversion is achieved upon FTIR and optical tunneling simultaneously. The intensity transmittances of \( p \)- and \( s \)-polarized light at any principal angle are given by 

\[
\tau_p = \tan \delta_p / \tan \delta_s \quad \text{and} \quad \tau_s = - \tan \delta_p / \tan \delta_s ,
\]

respectively. The efficiency of linear-to-circular polarization conversion in optical tunneling is maximum at \( \phi_p \). © 2011 Optical Society of America

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1. INTRODUCTION

At a dielectric-conductor planar interface, a principal angle is defined as an angle of incidence at which incident linearly polarized monochromatic light of the proper azimuth, called the principal azimuth, is reflected circularly polarized [1-3]. Depending on the value of the relative complex dielectric function of the two media, one, two, or three principal angle–principal azimuth pairs may exist [2,3]. For light reflection by a transparent thin film on an absorbing substrate, there is a continuum of principal angles and an associated range of principal azimuths [4].

A previous paper [5] presented a detailed analysis of the phase shifts that \( \lambda \) experience in frustrated total internal reflection (FTIR) and optical tunneling at angles of incidence \( \phi \) by the critical angle by a thin film of low refractive index \( n_1 \) and thickness \( d \), which is embedded in a transparent bulk medium of high refractive index \( n_0 \) (typically a uniform air gap, \( n_1 = 1 \), between parallel plane faces of two transparent prisms; Fig. 1). In a related paper [6], Azzam and Spinu described an FTIR, 2–12 \( \mu \)m wavelength-tunable circular-polarization beam splitter that uses a variable-thickness air gap between two Ge prisms.

In this paper, closed-form solutions for the principal angles, principal azimuths, and film thicknesses that produce linear-to-circular polarization conversion in FTIR and optical tunneling by an embedded low-index thin film are obtained. The principal angle condition is considered analytically and graphically in Section 2. At a given principal angle \( \phi \) and given refractive index ratio \( N = n_0 / n_1 \), an explicit solution for the normalized film thickness \( d / \lambda \) that produces dual quarter-wave retardation (QWR) in reflection and transmission is derived in Section 3. In Section 4, the associated principal azimuths of reflection and transmission are obtained in terms of the 01 interface Fresnel reflection phase shifts for the \( p \) and \( s \) polarizations. Section 5 is devoted to the special case of dual circular polarization in reflection and transmission at the unique incidence angle of equal tunneling of the \( p \) and \( s \) polarizations. In Section 6, the intensity reflectances and transmittances of \( p \)- and \( s \)-polarized light in FTIR and optical tunneling are obtained as functions of the principal angle \( \phi \) and refractive index ratio \( N \). Section 7 gives a brief summary of the paper. In Appendix A, alternate expressions of the principal azimuths of reflection and transmission are presented, and, in Appendix B, the condition of maximum linear-to-circular polarization conversion in optical tunneling is considered.

2. PRINCIPAL ANGLES OF FTIR AND OPTICAL TUNNELING

The changes of polarization that accompany FTIR and optical tunneling are determined by the ratios of complex-amplitude reflection \( R \) and transmission \( T \) coefficients of \( p \) and \( s \)-polarized light that account for coherent multiple-plane-wave interference within the embedded layer. These ellipsometric functions [7] are expressed as 

\[
\rho_p = R_p / R_s = \tan \psi_p , \exp(j \Delta_p) , \\
\rho_s = T_p / T_s = \tan \psi_s , \exp(j \Delta_s) .
\]
In [5], it is shown that

\[ \Delta_r = \Delta_t, \]  
\[ \tan \psi_p / \tan \psi_s = \sin \delta_r / \sin \delta_p = (N^2 + 1) \sin^2 \phi - 1. \]  

