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Phase shifts in frustrated total internal reflection and optical tunneling by an embedded low-index thin film

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Phase shifts in frustrated total internal reflection and optical tunneling by an embedded low-index thin film

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Simple and explicit expressions for the phase shifts that \( p \) - and \( s \)-polarized light experience in frustrated total internal reflection (FTIR) and optical tunneling by an embedded low-index thin film are obtained. The differential phase shifts in reflection and transmission \( \Delta_s, \Delta_p \) are found to be identical, and the associated ellipsometric parameters \( \psi, \phi \) are governed by a simple relation, independent of film thickness. When the Fresnel interface reflection phase shifts for the \( p \) and \( s \) polarizations or their average are quarter-wave, the corresponding overall reflection phase shifts introduced by the embedded layer are also quarter-wave for all values of film thickness. In the limit of zero film thickness (i.e., for an ultrathin embedded layer), the reflection phase shifts are also quarter-wave independent of polarization \( (p \) or \( s \)) or angle of incidence (except at grazing incidence).

Finally, variable-angle FTIR ellipsometry is shown to be a sensitive technique for measuring the thickness of thin uniform air gaps between transparent bulk media. © 2006 Optical Society of America

1. INTRODUCTION

Frustrated total internal reflection (FTIR) and optical tunneling have been studied extensively in the past and have found many important applications. (See, for example, the concise review by Zhu et al.\(^1\) and references cited therein.) However, most previous studies of FTIR have been limited to considerations of the power fractions that appear in reflection and transmission. A notable exception is the work of Carneglia and Mandel\(^2,3\) who measured the phase shifts associated with evanescent waves in FTIR using a holographic technique.

In this paper a detailed analysis is provided of the phase shifts that accompany FTIR and optical tunneling by a low-index thin film that is embedded in a high-index medium, and numerous interesting new results are obtained. This extends previous work that dealt with total internal reflection phase shifts at a single interface between two transparent media.\(^4\) Differential reflection and transmission phase shifts are readily measurable by ellipsometry.\(^5\)

In Section 2 simple expressions are derived for the overall reflection and transmission phase shifts introduced by an embedded layer in terms of the Fresnel interface reflection phase shifts and a normalized film thickness. Section 3 considers the ratios of complex-amplitude reflection and transmission coefficients for the \( p \) and \( s \) polarizations for a layer of any thickness and in the limit of zero and infinite thickness. In Section 4 the results of Sections 2 and 3 are applied to a uniform air gap between two glass prisms. The special case of light reflection and transmission at the critical angle is considered in Section 5. Section 6 gives a brief summary of the paper. Finally, in Appendix A, a new expression for the ratio of slopes of the \( p \) and \( s \) interface reflection phase shifts as functions of angle of incidence is derived.

2. REFLECTION AND TRANSMISSION PHASE SHIFTS

Consider a uniform layer (medium 1) of thickness \( d \) and low refractive index \( n_1 \) that is embedded in a bulk medium (0) of high refractive index \( n_0 \) (Fig. 1). If monochromatic light of wavelength \( \lambda \) is incident on the layer from medium 0 at an angle \( \phi \) greater than the critical angle, \( \phi_0 = \arcsin(n_1/n_0) \), FTIR takes place and some of the light tunnels across the thin film as a transmitted wave. Total internal reflection is restored if \( d \gg \lambda \).

The overall complex-amplitude reflection and transmission coefficients of the embedded layer for \( p \)- and \( s \)-polarized incident light are given by\(^5\)

\[
R_\nu = |R_{\nu 01}| \exp(j \Delta_{\nu 01}) = r_{01\nu} \frac{(1 - X)}/(1 - r_{01\nu}^2 X),
\]

\[
T_\nu = |T_{\nu 01}| \exp(j \Delta_{\nu 01}) = (1 - r_{01\nu}^2) X^{1/2}/(1 - r_{01\nu}^2 X),
\]

\( \nu = p, s, \) \( (1) \)

where \( r_{01\nu} \) is the Fresnel reflection coefficient of the 01 interface for the \( \nu \) polarization \( (\nu = p, s) \). Above the critical angle, \( r_{01\nu} \) is a pure phase factor,\(^6\)

\[
r_{01\nu} = \exp(j \delta_{01\nu}),
\]

and the interface reflection phase shifts \( \delta_{01\nu}(\nu = p, s) \) are given by\(^4\)

\[
\delta_p = 2 \arctan(NU/\cos \phi),
\]

\[
\delta_s = 2 \arctan(U/N \cos \phi),
\]

