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Poincaré sphere representation of the fixed-polarizer rotating-retarder optical system

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The trajectory of the polarization state of a monochromatic light beam after it passes through a fixed linear polarizer and a rotating linear retarder of arbitrary retardance Δ is determined on the Poincaré sphere. The three-dimensional figure-8 contour is shown to be the line of intersection of a right-circular cylinder with the sphere. The cylinder is parallel to the polar (s_3) axis, touches the sphere at the equator (at the point that represents the linear polarization transmitted by the fixed polarizer), and has a radius $r = \sin^2(\Delta/2)$. Projections of the trajectory in the coordinate planes of the normalized Stokes parameter space (s_1, s_2, s_3) are also determined. © 2000 Optical Society of America [S0740-3232(00)02411-X]

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The combination of a linear polarizer and linear retarder is a versatile optical system that is commonly used for polarization-state generation (PSG) and polarization-state detection[1–4] (PSD). Rotation of these elements around the light beam axis creates a variable elliptic polarizer or analyzer. The retardance (differential phase shift) Δ of the retarder is often selected as quarter wave ($\Delta = 90°$). This choice is essential for complete PSG, to produce all possible polarization states (including the right- and left-circular polarizations) that correspond to all points on the surface of the Poincaré sphere.[5–7] However, it has been shown recently[8] that for PSD, other choices of Δ may be optimal ($\Delta = 132°$).

When the polarizer is set at a fixed azimuth and the retarder is rotated, the point that represents the emergent polarization state in PSG (or the elliptic analyzer in PSD) traces a three-dimensional, nonplanar, figure-8 contour on the Poincaré sphere. A sketch of such a contour on the Poincaré sphere for a rotating quarter-wave retarder (QWR) appears in an early paper by Rajagopalan and Ramaseshan.[9] A computer-generated family of such curves, for different values of Δ, are presented by Sabatke *et al.*[8] Stereographic projections of the QWR contour onto a tangent (complex) plane are given by Azzam *et al.*[3,10]

In this communication, the exact nature of this contour is determined as the line of intersection of a right-circular cylinder with the Poincaré sphere. The cylinder axis is parallel to the polar (s_3) axis, touches the sphere at the equator (at the point that represents the linear polarization transmitted by the fixed polarizer), and has a radius $r = \sin^2(\Delta/2)$. (Thus $r = 0$, 1/2, and 1, when $\Delta = 0$, $\pi/2$, and π, respectively.) Normal projections of the trajectory in the coordinate planes of the normalized Stokes parameter space are also determined.

Figure 1 shows a PSG system that consists of an ideal linear polarizer with transmission-axis azimuth P and a linear retarder of retardance Δ and fast-axis azimuth C. P and C are measured from the x axis of a transverse xy

reference coordinate system for a monochromatic light beam traveling in the direction of the z axis. When the polarizer is fixed, we set $P = 0$ for simplicity and without loss of generality. With the Stokes–Mueller calculus,[1,3] the normalized Stokes parameters of the output light are obtained as

$$s_0 = 1, \qquad s_1 = \cos^2 2C + \sin^2 2C \cos \Delta,$$

$$s_2 = \sin 2C \cos 2C(1 - \cos \Delta), \qquad s_3 = \sin 2C \sin \Delta. \tag{1}$$

The normalized Stokes vector $\mathbf{s} = (s_1, s_2, s_3)$ is of unit length, and its tip traces a figure-8 contour on the Poincaré sphere as the retarder is rotated (and C is changed) for a given Δ. To reveal the nature of this contour, the normalized Stokes parameters s_1 and s_2 are rewritten in the following form

$$s_1 = \cos^2(\Delta/2) + \sin^2(\Delta/2)\cos 4C, \tag{2a}$$

$$s_2 = \sin^2(\Delta/2)\sin 4C. \tag{2b}$$

C is readily eliminated between Eqs. (2a) and (2b) so that

$$[s_1 - \cos^2(\Delta/2)]^2 + s_2{}^2 = \sin^4(\Delta/2). \tag{3}$$

Equation (3), which represents the normal projection of the contour in the equatorial (s_1, s_2) plane, is that of a circle with center at

$$(s_1, s_2) = [\cos^2(\Delta/2), 0], \tag{4}$$

and radius of

$$r = \sin^2(\Delta/2). \tag{5}$$

It follows that the trajectory of the polarization state at the output of the fixed-polarizer rotating-retarder system of Fig. 1 is the line of intersection with the Poincaré sphere of a right-circular cylinder whose axis and radius are given by Eqs. (4) and (5), respectively.