In Eq. (3), \( N = n_0/n_1 \) is the high-to-low index ratio and \( \delta_p, \delta_s \) are 01 interface Fresnel reflection phase shifts for the p and s polarizations. Starting from results already obtained in [5], \( \rho \) is written as

\[ \rho_r = \frac{\tanh x \cos \delta_s - j \sin \delta_s}{\tanh x \cos \delta_p - j \sin \delta_p}, \]  
\[ x = 2\pi n_1(d/\lambda)(N^2 \sin^2 \phi - 1)^{1/2}. \]  

At a principal angle,

\[ \Re \rho_r = 0, \]  
and substitution of \( \rho_r \) from Eq. (4) in Eq. (5) leads to

\[ \tanh^2 x \cos \delta_p \cos \delta_s + \sin \delta_p \sin \delta_s = 0, \]  
\[ -\tanh^2 x = \tan \delta_p \tan \delta_s. \]  

Equation (8) is the simplest possible form of the principal-angle condition in FTIR by an embedded low-index thin film. (This result is also obtained by setting the denominator of the right-hand side of Eq. (8) in [5] equal to zero.) For specified values of \( n_1, N \), Eq. (8) represents the constraint on \( d/\lambda, \phi \) such that the overall differential reflection and transmission phase shifts are quarter-wave, i.e., \( \Delta_r = \Delta_t = \pi/2 \).

From the known expressions of the 01 interface Fresnel reflection phase shifts \( \delta_n (n = p, s) \) [5], the right-hand side of Eq. (8) can be cast as a function of \( \phi \) of the form

\[ \tan \delta_p \tan \delta_s = \frac{(\sin^2 \phi_p - \sin^2 \phi_s)(1 - \sin^2 \phi)}{(\sin^2 \phi - \sin^2 \phi_p)(\sin^2 \phi - \sin^2 \phi_s)}. \]  

In Eq. (9), \( \phi_c = \sin^{-1}(1/N) \) is the critical angle and \( \phi_p, \phi_s \) are the angles of incidence at which \( \delta_p = \pi/2 \) and \( \delta_s = \pi/2 \), respectively [5].

As an example, consider a uniform air gap \( n_1 = 1 \) between two Ge prisms \( N = 4 \) in the IR. Figure 2 shows the product of tangents given by Eq. (9) as a function of \( \phi \) for \( \phi_c \leq \phi \leq 90^\circ \) as a continuous line. Singularities of this function appear at \( \phi_p = 14.90^\circ \) and \( \phi_s = 46.73^\circ \) as expected. In Fig. 2, a family of curves (dashed curves) that represent the left-hand side of Eq. (8) is also plotted versus \( \phi \) for discrete values of \( d/\lambda = 0.005, 0.02 \) to 2.00 in equal steps of 0.02, and 10. Solutions of Eq. (8) correspond to points of intersection of the negative branch of the product-of-tangents function \( (\phi_p < \phi < \phi_s) \) and the dashed lines that represent \(-\tanh^2 x \) (e.g., points A and B for \( d/\lambda = 0.08 \)). Principal angles of the Ge–air–Ge system cease to exist for very thin films with \( d/\lambda < 0.059 \). At large film thicknesses \( e.g., d/\lambda \geq 10 \), optical tunneling is negligible and TIR at the 01 interface is restored. In this large-thickness limit, \( \tanh^2 x = 1 \) and Eq. (8) reduces to

\[ \tan \delta_p \tan \delta_s = -1. \]  

Equation (10) indicates that \( \delta_p - \delta_s = \pi/2 \), which is the principal-angle condition of TIR at the 01 interface. The full range of principal angles \( \phi_1 < \phi < \phi_2 \) is defined in Fig. 2 by the points of intersection \( P_1 \) and \( P_2 \) at which Eq. (10) is satisfied. The corresponding limiting angles \( \phi_1 \) and \( \phi_2 \) are given by [8]

\[ \sin^2 \phi_{1,2} = [(N^2 + 1) \mp (N^4 - 6N^2 + 1)^{1/2}] / 4N^2. \]  

Acceptable solutions of Eq. (11) exist if

\[ N > \sqrt{2} + 1 = 2.414. \]  

For the Ge–air interface \( N = 4, \phi_1 = 15.04^\circ \) (which is slightly above the critical angle \( \phi_c = 14.48^\circ \) and \( \phi_2 = 42.93^\circ \).