\( (3) \)

in which...
Equations (8) and (9) are concise and elegant expressions for the overall incident and transmission phase shifts for the $p$ polarization in terms of the Fresnel interface reflection phase shifts $\delta_i$ and the normalized film thickness $x$ [Eq. (6)]. From Eqs. (8) and (9) it immediately follows that

$$\tan \Delta_r \tan \Delta_p = -1, \quad \Delta_r - \Delta_p = \pm \pi/2. \quad (10)$$

Equation (11) indicates that the overall reflection and transmission phase shifts for the $p$ polarization ($\nu=p,s$) differ by a quarter-wave, in agreement with a recent result obtained by Efimov et al.\(^6\)

The (ellipsometric) differential reflection and transmission phase shifts are given by

$$\Delta_r = \Delta_{rp} - \Delta_{rs}, \quad \Delta_t = \Delta_{tp} - \Delta_{ts}. \quad (12)$$

From the trigonometric identity for the tangent of the difference of two angles and Eqs. (8), (9), and (12), we obtain

$$\tan \Delta_r = \tan \Delta_p = \tanh x(\tan \delta_p - \tan \delta_i)/(\tanh^2 x + \tan \delta_p \tan \delta_i). \quad (13)$$

$$\Delta_r = \Delta_t. \quad (14)$$

Equation (14) also follows readily from Eqs. (11) and (12).

The average phase shift on reflection for the $p$ and $s$ polarizations is given by

$$\Delta_{rp} = (\Delta_{rp} + \Delta_{rs})/2. \quad (15)$$

From Eqs. (8), (9), and (15) and the trigonometric identity for the tangent of the sum of two angles, we obtain

$$\tan(2\Delta_{rp}) = \tanh x(\tan \delta_p + \tan \delta_i)/(\tanh^2 x - \tan \delta_p \tan \delta_i). \quad (16)$$

### 3. RATIOS OF COMPLEX REFLECTION AND TRANSMISSION COEFFICIENTS FOR THE $p$ AND $s$ POLARIZATIONS

The ellipsometric functions in reflection and transmission are defined by\(^5\)

$$\rho_p = R_p/R_s = \tan \psi_p \exp(j\Delta_p),$$

$$\rho_s = T_p/T_s = \tan \psi_s \exp(j\Delta_s). \quad (17)$$

By substitution of Eqs. (1) in Eqs. (17), we obtain

$$\rho_p = (r_{01p}/r_{01s})[(1 - r_{01s}^2 X)/(1 - r_{01p}^2 X)],$$

$$\rho_s = [(1 - r_{01p}^2)/(1 - r_{01s}^2)][(1 - r_{01s}^2 X)/(1 - r_{01p}^2 X)]. \quad (18)$$

In the limit of zero layer thickness, $d=0$ and $X=1$, Eqs. (18) become

$$\rho_p(X=1) = (r_{01p}/r_{01s})[(1 - r_{01s}^2)/(1 - r_{01p}^2)],$$

$$\rho_p(X=1) = 1. \quad (19)$$

And in the limit of infinite layer thickness, $d=\infty$ and $X=0$, Eqs. (18) reduce to

$$\rho_p(X=0) = (r_{01p}/r_{01s}),$$

$$\rho_s(X=0) = (1 - r_{01p}^2)/(1 - r_{01s}^2). \quad (20)$$

Whereas the second of Eqs. (19) and the first of Eqs. (20) are intuitively apparent, the remaining two equations are not.

For a layer of any thickness $d$, the ratio

$$\Gamma = \rho_p/\rho_s = (\tan \psi_p/\tan \psi_s)\exp[j(\Delta_p - \Delta_s)]$$

$$= (r_{01s} - r_{01s}^{-1})/(r_{01p} - r_{01p}^{-1}), \quad (21)$$

which is independent of film thickness. Under FTIR conditions, substitution of the interface Fresnel reflection coefficients from Eq. (2) into Eq. (21) gives the surprisingly simple result

$$\Gamma = \rho_p/\rho_s = \sin \delta_p/\sin \delta_p. \quad (22)$$

Between the critical angle and grazing incidence, $\phi_c < \phi < 90^\circ$, $\sin \delta_p, \sin \delta_i > 0$, and the right-hand side of Eq. (22) is a positive real number. It follows from Eqs. (21) and (22) that
\[ \Delta_r - \Delta_i = 0, \]  
\[ \tan \phi_r / \tan \phi_i = \sin \delta_i / \sin \delta_r. \]  
Equation (23) is the same as Eq. (14). By use of Eqs. (3) and some trigonometric manipulations, Eq. (24) can be transformed to

\[ \tan \phi_r / \tan \phi_i = \sin \delta_i / \sin \delta_r = (N^2 + 1) \sin^2 \phi - 1. \]  

In Appendix A we also show that

\[ \sin \delta_i / \sin \delta_r = \delta_i' / \delta_r', \]  
where \( \delta_i' \) and \( \delta_r' \) are the derivatives (slopes) of the interface reflection phase shifts with respect to the angle of incidence \( \phi \). Equation (26) represents a new interesting relation between the total internal reflection phase shifts for the \( p \) and \( s \) polarizations at a dielectric–dielectric interface.  