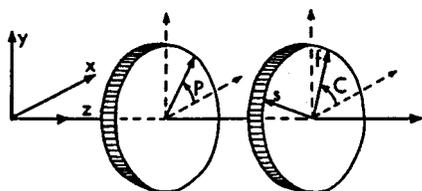Figure 2 shows the cylinder–sphere intersection for QWR ($\Delta = 90°$). For this retardance value the cylinder

Fig. 1. Polarization-state generator that consists of a linear po-
larizer and linear retarder. *P* and *C* are the orientation angles
of the transmission axis of the polarizer and the fast axis of the
retarder, respectively, relative to the reference *x* direction.

teresting to note that the line of intersection represents
the locus of all polarization states with equal azimuth and
ellipticity angles, i.e., equal longitude and latitude on the
sphere.

Figure 3 shows the normal projection of Fig. 2 in the
equatorial (s_1, s_2) plane. The circle of intersection of the
cylinder with the equatorial plane is represented by Eq.
(3), where $\Delta = 90°$.

For completeness, we also determine the two remaining
projections of the trajectory of output states in the (s_1, s_3)
and (s_2, s_3) coordinate planes. Elimination of *C* between
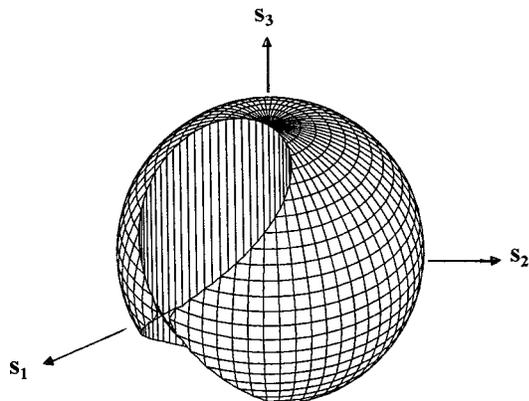these normalized Stokes parameter pairs in Eqs. (1) gives

Fig. 2. Locus of the polarization state of light leaving a fixed-
polarizer rotating-retarder system (Fig. 1) is represented by the
line of intersection of a right-circular cylinder with the Poincaré
sphere. The diameter of the cylinder is half the diameter of the
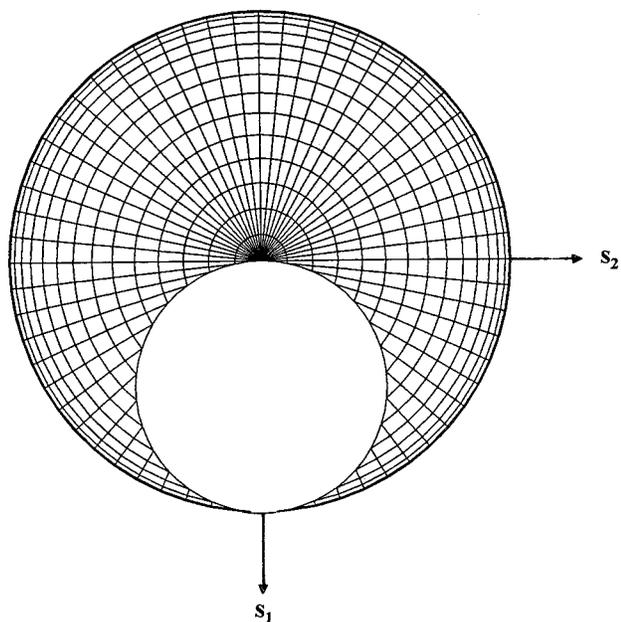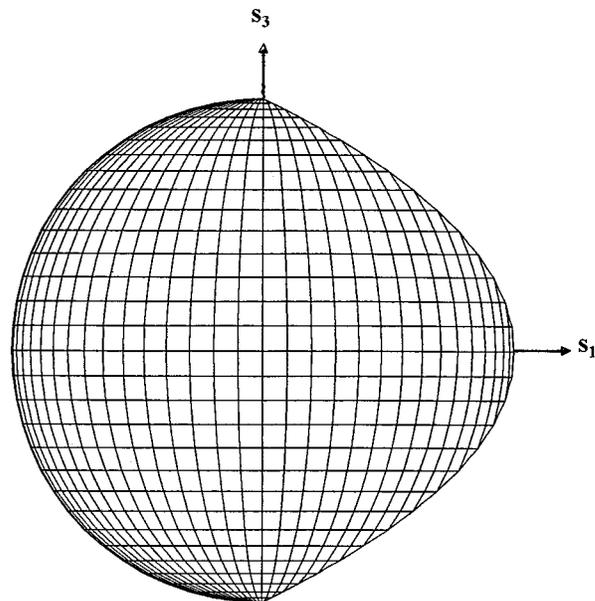sphere for QWR ($\Delta = 90°$).