3. FILM THICKNESS FOR QWR AT A GIVEN PRINCIPAL ANGLE

For an index ratio \( N > 2.414 \) and a principal angle \( \phi \) in the range \( \phi_1 < \phi < \phi_2 \) defined by Eq. (11), the value of \( \tanh x \) that satisfies the principal-angle condition is determined by Eqs. (8) and (9). From \( \tanh x \),

\[ X = \exp(-2x) = (1 - \tanh x)/(1 + \tanh x) \]  
is calculated. Next, the normalized film thickness that produces QWR in reflection and transmission \( (\Delta_r = \Delta_t = \pi/2) \) is obtained from Eqs. (5) and (13) as

\[ d/\lambda = \text{some value} \]  

![Fig. 2. (Color online) Graphical construction that illustrates the range of possible solutions of Eq. (8).](image-url)
As an example, the above algorithm is applied to a uniform air gap between two IR-transparent Ge prisms \( n_1 = 1, N = 4 \) at \( \phi = 30^\circ \); this gives \( \delta_p = 165.75^\circ, \delta_s = 53.13^\circ, \tan x = (- \tan \delta \tan \delta_s)^{-1/2} = 0.581914, X = 0.264291, \) and \( \frac{d}{\lambda} = 0.061138 \).

Use of the same algorithm for the Ge–air–Ge system over the full range of principal angles from \( \phi_1 = 15.04^\circ \) to \( \phi_2 = 42.93^\circ \), \( \frac{d}{\lambda} \) at which \( \Delta_r = \Delta_t = \pi/2 \) is obtained as a function of principal angle \( \phi \), as shown in Fig. 3. Except for a change of scale of the ordinate axis, this graph is the same as the one obtained by Azzam and Spinn (Fig. 2 in [6]) using an iterative numerical technique.

4. REFLECTION AND TRANSMISSION PRINCIPAL AZIMUTHS AT A GIVEN PRINCIPAL ANGLE

At a principal angle, \( \cos \Delta_r = 0, \sin \Delta_r = 1, \text{Re}\rho_r = 0 [\text{Eq. (9)}], \) and \( \rho_r \) reduce to

\[
\rho_r = j \tan \psi_r. \tag{15}
\]

In Eq. (15), \( \psi_r \) is the associated principal azimuth (the angle between the electric-field vector of incident linearly polarized light and the plane of incidence) that produces circularly polarized reflected light. From Eqs. (4) and (15), we obtain

\[
\text{Im}\rho_r = \tan \psi_r = \frac{-\tan^2 x \sin(\delta_p - \delta_s)}{\tan^2 x \cos^2 \delta_p + \sin^2 \delta_p}. \tag{16}
\]

Substitution of \( \tan^2 x = -\tan \delta \tan \delta_s \) [Eq. (8)] in Eq. (16) and use of trigonometric identities lead to an explicit expression for the principal azimuth \( \psi_r \) in terms of the 01 interface TIR phase shifts \( \delta_v (\nu = p, s) \):

\[
\tan^2 \psi_r = -\sin 2\delta_p / \sin 2\delta_s. \tag{17}
\]

At the same principal angle, the principal azimuth \( \psi_t \) that produces circular polarization of the transmitted (instead of reflected) light is derived from Eqs. (3) and (17) as

\[
\tan^2 \psi_t = -\tan \delta_p / \tan \delta_s. \tag{18}
\]

An air gap between two Ge prisms \( n_1 = 1, N = 4 \) with thickness \( \frac{d}{\lambda} = 0.061138 \) has a principal angle \( \phi = 30^\circ \) (Section 2). Substitution of \( \delta_p = 165.75^\circ \) at \( \delta_s = 53.13^\circ, \phi = 30^\circ \) in Eqs. (17) and (18) gives \( \psi_r = 54.816^\circ, \psi_t = 23.578^\circ \). Calculation of \( \psi_r, \psi_t \) as functions of \( \phi \) over the full range of principal angles \( \phi_1 \leq \phi \leq \phi_2 \) of the Ge–air–Ge system produces the two curves shown in Fig. 4.

