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Maximum minimum reflectance of parallel-polarized light at interfaces between transparent and absorbing media

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The pseudo-Brewster angle ϕ_{pB}, of minimum reflectance $\mathcal{R}_{pm}$ for the parallel (p) polarization, of an interface between a transparent and an absorbing medium is determined by $\mathrm{Im}\{(\epsilon - u)[1 - (1 + \epsilon^{-1})u]^2\} = 0$, where ϵ is the complex ratio of dielectric constants of the media and $u = \sin^2\phi_{pB}$. It is shown that, for a given value of the normal-incidence amplitude reflectance $|r|$, there is an associated normal-incidence phase shift, $\delta = \delta_{mm}$, that leads to maximum minimum parallel reflectance, $\mathcal{R}_{pmm}$. We determine δ_{mm}, $\mathcal{R}_{pmm}$, ϕ_{pBmm} as functions of $|r|$. We find that, as $|r|$ increases from 0 to 1, δ_{mm} decreases from 90° to 0, $\mathcal{R}_{pmm}/|r|^2$ increases from 0 to 1, and the associated ϕ_{pBmm} decreases from 45° to 0, all monotonically.

1. INTRODUCTION

Perhaps the most striking feature of the reflection of a plane wave of monochromatic light (or any other electromagnetic radiation) at an interface between two transparent media is the complete extinction of the reflected wave at a certain angle of incidence, the Brewster angle[1] ϕ_B, when the incident light is parallel (p or TM) polarized. If ϵ_0 and ϵ_1 are the dielectric constants at a given wavelength of the media of incidence and refraction, respectively, ϕ_B is given by

$$\phi_B = \tan^{-1}\epsilon^{1/2}, \tag{1}$$

where

$$\epsilon = \epsilon_1/\epsilon_0. \tag{2}$$

When the medium of refraction is absorbing (and ϵ becomes complex), the reflectance of the interface for incident p-polarized light is nonzero at all angles of incidence but reaches a minimum at the so-called pseudo-Brewster angle[2] ϕ_{pB}. In this Letter we derive a new equation for ϕ_{pB} in terms of complex ϵ. The correct relation, which replaces Eq. (1), between ϕ_{pB} and the complex relative refractive index

$$N = N_1/N_0 = \epsilon^{1/2} \tag{3}$$

was first found by Humphreys-Owen[3] after it had eluded others for many years.[4]

Subsequently we present and analyze a condition of maximum minimum parallel reflectance for interfaces between transparent and absorbing media. We were led to this condition by the following reasoning. Let

$$r = |r|e^{j\delta} \tag{4}$$

be the interface normal-incidence complex reflection coefficient. For a given value of the amplitude reflectance, $|r|$ = constant, the oblique-incidence parallel reflectance $\mathcal{R}_p$ goes to zero at exact Brewster angles $\phi_B(0)$ and $\phi_B(\pi)$ when the normal-incidence phase shift δ equals 0 and π, respectively. This represents light reflection from opposite sides of a given interface between two transparent media; in this case, $\phi_B(\pi) = 90° - \phi_B(0)$. When $\delta \neq 0$ or $\delta \neq \pi$, $\mathcal{R}_p$ reaches a nonzero minimum $\mathcal{R}_{pm}$ at a pseudo-Brewster angle ϕ_{pB}. If we allow δ to vary continuously from 0 to π with $|r|$ = constant, $\mathcal{R}_{pm}$ must go from 0 (at $\delta = 0$) to a maximum $\mathcal{R}_{pmm}$ at a certain $\delta = \delta_{mm}$ and back to 0 (at $\delta = \pi$). The subscript mm denotes maximum minimum here and throughout.

The condition of maximum minimum parallel reflectance is verified by direct computation assuming different values of $|r|$. $\mathcal{R}_{pm}$ and ϕ_{pB} are determined as functions of δ and $|r|$, and $\mathcal{R}_{pmm}$, δ_{mm}, and ϕ_{pBmm} are computed and plotted versus $|r|$.

We adopt the $e^{j\omega t}$ time dependence and the Nebraska (Muller) conventions.[5] At normal incidence the reflection coefficients for the p and s polarizations differ in sign, i.e., $r_s = -r_p = r$.

