Relationship between the p and s Fresnel reflection coefficients of an interface independent of angle of incidence

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The Fresnel reflection coefficients \( r_p \) and \( r_s \) of \( p \)- and \( s \)-polarized light at the planar interface between two linear isotropic media are found to be interrelated by \( (r_p - r_s)/(1 - r_p r_s) = \cos 2\phi \), independent of the angle of incidence \( \phi \), where \( \tan^2 \beta = c \) and \( \epsilon \) is the (generally complex) ratio of dielectric constants of the media of refraction and incidence. This complements another relation (found earlier), \( (r_p^2 - r_s^2)/(r_s - r_p r_s) = \cos 2\phi \), which is valid at a given \( \phi \) independent of \( \epsilon \) (i.e., for all possible interfaces). Taken together, these two equations specify \( r_p \) and \( r_s \) completely and can be used to replace the original Fresnel equations.

\[
(1 - r_p) = f(\epsilon, \phi), \tag{1}
\]
\[
r_s = g(\epsilon, \phi), \tag{2}
\]

where \( \phi \) is the angle of incidence and \( \epsilon \) is the ratio of the dielectric constant of the medium of incidence (real). By eliminating \( \epsilon \) between Eqs. (1) and (2), we previously obtained

\[
r_p = r_s(r_s - \cos 2\phi)/(1 - r_s \cos 2\phi). \tag{3}
\]

Equation (3) is a direct relation between \( r_p \) and \( r_s \) that is valid for all possible interfaces at a given angle of incidence \( \phi \). Its properties and the insight that one can derive from it are discussed in Refs. 2 and 3.

It is apparent that it should also be possible to eliminate \( \phi \) between Eqs. (1) and (2) and obtain a second direct relation between \( r_p \) and \( r_s \) that is valid for a given interface (or a given \( \epsilon \)) at all angles of incidence. After a few algebraic steps, we get

\[
(r_s - r_p)/(1 - r_p r_s) = c, \tag{4}
\]

where

\[
c = (1 - \epsilon)/(1 + \epsilon). \tag{5}
\]

Equation (4) is, to our knowledge, new and can be verified by direct substitution from Eqs. (1) and (2). Equation (4) can be rearranged to read as

\[
r_p = (r_s - c)/(1 - cr_s), \tag{6}
\]
or

\[
r_s = (r_p + c)/(1 + cr_p), \tag{7}
\]

each of which is in the form of a bilinear transformation. (For a given complex \( \epsilon \), as \( \phi \) increases from 0° to 90°, \( r_p \) and \( r_s \) trace trajectories in the complex plane that are images of each other through such a transformation.)

It is instructive to consider some special cases. At normal incidence (\( \phi = 0 \)), \( r_p = -r_s \) (in the Nebraska conventions), and Eq. (4) gives \( 2r_s = c(1 + r_s^2) \). When this equation is solved for \( r_s \), one retrieves the known result \( r_s(0) = (1 - \epsilon^2)/(1 + \epsilon^2) \). At grazing incidence, \( r_p = r_s \), and Eq. (4) takes the form \( 0/(1 - r_s^2) = c \neq 0 \), from which one correctly obtains \( r_s^2 = 1 \).

At \( \phi = 45^\circ \), \( r_p = r_s^2 \); substitution of this result into Eq. (4) gives \( r_s/(r_s^2 + r_s + 1) = (1 - \epsilon)/(1 + \epsilon) \). By solving the latter equation for \( \epsilon \), one obtains \( \epsilon = (1 + r_s^2)/(1 + r_s)^2 \), a nice-looking result that has proved to be useful recently.

For a dielectric–dielectric interface, \( r_p = 0 \) when light is incident at the Brewster angle \( \phi = \phi_B \). Setting \( r_p = 0 \) in Eq. (4) gives

\[
r_s(\phi_B) = c = (1 - \epsilon)/(1 + \epsilon). \tag{8}
\]

Equation (8) is a simple reduced form of the Fresnel reflection coefficient for the unextinguished \( s \) polarization at the Brewster angle. Equation (8) also provides some meaning for the constant \( c \). By combining Eqs. (6) and (8), we obtain

\[
r_p(\phi) = r_s(\phi_B)/r_s(\phi_B), \tag{9}
\]

Equation (9) is interesting in that it expresses the reflection coefficient of a dielectric–dielectric interface for the \( p \) polarization at any angle \( \phi \) in terms of that for the \( s \) polarization at the same angle \( \phi \) and at the Brewster angle \( \phi_B \).

For a dielectric–dielectric interface under conditions of total internal reflection (TIR; \( \epsilon < 1 \) and \( c > 0 \)), we can write

\[
r_p = \exp(j\delta_p), \quad r_s = \exp(j\delta_s), \tag{10}
\]

where \( \delta_p \) and \( \delta_s \) are the phase shifts for \( p \) and \( s \) polarization, respectively.
where $\delta_p$ and $\delta_s$ are the phase shifts that the $p$- and $s$-polarized components of the electric vector experience on TIR. By substituting Eqs. (10) into Eq. (6) and noting that $c$ is real, one gets after some manipulations

$$\tan \delta_p = \frac{(1 - c^2) \sin \delta_s}{2c + (1 + c^2) \cos \delta_s}.$$  

Equation (11) ties directly the TIR phase shifts $\delta_p$ and $\delta_s$ for a given interface at all angles of incidence, from the critical angle $\phi_c = \sin^{-1} \sqrt{\frac{1}{c^2}}$ to grazing incidence $\phi = 90^\circ$.

Before concluding, we note that Eq. (3) can be rewritten as

$$\frac{r_s^2 - r_p}{r_s - r_p r_p} = \cos 2\phi,$$  

which bears resemblance to Eq. (4). To create more symmetry between the two independent relations between $r_p$ and $r_s$ at constant $\epsilon$ [Eq. (4)] and at constant $\phi$ [Eq. (12)], we introduce a (generally complex) angle $\beta$ such that

$$\epsilon = \tan^2 \beta.$$  

(It is interesting to note that $\beta$ reduces to the usual Brewster angle, $\beta = \phi_B$, when $\epsilon$ is real, i.e., for an interface between two transparent media.) With Eq. (13), $c$ of Eq. (5) becomes

$$c = \cos 2\beta,$$  

and Eq. (4) now reads as

$$\frac{r_s - r_p}{(1 - r_s)^2} = \cos 2\beta.$$  

The similarity in structure between Eqs. (12) and (15) is remarkable. These two equations, taken together, specify $r_p$ and $r_s$ completely and can be used to replace the original Fresnel equations.

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REFERENCES AND NOTES

1. See, for example, M. Born and E. Wolf, *Principles of Optics* (Pergamon, New York, 1975), Sec. 1.5.2.
4. The simplest way is to form the ratios $U_\nu = (1 - r_\nu)/(1 + r_\nu)$, where $\nu = p, s$. It follows immediately that $U_s = \epsilon U_p$, from which Eq. (4) is obtained.
8. With $c = \cos 2\beta$, Eq. (6) also becomes similar to Eq. (3), except for the multiplicative factor $r_s$ that appears in the right-hand side of Eq. (3).