In Appendix A, we give alternate explicit expressions of \( \psi_r, \psi_t \) as functions of \( \phi \) for a given \( N \) and locate the angular positions of the minimum and maximum of the \( \psi_r, \text{versus-}\phi \) curve.

5. DUAL CIRCULAR POLARIZATION OF REFLECTED AND TRANSMITTED LIGHT AT ONE PRINCIPAL ANGLE

From Eq. (3), the principal azimuths of FTIR and optical tunneling are equal, \( \psi_r = \psi_t \), at one principal angle \( \phi_e \), given by

\[
\sin^2 \phi_e = 2/(N^2 + 1). \tag{19}
\]

\( \phi_e \) of Eq. (19) is the incidence angle of equal tunneling of \( p \)- and \( s \)-polarized light [9,10] so that \( \psi_r = \psi_t = 45^\circ \). In Fig. 4, for \( N = 4, \psi_r = \psi_t = 45^\circ \) at \( \phi_e = \sin^{-1}(2/17)^{1/2} \approx 20.06^\circ \).

It is worthwhile to recall that \( \phi_e \) is the angle of incidence at which phase difference \( (\delta_p - \delta_s) \) is maximum and the average phase shift is \( (\delta_p + \delta_s)/2 = \pi/2 \) [8].

Also, at \( \phi_e \), the Fresnel reflection phase shifts at the 01 interface [8] simplify to

\[
\tan(\delta_p/2) = 1/N, \quad \tan(\delta_s/2) = N. \tag{20}
\]

Equation (20) indicates that \( \delta_p \) and \( \delta_s \) at \( \phi_e \) are equal to double the Brewster angles of internal and external reflection at the 01 interface, respectively.

Still another curious property of \( \phi_e \) is that, for a given \( N \), the product of tangents given by Eq. (9) reaches a maximum at that angle. This can be proved by substituting \( \sin^2 \phi = u \) in Eq. (9) and setting the derivative of the right-hand side with respect to \( u \) equal to zero. By use of Eq. (20) and the trigonometric identity \( \tan x = 2\tan(x/2)/[1 - \tan^2(x/2)] \), the maximum value of \( \tan \delta_p \tan \delta_s \) at \( \phi_e \) is obtained:
The intensity reflectances and transmittances of \( p \)- and \( s \)-polarized light at a given principal angle

The intensity reflectances and transmittances of \( p \)- and \( s \)-polarized light are obtained by taking the squared absolute value of the corresponding complex-amplitude reflection and transmission coefficients \( R_p, R_s, T_p, T_s \) given in [5]. The resulting expressions are functions of the film thickness parameter \( x \) [Eq. (5)] and the 01 interface Fresnel reflection phase shifts \( \delta_y \) \((\nu = p, s)\):

\[
R_y = |R_y|^2 = \frac{\cosh 2x - 1}{\cosh 2x - \cos 2\delta_y}, \quad \nu = p, s; \tag{26}
\]

\[
\tau_y = 1 - |R_y|^2 = \frac{1 - \cos 2\delta_y}{\cosh 2x - \cos 2\delta_y}, \quad \nu = p, s. \tag{27}
\]