2. NEW DERIVATION FOR THE PSEUDO-BREWSTER ANGLE ϕ_{pB} OF AN INTERFACE WITH KNOWN ϵ

The simplicity of the following derivation of the pseudo-Brewster angle ϕ_{pB} in terms of the complex relative dielectric constant $\epsilon = \epsilon_1/\epsilon_0$ of an interface, compared with that of Ref. 3, results from stating the condition of minimum parallel reflectance in terms of the complex amplitude-reflection coefficient r_p instead of the real intensity reflectance

$$\mathcal{R}_p = |r_p|^2. \tag{5}$$

Thus, if we write

$$r_p = |r_p|e^{j\delta_p} \tag{6}$$

at any angle of incidence ϕ and take the derivative with respect to ϕ (indicated by a prime superscript) of the natural logarithm of both sides, we get

$$r_p'/r_p = (|r_p|'/|r_p|) + j\delta_p'. \tag{7}$$

The condition for minimum parallel reflectance is that

$$\mathcal{R}_p' = 0, \tag{8}$$

or, equivalently,

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$$2|r_p||r_p|' = 0 \tag{9}$$

if Eq. (5) is used. Because $|r_p| \neq 0$ at any angle of incidence when ϵ is complex, Eq. (9) requires that

$$|r_p|' = 0. \tag{10}$$

With $|r_p| \neq 0$, the condition for minimum $\mathcal{R}_p$ takes its most convenient form when Eq. (10) is substituted into Eq. (7), namely,

$$\mathrm{Re}(r_p'/r_p) = 0. \tag{11}$$

Equation (11) locates the pseudo-Brewster angle, $\phi = \phi_{\mathrm{pB}}$.

To proceed from Eq. (11), we must write r_p as a function[6] of ϵ and ϕ:

$$r_p = (1 - X)/(1 + X), \tag{12}$$

$$X = (\epsilon - \sin^2\phi)^{1/2}/\epsilon \cos \phi. \tag{13}$$

Differentiation of Eqs. (12) and (13) gives

$$\begin{aligned} r_p'/r_p &= -2X'/(1 - X^2) \\ &= \frac{2\epsilon(1-\epsilon)\sin\phi}{(\epsilon - \sin^2\phi)^{1/2}(\epsilon^2\cos^2\phi - \epsilon + \sin^2\phi)}. \end{aligned} \tag{14}$$

By substituting Eq. (14) into Eq. (11) and by using the convenient change of variable

$$u = \sin^2\phi, \tag{15}$$

we obtain

$$\mathrm{Re}\left[(\epsilon - u)^{1/2}\left(1 - \frac{\epsilon + 1}{\epsilon}u\right)\right] = 0. \tag{16}$$

In reaching Eq. (16) we used the fact that, if $\mathrm{Re}(rz) = 0$, where r and z are real and complex, respectively, $\mathrm{Re}(1/z) = 0$.

For a given complex ϵ (of a given interface), the u that satisfies Eq. (16) determines the pseudo-Brewster angle

$$\phi_{\mathrm{pB}} = \sin^{-1}u^{1/2}. \tag{17}$$

In the special case when ϵ is real (i.e., for an interface between two transparent media), Eq. (16) has the following solution[7] for u:

$$u = \epsilon/(\epsilon + 1). \tag{18}$$

From Eqs. (18) and (15), we get

$$\epsilon = u/(1 - u) = \tan^2\phi, \tag{19}$$

which is the correct Brewster law, as expected.

An alternative preferable form of the equation for the pseudo-Brewster angle is

$$\mathrm{Im}\left[(\epsilon - u)\left(1 - \frac{\epsilon + 1}{\epsilon}u\right)^2\right] = 0. \tag{20}$$

To obtain Eq. (20) from Eq. (16) we used the fact that, when $\mathrm{Re}\, z = 0$, $\mathrm{Im}\, z^2 = 0$.

Equation (20) can be used to prove the existence of the pseudo-Brewster angle for any interface, i.e., for any given ϵ. We write Eq. (20) as $\mathrm{Im}[F(u)] = 0$. A solution of this equation for u (hence for the pseudo-Brewster angle) exists if the trajectory of $F(u)$ in the complex plane, as u increases from 0 to 1, intersects the real axis. Such intersection is guaranteed because the end points of this trajectory, $F(0) = \epsilon$ and $F(1) = \epsilon^{-1} - 1$, lie on opposite sides of the real axis for any complex ϵ.