Fig. 4. Normal projection of Fig. 2 on the (s_1, s_3) coordinate
plane. The trajectory of the polarization states shown in Fig. 2
also lies on a parabolic cylinder parallel to the s_2 axis.

Fig. 3. Normal projection of Fig. 2 on the (s_1, s_2) coordinate
plane, where (s_1, s_2, s_3) are the normalized Stokes parameters.

radius is half the sphere radius and the intersection con-
tour passes through the north and south poles of the
sphere, which correspond to the right- and left-circular
polarizations, respectively. The grid of longitudes and
latitudes on the Poincaré sphere represent the equiazi-
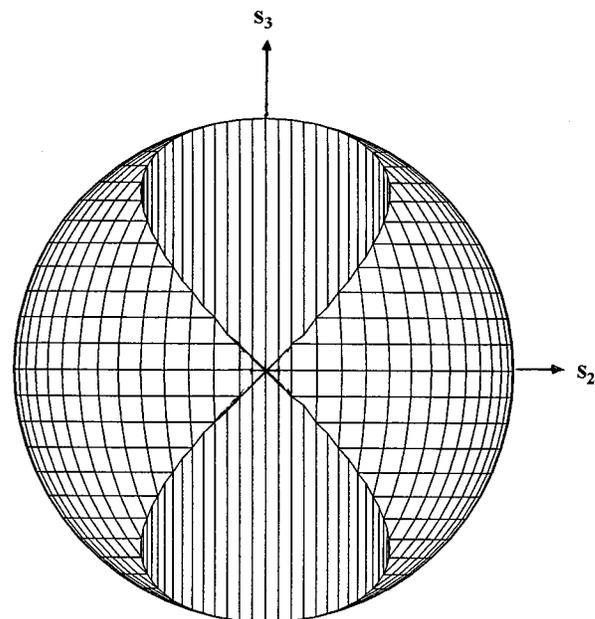muth and equiellipticity contours, respectively.[3] It is in-

Fig. 5. Normal projection of Fig. 2 on the (s_2, s_3) coordinate
plane showing a head-on view of the figure-8 contour.

$$s_3{}^2 = [2\cos^2(\Delta/2)](1 - s_1), \qquad (6)$$

$$s_3{}^4 - (\sin^2\Delta)s_3{}^2 + [4\cos^2(\Delta/2)]s_2{}^2 = 0. \qquad (7)$$

Equation (6) shows another interesting result, namely, that the trajectory of output states is also the line of intersection of a parabolic cylinder parallel to the s_2 axis with the Poincaré sphere. Equation (7) is that of the projected figure-8 contour in the (s_2, s_3) plane.

Figures 4 and 5 represent the two normal projections of Fig. 2 in the (s_1, s_3) and (s_2, s_3) coordinate planes, respectively, and show the projected parabola and figure-8 contours for $\Delta = 90°$.

The foregoing results apply equally when the direction of propagation of light in Fig. 1 is reversed and the system functions as an elliptic analyzer or polarimeter. In this case the normalized Stokes parameters are those of the input elliptic polarization state of maximum transmission through the system for given orientations of the optical elements.

In conclusion, a detailed analysis of the fixed-polarizer rotating-retarder optical system has been presented. The three-dimensional figure-8 locus of (output or input) polarization states characteristic of this common system is shown to be the line of intersection of the Poincaré sphere with a right-circular cylinder whose axis is parallel to the polar axis of the sphere and whose radius is equal to the squared sine of half the retardance of the rotating retarder. These results complement those obtained by others[8–10] concerning this locus.

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