From Eq. (27), the principal-angle condition of Eq. (8), and the identity

\[
\cosh 2x = (1 + \tanh^2 x)/(1 - \tanh^2 x), \tag{28}
\]

the following expressions of the \( p \) and \( s \) intensity transmittances are obtained:

\[
\tau_p = \tan \delta_p / \tan(\delta_p - \delta_s), \quad \tau_s = -\tan \delta_s / \tan(\delta_p - \delta_s). \tag{29}
\]

Note that, from Eq. (22), \( \tau_p / \tau_s = \tan^2 \psi_t = -\tan \delta_p / \tan \delta_s \) in agreement with Eq. (15). The corresponding intensity reflectances are given by

\[
R_y = 1 - \tau_y, \quad \nu = p, s. \tag{30}
\]

Substitution of \( \delta_p, \delta_s, (\delta_p - \delta_s) \) as functions of \( N \) and \( \phi \) from [8] in Eqs. (29) yields

\[
\tau_p = \frac{2N^2 \sin^4 \phi - (N^2 + 1) \sin^2 \phi + 1}{(N^2 + 1) \sin^4 \phi + (N^2 + 1) \sin^2 \phi}, \tag{31}
\]

\[
\tau_s = \frac{2N^2 \sin^4 \phi - (N^2 + 1) \sin^2 \phi + 1}{2N^2 \sin^4 \phi - (N^2 + 1) \sin^2 \phi}. \tag{32}
\]

Equations (31) and (32) are valid over the full range of principal angles \( \phi_1 \leq \phi \leq \phi_2 \) defined by Eq. (11) and give exact mathematical representation of the families of curves presented in Fig. 5 in [6]. Both transmittances are zero at the limiting angles \( \phi_1 \) and \( \phi_2 \) given by Eq. (11) as can be verified by setting the common numerator of Eqs. (31) and (32) equal to zero.

For a given \( N \), the transmittance \( \tau_p(\phi) \) of Eq. (31) reaches a maximum at a principal angle of incidence given by

\[
\sin^2 \phi_{p \text{max}} = (N^2 + 1)/(N^2 - 1)^2. \tag{33}
\]

Likewise, for a given \( N \), the transmittance \( \tau_s(\phi) \) of Eq. (32) reaches a maximum at a principal angle of incidence given by

\[
\sin^2 \phi_{s \text{max}} = (N^2 + 1)/4N^2. \tag{34}
\]

Equations (33) and (34) are obtained by substituting \( \sin^2 \phi = u \) in Eqs. (31) and (32) and setting the derivative of the right-hand side of each equation with respect to \( u \) equal to zero.

For a given \( N \), the maximum transmittances at \( \phi_{p \text{max}} \) and \( \phi_{s \text{max}} \) are equal,

\[
\tau_{p \text{max}} = \tau_{s \text{max}} = \frac{N^4 - 6N^2 + 1}{N^4 + 2N^2 + 1}, \tag{35}
\]

and the corresponding principal azimuths are related by

\[
\psi_p(\phi_{p \text{max}}) = 90^\circ - \psi_s(\phi_{p \text{max}}), \tag{36}
\]

\[
\psi_p(\phi_{p \text{max}}) = \arctan \left[ \frac{N^6 - 3N^4 - 3N^2 + 1}{2N^2(N^2 + 1)} \right]^{1/2}. \tag{37}
\]

Fig. 5. (Color online) Normalized thickness \( d/\lambda \) of a low-index embedded layer that produces circular polarization in FTIR and optical tunneling at \( \phi_1 \) and \( \psi_t = \psi_s = 45^\circ \) [Eq. (24)] is plotted as a function of the refractive index ratio \( N \) over the range 2.5 \( \leq N \leq 6.0 \). Equal transmittance of \( p \)- and \( s \)-polarized light \( \tau_p(\phi_1) = \tau_s(\phi_2) = \tau \) [Eq. (30)] is also shown as a function of \( N \).
For the Ge–air–Ge system in the IR, $N = 4$ and Eqs. (33)-(37) give

$$\phi_{p, \text{max}} = 15.95^\circ, \quad \phi_{s, \text{max}} = 31.02^\circ;$$
$$\tau_{p, \text{max}} = \tau_{s, \text{max}} = 0.5571; \quad \psi_1(\phi_{p, \text{max}}) = 67.84^\circ;$$
$$\psi_1(\phi_{s, \text{max}}) = 22.16^\circ.$$ (38)

The analytical expressions presented here fully explain the results previously obtained by Azzam and Spinu using a numerical technique (see Fig. 5 in [6]).