If we substitute

$$\epsilon = \epsilon_r + j\epsilon_i, \tag{21a}$$

$$1/\epsilon = \bar{\epsilon} = \bar{\epsilon}_r + j\bar{\epsilon}_i, \tag{21b}$$

where

$$\bar{\epsilon}_r = \epsilon_r/(\epsilon_r^2 + \epsilon_i^2), \qquad \bar{\epsilon}_i = -\epsilon_i/(\epsilon_r^2 + \epsilon_i^2) \tag{21c}$$

and ϵ_i is negative in the Nebraska (Muller) conventions,[5] Eq. (20) can be expanded to give a cubic equation in u:

$$\alpha_3u^3 + \alpha_2u^2 + \alpha_1u + \alpha_0 = 0, \tag{22}$$

with coefficients

$$\begin{aligned} \alpha_0 &= -\epsilon_i, \\ \alpha_1 &= 2\epsilon_i, \\ \alpha_2 &= -(\epsilon_i + 3\bar{\epsilon}_i), \\ \alpha_3 &= 2\bar{\epsilon}_i(1 + \bar{\epsilon}_r). \end{aligned} \tag{23}$$

Use of Eqs. (21c) and multiplication by a common factor lead to the following equivalent set of coefficients:

$$\begin{aligned} \alpha_0 &= |\epsilon|^4, \\ \alpha_1 &= -2|\epsilon|^4, \\ \alpha_2 &= |\epsilon|^4 - 3|\epsilon|^2, \\ \alpha_3 &= 2\epsilon_r + 2|\epsilon|^2, \end{aligned} \tag{24}$$

where $|\epsilon|^2 = \epsilon_r^2 + \epsilon_i^2$. Equation (22), with coefficients given by Eqs. (24), becomes identical to the corresponding cubic equation derived by Hymphreys-Owen[3] when the substitution $\epsilon = N^2 = (n - jk)^2$ is made.

3. CONDITION OF MAXIMUM MINIMUM PARALLEL REFLECTANCE

In terms of the complex normal-incidence reflection coefficient, $|r|e^{j\delta}$, the complex relative dielectric constant ϵ of the interface is given by

$$\epsilon = \left(\frac{1 - |r|e^{j\delta}}{1 + |r|e^{j\delta}}\right)^2. \tag{25}$$

For a given value of the normal-incidence amplitude reflectance, $|r| = 0.1, 0.2, \ldots, 0.9$, we let the associated normal-incidence phase shift δ take values from 0 to 180° in equal steps of 1°. ϵ is computed from Eq. (25) and the coefficients of the cubic equation are determined by Eqs. (24). The cubic Eq. (22) is solved explicitly and exactly,[8] and only one real root is always found in the interval $0 \leq u \leq 1$, from which the pseudo-Brewster angle ϕ_{pB} is calculated by using Eq. (17). Figure 1 shows ϕ_{pB} as a function of δ with $|r|$ as a parameter marked by each curve. As we have already noted in Section 1, for a given $|r|$, $\delta = 0$ and $\delta = 180°$ represent the limiting cases of internal and external reflection, respectively, at a dielectric–dielectric interface, with associated exact Brewster angles that sum to 90°.

In Fig. 1 all curves appear to pass through a common point, which leads to the interesting conclusion that a pseudo-

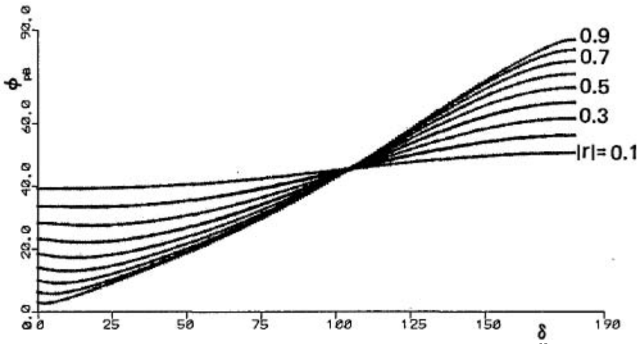

Fig. 1. Pseudo-Brewster angle ϕ_{pB} as a function of the normal-incidence reflection phase shift δ for different constant values of the normal-incidence amplitude reflectance, $|r| = 0.1, 0.2, \ldots, 0.9$, as a parameter. Both δ and ϕ_{pB} are in degrees.

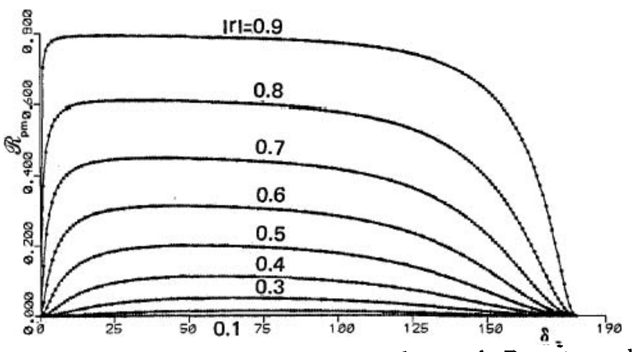

Fig. 2. Minimum parallel reflectance at the pseudo-Brewster angle $\mathcal{R}_{pm}$ as a function of the normal-incidence reflection phase shift δ (in degrees) for different constant values of the normal-incidence amplitude reflectance, $|r| = 0.1, 0.2, \ldots, 0.9$, as a parameter.