4. FRUSTRATED TOTAL INTERNAL REFLECTION PHASE SHIFTS FOR AN AIR GAP BETWEEN TWO GLASS PRISMS

As a specific example, we consider FTIR and optical tunneling by a uniform air gap \( (n_0 = 1) \) between two glass prisms \( (n_1 = 1.5) \).

Figure 2 shows the reflection phase shift for the \( p \) polarization \( \Delta_{rp} \) as a function of the angle of incidence \( \phi \) for \( d/\lambda = 0.005, 0.02 \) to 0.20 in equal steps of 0.02, and 10. Note that \( \phi \) covers the full range from 0 to 90°. This range includes both FTIR \( (\phi > \phi_c = 41.181°) \) and partial internal reflection \( (\phi < \phi_c) \). The curve of \( \Delta_{rp} \) versus \( \phi \) for \( d/\lambda = 10 \) exhibits many oscillations below the critical angle that are not shown in Fig. 2.

A striking feature of Fig. 2 is that all curves pass through a common point \( A \) at which \( \Delta_{rp} = 90° \). This occurs at the angle of incidence \( \phi_p \) at which the interface reflection phase shift \( \delta_i = 90° \), as can be seen from Eq. (8). According to Ref. 4, \( \phi_p \) is determined by

\[ \sin^2 \phi_p = (N^2 + 1)/(N^4 + 1), \]  
which gives \( \phi_p = 53.248° \) for \( N = 1.5 \). Therefore the FTIR phase shift for the \( p \) polarization is constant at quarter-wave \( (\Delta_{rp} = 90°) \), independent of film thickness, when light is incident at the special angle \( \phi_p \). At the same angle of incidence, the corresponding transmission phase shift is zero, \( \Delta_{rt} = 0 \), independent of film thickness, as can be inferred from Eq. (9).

Similar results are obtained for the \( s \) polarization as shown in Fig. 3. Again, all curves pass through a common point \( B \) at which \( \Delta_{rs} = 90° \). This occurs at the angle of incidence \( \phi_s \) at which the interface reflection phase shift \( \delta_i = 90° \), as one expects from Eq. (8). According to Ref. 4, \( \phi_s \) is determined by

Fig. 2. Reflection phase shift for the \( p \) polarization \( \Delta_{rp} \) as a function of the angle of incidence \( \phi \) for \( d/\lambda = 0.005, 0.02 \) to 0.20 in equal steps of 0.02, and 10. All curves pass through a common point \( A \) at which \( \Delta_{rp} = 90° \). These results are calculated for an air gap of thickness \( d \) between two glass prisms \( (N = 1.5) \).

Fig. 3. Reflection phase shift for the \( s \) polarization \( \Delta_{rs} \) as a function of the angle of incidence \( \phi \) for \( d/\lambda = 0.005, 0.02 \) to 0.20 in equal steps of 0.02, and 10. All curves pass through a common point \( B \) at which \( \Delta_{rs} = 90° \). These results are calculated for an air gap of thickness \( d \) between two glass prisms \( (N = 1.5) \).

Fig. 4. Average reflection phase shift \( \Delta_{ra} = (\Delta_{rp} + \Delta_{rt})/2 \) as a function of the angle of incidence \( \phi \) for \( d/\lambda = 0.005, 0.02 \) to 0.20 in equal steps of 0.02, and 10. As in Figs. 2 and 3, all curves pass through a common point \( C \) at which \( \Delta_{ra} = 90° \). These results are calculated for an air gap of thickness \( d \) between two glass prisms \( (N = 1.5) \).
which gives $\phi_s = 58.194^\circ$ for $N=1.5$. Therefore the s-polarization FTIR phase shift is constant at quarter-wave ($\Delta_{r_s}=90^\circ$), independent of film thickness, when light is incident at an angle $\phi_s$. At the same angle, the corresponding transmission phase shift is zero, $\Delta_{t_s}=0$, independent of film thickness, as can be seen from Eq. (9).