When the denominators of the right-hand sides of Eqs. (31) and (32) are equal, the $p$ and $s$ transmittances become equal, and the condition of equal tunneling of the $p$ and $s$ polarizations [Eq. (19)] at $\phi = \phi_c$ is recovered. At $\phi_c$, the equal throughputs for the $p$ and $s$ polarizations are given by

$$\tau_p(\phi_c) = \tau_s(\phi_c) = 1 - \frac{1}{2} \left( \frac{N^2 + 1}{N^2 - 1} \right)^2.$$ (39)

The same transmittance of $p$- and $s$-polarized light at $\phi = \phi_c$ [$\tau_p(\phi_c) = \tau_s(\phi_c) = \tau$] is plotted as a function of $N$ in Fig. 5. For $N = 4$, Eq. (39) gives $\tau_p(\phi_c) = \tau_s(\phi_c) = 0.35778$ in agreement with [6].

Another curious result of this section is that $\phi_{s, \text{max}}$ given by Eq. (34) is also the angle of incidence at which $\delta_p = 3\delta_s$ [8].

In Appendix B, it is shown that the efficiency of linear-to-circular polarization conversion upon optical tunneling is maximum at $\phi = \phi_c$.

7. CONCLUSION

Highlights of this paper are summarized as follows.

1. The principal-angle condition of FTIR and optical tunneling by an embedded low-index thin film is given in concise form by Eq. (8). For selected refractive indices $n_i, N$, Eq. (8) represents the constraint on the normalized film thickness and principal angle $(d/\lambda, \phi)$ such that the differential phase shifts in reflection and transmission are quarter-wave. Figure 2 illustrates the domain of $d/\lambda, \phi$ for which acceptable solutions of Eq. (8) exist when $n_i = 1, N = 4$.

2. For given values of $n_i, N, d/\lambda$ that leads to QWR in reflection and transmission (i.e., $\Delta_r = \Delta_t = \pi/2$) at a principal angle $\phi$ is explicitly determined by Eq. (14).

3. Equations (17) and (18) determine the reflection and transmission principal azimuths $\psi_r, \psi_t$ in terms of the 01 interface Fresnel reflection phase shifts $\delta_1 (\nu = p, s)$.

4. At the angle $\phi_c$ given by Eq. (19), $\psi_r = \psi_t = 45^\circ$, and circular polarization in FTIR and optical tunneling is achieved simultaneously at thickness-to-wavelength ratio $d/\lambda$ given by Eq. (24).

5. At a given principal angle, the throughputs for the $p$ and $s$ polarizations in optical tunneling are given by Eqs. (29), (31), and (32). These transmittances have maxima at principal angles given by Eqs. (33) and (34), respectively.

6. The efficiency of linear-to-circular polarization conversion upon optical tunneling is maximum [Eq. (37)] at $\phi = \phi_c$.