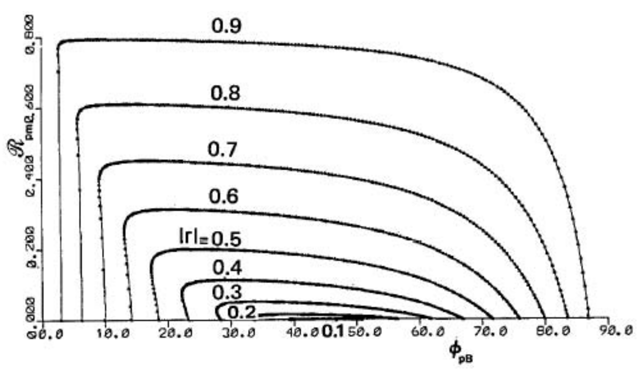

Fig. 3. Minimum parallel reflectance $\mathcal{R}_{pm}$ versus the pseudo-Brewster angle ϕ_{pB} (in degrees) for different constant values of the normal-incidence amplitude reflectance, $|r| = 0.1, 0.2, \ldots, 0.9$, as a parameter.

Brewster angle of 45° corresponds to a normal-incidence phase shift that is restricted to a brief interval $99° \lesssim \delta \lesssim 105°$ for $0.1 \leq |r| \leq 0.9$.

After ϕ_{pB} is calculated for a given ϵ, the associated minimum parallel reflectance $\mathcal{R}_{pm}$ is determined from Eqs. (5), (12), and (13). In Fig. 2 $\mathcal{R}_{pm}$ is plotted versus δ with $|r|$ as a parameter. $\mathcal{R}_{pm}$ equals 0 when $\delta = 0$, $\delta = 180°$ (corresponding to extinction of the reflected wave at exact Brewster angles) and reaches a maximum, $\mathcal{R}_{pmm}$, at a certain phase $0 < \delta_{mm} < 180°$. The peak of each $\mathcal{R}_{pm}$-versus-δ curve is broad.

From the data of Figs. 1 and 2, δ can be eliminated, and $\mathcal{R}_{pm}$ is related to ϕ_{pB} at constant $|r|$. The results appear in Fig. 3.

Figures 1–3 can be used as nomograms for approximate calculation[9] of complex ϵ from measured ϕ_{pB} and $\mathcal{R}_{pm}$. For example, such data locate a point in Fig. 3 from which $|r|$ can be read by interpolation. Next, $|r|$ and ϕ_{pB} locate a point in Fig. 1; hence the normal-incidence phase shift δ is determined. Finally, from $|r|e^{j\delta}$, ϵ is calculated by using Eq. (25). Of course, nomograms with larger numbers of curves can be computer generated for higher accuracy. Alternatively, the approximate ϵ can be improved by numerical iteration to minimize the difference between the measured and computed (ϕ_{pB}, $\mathcal{R}_{pm}$).

Because we are particularly interested in the condition of maximum minimum parallel reflectance, the normal-incidence phase shift required to achieve this condition at a given $|r|$, δ_{mm}, was determined. Figure 4 shows δ_{mm} as a function of $|r|$. δ_{mm} decreases from 90° to 0 as $|r|$ increases from 0 to 1. The associated maximum minimum parallel reflectance,

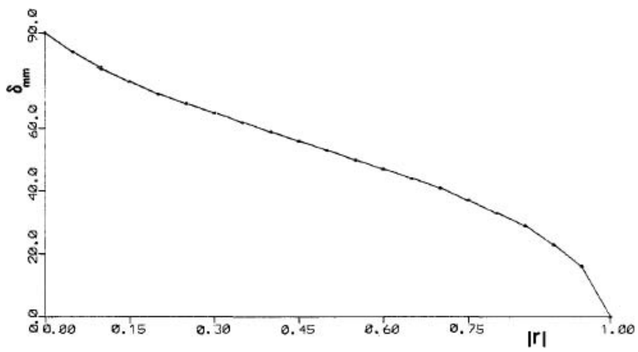

Fig. 4. Normal-incidence reflection phase shift δ_{mm} (in degrees) that leads to maximum minimum parallel reflectance for a given normal-incidence amplitude reflectance $|r|$ plotted here versus $|r|$.