Another important result can be inferred from Figs. 2 and 3. In the limit of zero film thickness (i.e., for an ultrathin embedded layer), the reflection phase shift is quarter-wave independent of polarization ($p$ or $s$) or angle of incidence (except at grazing incidence). This is also predicted by Eq. (8), which shows that as $x$ and $d$ go to zero, $\tan x$ goes to zero, and $\Delta_{r_s}=90^\circ$ for all values $\delta_s = 0, \pi$.

The results for the average phase shift $\Delta_{ra}=(\Delta_{rp}+\Delta_{r_s})/2$ as a function of the angle of incidence $\phi$ are shown in Fig. 4. As in Figs. 2 and 3, all curves in Fig. 4 pass through a common point $C$ at which $\Delta_{ra}=90^\circ$. This occurs at the angle of incidence $\phi_a$ at which the average interface reflection phase shift is quarter-wave, $\delta_a=(\delta_p + \delta_s)/2=90^\circ$. Under this condition, $(\tan \delta_p + \tan \delta_s) = 0$, and $\Delta_{ra}=90^\circ$, according to Eq. (16). From Ref. 4, $\phi_a$ is determined by

$$\sin^2 \phi_s = 2/(N^2 + 1),$$

which gives $\phi_a = 52.671^\circ$ for $N=1.5$. Therefore the average of the FTIR phase shifts for the $p$ and $s$ polarizations remains constant at quarter-wave ($\Delta_{r_a}=90^\circ$), independent of film thickness, when light is incident at the angle $\phi_a$. At the same angle, the corresponding average transmission phase shift is zero, $\Delta_{t_a}=0$, independent of film thickness, as can be inferred from Eq. (11).

These results add new insight to the significance of the special angles $\phi_p, \phi_s$, and $\phi_a$ in the present context of FTIR by an embedded low-index film.

Figure 5 shows the differential reflection phase shift $\Delta_r=(\Delta_{rp}-\Delta_{r_s})$ as a function of the angle of incidence $\phi$ for $d/\lambda=0.005, 0.02$ to 0.20 in equal steps of 0.02, and 10. It is apparent that the angular response of this differential phase is a sensitive function of $d/\lambda$, which suggests that ellipsometry$^5$ would be an excellent technique for measuring air-gap thickness. A salient feature of the $\Delta_r$-versus-$\phi$ curve is the magnitude and the location of its peak. As $d/\lambda$ decreases, the maximum value of $\Delta_r=(\Delta_{rp}-\Delta_{r_s})$ decreases and its location shifts toward higher angles.

Finally, Figs. 6 and 7 show the ellipsometric parameters $\psi_r$ and $\psi_s$ as functions of the angle of incidence $\phi$ for $d/\lambda=0.005, 0.02$ to 0.20 in equal steps of 0.02, and 10. In Fig. 6, notice that $\psi_r=0$, independent of $d/\lambda$, at the Brewster angle of internal reflection of the glass–air interface, $\phi_r= \arctan(1/1.5)=33.690^\circ$. Also, all curves in Figs. 6 and 7 pass through a common point $D$ at which $\psi_r=0$, independent of $d/\lambda$. The common angle of incidence at point $D$ in Figs. 6 and 7 is given by Eq. (29). If Eq. (29) is substituted into Eq. (25), we obtain $\tan \psi_r = \tan \psi_s = 1$. This confirms that $\phi_a$ is the angle of equal reflection and equal
tunneling for the $p$ and $s$ polarizations.\textsuperscript{7,8} Note that $\phi_0$ also happens to be the angle of maximum differential reflection phase shift\textsuperscript{4} $\Delta$, in the limit of $d/\lambda \gg 1$.

At grazing incidence ($\phi = 90^\circ$), $\psi_G = 45^\circ$, $\tan \psi_G = 1$, and Eq. (25) gives

$$
\psi_G = \arctan(1/N^2),
$$

(30)

independent of film thickness. For $N = 1.5$, Eq. (30) gives $\psi_G = 23.962^\circ$, which corresponds to point G in Fig. 7.

5. REFLECTION AND TRANSMISSION PHASE SHIFTS AT THE CRITICAL ANGLE

It is interesting to consider the reflection and transmission phase shifts by an embedded low-index layer at the critical angle, $\phi_c = \arcsin(1/N)$. At this angle, Eqs. (4) show that $U = 0$. Substitution of $U = 0$ in Eqs. (3) and (6) yields

$$
\delta_r = \tan \delta_s = 0,
$$

$$
x = \tanh x = 0.
$$

(31)

If Eqs. (31) are used in Eqs. (8) and (9), the indeterminate forms

$$
\tan \Delta_{rs} = 0/0,
$$

$$
\tan \Delta_{sp} = 0/0
$$

(32)