APPENDIX A

By substituting the Fresnel interface reflection phase shifts $\delta_1 (\nu = p, s)$ as functions of $N, \phi$ from [8] in Eqs. (18) and (17), we obtain

$$\tan^2\psi_t = -\left( \frac{u_p}{u_s} \right) \left( \frac{u - u_s}{u - u_p} \right).$$ (A1)

$$\tan^2\psi_r = -\left( \frac{4u_p}{u_s u_p} \right) \left( \frac{(u - u_s)(u - 0.5u_e)^{1/2}}{(u - u_p)} \right).$$ (A2)

In Eq. (A1) and (A2), $u = \sin^2\phi$ and $u_p, u_s, u_e$ are the values of $u$ evaluated at the special angles $\phi_p, \phi_s, \phi_e$ defined in [8] that depend on $N$ only. Equations (A1) and (A2) provide explicit expressions for the principal azimuths $\psi_r, \psi_t$ as functions of the principal angle $\phi$ for any given $N$. For $N = 4$, the curves of $\psi_r, \psi_t$ versus $\phi$ are plotted in Fig. 4.

In Fig. 4, it is apparent that $\psi_t$ decreases monotonically as $\phi$ increases, whereas $\psi_r$ exhibits a minimum and a maximum as a function of $\phi$. The angular positions of the minimum and maximum of $\psi_t$ are determined by setting the first derivative of the right-hand side of Eq. (A2) with respect to $u$ equal to zero. This gives a quadratic equation in $u$ whose roots are

$$u_1 = (1/4) \left( (3u_p + u_s) \pm \sqrt{(3u_p + u_s)^2 + 4u_e(u_s - u_p) - 16u_p u_s} \right).$$ (A3)

For $N = 4$, $u_p = 17/257, u_s = 17/32, u_e = 2/17$, and Eq. (A3) gives $u_+ = 0.073719, \phi_e = 15.7542^\circ; u_+ = 0.291128, \phi_e = 32.6539^\circ$, which exactly locate the angular positions of the minimum and maximum of $\psi_t$ in Fig. 4.

APPENDIX B

At a given principal angle–principal azimuth pair $(\phi, \psi_1)$, incident linearly polarized light of intensity $I_1$ is partially transmitted as circularly polarized with intensity $I_t$, given by

$$I_t = I_1 (\tau_p \cos^2 \psi_1 + \tau_s \sin^2 \psi_1).$$ (B1)

The efficiency of linear-to-circular polarization conversion in optical tunneling is given by

$$\eta_{\text{LTC}} = I_t/I_1 = (\tau_p \cos^2 \psi_1 + \tau_s \sin^2 \psi_1).$$ (B2)

Given that $\tau_p/\tau_s = \tan^2 \psi_1$, Eq. (B2) becomes

$$\eta_{\text{LTC}} = I_t/I_1 = 2\tau_p \cos^2 \psi_1 = 2\tau_s \sin^2 \psi_1.$$ (B3)

Substitution of Eqs. (18) and (29) and the Fresnel interface reflection phase shifts $\delta_1 (\nu = p, s)$ as functions of $N, \phi$ in Eq. (B3) leads to

$$\eta_{\text{LTC}} = \left[ u_p/(u_p - u_s) \right] [2 - 2u_s u_1^{-1} + u_s u_1^{-2}].$$ (B4)

In Eq. (B4), $u = \sin^2\phi$ and $u_p, u_s, u_e$ are the values of $u$ evaluated at the angles $\phi_1, \phi_p, \phi_s$ defined in [8]. By setting

$$d\eta_{\text{LTC}}/du = 0$$ (B5)

in Eq. (B4), the value of $u$ at which $\eta_{\text{LTC}}$ is maximum is obtained:

$$u = u_1/u_s = 2/(N^2 + 1).$$ (B6)

Equations (19) and (B6) confirm that $\eta_{\text{LTC}}$ is maximum at the angle $\phi_c$ of equal tunneling of the $p$ and $s$ polarizations.
\( \psi = 45^\circ \); the associated maximum value of \( \eta_{\text{LTC}} \) is obtained from Eqs. (B3) and (39) as

\[
\eta_{\text{LTC}}^{\max} = \tau_p(\phi_e) = \tau_s(\phi_e) \left( 1 - \frac{1}{2} \frac{N^2 + 1}{N^2 - 1} \right)^2.
\] (B7)

**REFERENCES**