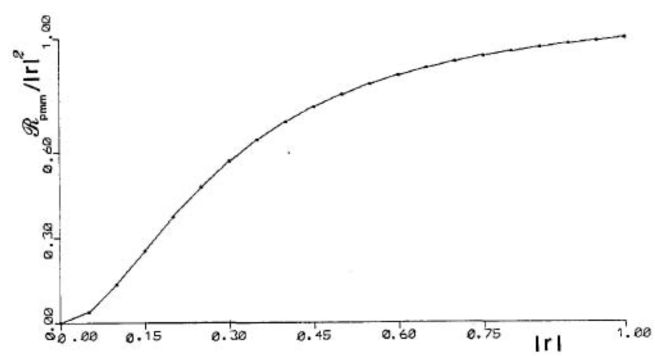

Fig. 5. Ratio of maximum minimum parallel reflectance $\mathcal{R}_{pmm}$ to normal-incidence reflectance $|r|^2$ plotted as a function of $|r|$.

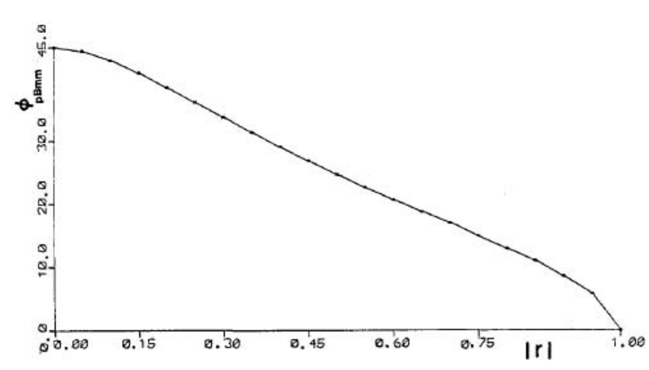

Fig. 6. Pseudo-Brewster angle (in degrees) ϕ_{pBmm} of maximum minimum parallel reflectance plotted versus the normal-incidence amplitude reflectance $|r|$.

$\mathcal{R}_{\rm pmm}$, normalized as a fraction of the normal-incidence intensity reflectance, i.e., $\mathcal{R}_{\rm pmm}/|r|^2$, is plotted versus $|r|$ in Fig. 5. We see that such a fraction increases from 0 to 1 as $|r|$ increases from 0 to 1. Finally, Fig. 6 shows $\phi_{\rm pBmm}$, associated with $\mathcal{R}_{\rm pmm}$, versus $|r|$. $\phi_{\rm pBmm}$ decreases monotonically from 45° to 0 as $|r|$ is increased from 0 to 1.

4. SUMMARY

If $|r|e^{j\delta}$ represents Fresnel's complex-amplitude normal-incidence reflection coefficient at an interface between a transparent and an absorbing medium, we find that, for a given $|r|$, the minimum reflectance for the parallel polarization $\mathcal{R}_{\rm pm}$ at the pseudo-Brewster angle $\phi_{\rm pB}$ reaches a maximum, $\mathcal{R}_{\rm pmm}$, at a certain normal-incidence phase shift $\delta = \delta_{\rm mm}$. As $|r|$ increases from 0 to 1, $\delta_{\rm mm}$ (for maximum minimum parallel reflectance) decreases from 90° to 0, $\mathcal{R}_{\rm pmm}/|r|^2$ increases from 0 to 1, and the associated $\phi_{\rm pBmm}$ decreases from 45° to 0, all monotonically.

These results are obtained after a new form of the equation for the pseudo-Brewster angle [Eq. (20)] is derived. The condition of maximum minimum parallel reflectance is verified through a graphical study of $\mathcal{R}_{\rm pm}$ as a function of δ with $|r|$ as a parameter. Furthermore, we plot $\phi_{\rm pB}$ versus δ and $\mathcal{R}_{\rm pm}$ versus $\phi_{\rm pB}$, with $|r|$ as a parameter. These graphs can be used as nomograms to determine the complex relative dielectric constant ϵ of an interface from measured $\phi_{\rm pB}$ and $\mathcal{R}_{\rm pm}$.

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7. The other solution of Eq. (16), $u = \epsilon$, is unacceptable when $\epsilon > 1$ or $\epsilon < 0$. When $0 < \epsilon < 1$, it yields the critical angle of total internal reflection, $\sin^{-1}\epsilon^{1/2}$.
8. See, for example, S. M. Selby, ed., *Standard Mathematical Tables*, 20th ed. (Chemical Rubber, Cleveland, Ohio, 1972), pp. 103–105.
9. This is Method F of Ref. 3.