(33)

are obtained. By applying L'Hôpital's rule\textsuperscript{9} to Eq. (8), we obtain

$$
\Delta_{rs}(\phi_c) = \arctan(\delta'_r/x') \delta_s,
$$

(34)

where $\delta'_r, x'$ are the angle-of-incidence derivatives of $\delta_s, x$. Evaluation of these derivatives (which are skipped here to save space) at the critical angle and substitution of the results in Eq. (34) give

$$
\Delta_{rs}(\phi_c) = \arctan[\pi^{-1}(d/\lambda)^{-1}N^2(N^2 - 1)^{-1/2}],
$$

(35)

$$
\Delta_{sp}(\phi_c) = \arctan[\pi^{-1}(d/\lambda)^{-1}(N^2 - 1)^{-1/2}]
$$

(36)

for the $p$ and $s$ polarizations, respectively. Numerical values obtained from Eqs. (35) and (36) for $N = 1.5$ agree with the results shown in Figs. 2 and 3, respectively.

At the critical angle, the corresponding transmission phase shifts differ from the reflection phase shifts of Eqs. (35) and (36) by $\pi/2$ according to Eq. (11).

6. SUMMARY

Explicit and elegant expressions [Eqs. (8), (9), (13), and (16)] have been obtained for the phase shifts that $p$- and $s$-polarized light experience in FTIR and optical tunneling by an embedded low-index thin film. The differential phase shifts in reflection and transmission $\Delta_p, \Delta_s$ are identical [Eqs. (14) and (23)], so that incident linearly polarized light is reflected and transmitted as elliptically polarized light of the same handedness. The associated ellipsometric parameters $\psi_p, \psi_s$ are governed by a simple relation [Eq. (25)], which is independent of film thickness.

At the special angles of incidence at which the Fresnel interface reflection phase shifts for the $p$ and $s$ polarizations and their average are quarter-wave\textsuperscript{4} [Eqs. (27)–(29)], the corresponding overall reflection phase shifts introduced by the embedded layer are also quarter-wave for all values of film thickness. Furthermore, in the limit of zero film thickness (i.e., for an ultrathin embedded layer), the reflection phase shifts are quarter-wave independent of polarization ($p$ or $s$) or angle of incidence (except at grazing incidence).

Finally, it has been noted that variable-angle FTIR ellipsometry is particularly suited for measuring thin uniform air gaps between bulk dielectric prisms (Fig. 5).

APPENDIX A

The phase shifts that accompany total internal reflection at a dielectric–dielectric interface were considered in Ref. 4. The reflection phase shifts for $p$ and $s$ polarizations $\delta_p$ and $\delta_s$ [Eqs. (3)] increase monotonically from $0$ at the critical angle to $\pi$ at grazing incidence (see Fig. 1, Ref. 4). In this appendix, we obtain a simple expression for the ratio of the derivatives (slopes) of the $\delta_p$- and $\delta_s$-versus-$\phi$ curves.

We start with Eq. (8) of Ref. 4, which is repeated here:

$$
\tan(\delta_p/2) = N^2 \tan(\delta_s/2).
$$

(A1)

By taking the derivative of the natural logarithm of both sides of Eq. (A1) with respect to $\phi$ and applying some trigonometric identities, we obtain

$$
\frac{\delta_p}{\delta_s} = \frac{\delta_p'}{\delta_s'} = \frac{\sin \delta_p}{\sin \delta_s},
$$

(A2)

where $\delta_p', \delta_s'$ are the derivatives of the interface reflection phase shifts with respect to $\phi$.

At the critical angle, both sides of Eq. (A2) are indeterminate ($\pi/2 = 0/0$). However, by applying L'Hôpital's rule\textsuperscript{9} we obtain

$$
\frac{\delta_p'}{\delta_s'} \bigg|_{\psi_G} = N^2.
$$

(A3)

At grazing incidence, the right-hand side of Eq. (A2) is again indeterminate. By applying L'Hôpital's rule\textsuperscript{9} once more, we obtain

$$
\frac{\delta_p'}{\delta_s'} \bigg|_{\psi_G} = 1/N^2.
$$

(A4)

Equations (A3) and (A4) indicate that the ratio of slopes is reversed between the critical angle and grazing incidence. Finally, we note that

$$
\frac{\delta_p'}{\delta_s'} \bigg|_{\psi_G} = 1,
$$

(A5)

where $\phi_a$ is given by Eq. (29). Equation (A5) is equivalent to $(\delta_p - \delta_s) = 0$, so that $\phi_a$ is also the angle at which the interface differential reflection phase shift is maximum, as was noted in Ref. 4.

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