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# Frequency-mixing detection (FMD) of polarization-modulated light

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When a light beam whose polarization and intensity are weakly modulated at a frequency $\omega_m$ passes through a periodic analyzer of frequency $\omega_a(\ll\omega_m)$ and the transmitted flux is linearly detected, the resulting total signal $S_t$ consists of two components: (i) a periodic baseband signal $S_{bb}$ with harmonics of frequencies $n\omega_a(n = 0,1,2,...)$ and (ii) an amplitude-modulated-carrier signal $\delta S_{mc}$ with center (carrier) frequency $\omega_m$ and sideband frequencies at $\omega_m \pm n\omega_a(n = 1,2,...)$. In this paper we show that the average polarization of the beam is determined by a limited spectral analysis of $S_{bb}$, whereas the polarization and intensity modulation are determined by a limited spectral analysis of $\delta S_{mc}$, or the associated envelope signal $\delta S_e$, where $\delta S_{mc} = \delta S_e \cos\omega_m t$. The theory of this frequency-mixing detection (FMD) of polarization modulation is developed for an arbitrary periodic analyzer. The specific case of a rotating analyzer is considered as an example. Applications of FMD include the retrieval of information impressed on light beams as polarization modulation in optical communication systems, and the automation of modulated ellipsometry, AIDER (angle-of-incidence-derivative ellipsometry and reflectometry), and modulated generalized ellipsometry.


## I. INTRODUCTION

Consider a beam of totally polarized light propagating in the $z$ direction, and let $E_x$ and $E_y$ be the complex amplitudes of the projections of the electric vector along two transverse mutually orthogonal directions $x$ and $y$, $xyz$ being a right-handed Cartesian coordinate system. The beam is characterized by its total intensity

$$I = E_x E_x^* + E_y E_y^* , \tag{1}$$

and its ellipse of polarization, specified completely by the complex number

$$\chi = E_y/E_x = \tan\psi\, e^{j\Delta} , \tag{2}$$

where $\Delta$ and $\psi$ represent the relative phase and arctangent of the relative amplitude of the $y$ and $x$ components, respectively. We assume that the beam is both intensity and polarization modulated such that

$$I = \bar{I} + \delta I , \tag{3a}$$

$$\psi = \bar{\psi} + \delta\psi , \quad \Delta = \bar{\Delta} + \delta\Delta , \tag{3b}$$

and that the modulation is small:

$$|\delta I/\bar{I}| \ll 1 , \quad |\delta\psi| , |\delta\Delta| < \sim \pi/12 . \tag{4}$$

$\bar{I}$ and $(\bar{\psi}, \bar{\Delta})$ represent the quiescent intensity and polarization of the beam, respectively; $\delta I$ and $(\delta\psi, \delta\Delta)$ represent the associated intensity and polarization modulation. For simplicity, we further assume that the modulation is sinusoidal with time $t$ and of frequency $\omega_m \ll \omega_{\text{opt}}$, where $\omega_{\text{opt}}$ is the optical frequency. Therefore we can write

$$\delta X = \widehat{\delta X} \cos\omega_m t , \quad X = I,\ \psi, \text{ and } \Delta \tag{5}$$

where the caret indicates the amplitude of a sinusoidal quantity.

In this paper we describe a frequency-mixing technique for the simultaneous detection of the five param-

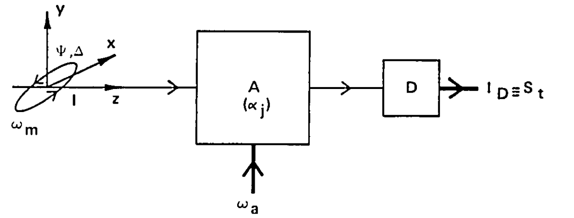


FIG. 1. A light beam of polarization ($\psi$, $\Delta$) and intensity ($I$) that are weakly modulated at a frequency $\omega_m$ is transmitted through an analyzer A one or more of whose parameters $\alpha_j$ is periodically swept at a frequency $\omega_a$. The interaction between the modulated beam and the periodic analyzer results in frequency mixing that is borne in the intensity variations of the light leaving the analyzer. The latter intensity variations are detected by a linear polarization-independent photodetector D, giving an electrical signal $\mathscr{I}_D$ (or $S_t$). By spectral analysis of $S_t$ (Fig. 3), the average polarization as well as the polarization and intensity modulation of the beam can be determined. $xyz$ represents a reference Cartesian coordinate system with the $z$ axis along the direction of propagation of the light beam.

eters that describe the sinusoidally modulated totally polarized light beam, namely, $\overline{\psi}$, $\overline{\Delta}$; $\widehat{\delta\psi}$, $\widehat{\delta\Delta}$, and $\widehat{\delta I/\overline{I}}$.

## II. PRINCIPLE OF FREQUENCY-MIXING DETECTION (FMD)

When a beam of totally polarized light, described by the three parameters $I$, $\psi$, and $\Delta$, passes through an arbitrary polarization analyzer A, Fig. 1, the transmitted intensity $I_t$ can generally be expressed as

$$I_t = If(\psi,\ \Delta,\ \alpha_j)\ , \tag{6}$$

where the function $f$ and its arguments $\alpha_j$ ($j=1, 2, \ldots, k$) are characteristic of the particular chosen analyzer. When the photodetector is linear, the detected signal $\mathscr{I}_D$ can be written as

$$\begin{aligned}\mathscr{I}_D &= cI_t \\ &= cIf(\psi,\ \Delta,\ \alpha_j)\ ,\end{aligned} \tag{7}$$

where $c$ is a multiplier that depends on the detector. Perturbations ($\delta I$, $\delta\psi$, $\delta\Delta$) of the intensity and polarization parameters of the beam produce a corresponding perturbation $\delta\mathscr{I}_D$ of the detected signal :

$$\delta\mathscr{I}_D = (\partial\mathscr{I}_D/\partial I)\,\delta I + (\partial\mathscr{I}_D/\partial\psi)\,\delta\psi + (\partial\mathscr{I}_D/\partial\Delta)\,\delta\Delta\ . \tag{8}$$

By use of Eq. (7), Eq. (8) becomes

$$\delta\mathscr{I}_D = cI[\,f\,(\delta I/I) + f_\psi\,\delta\psi + f_\Delta\,\delta\Delta]\ , \tag{9}$$

where

$$f_\psi = \partial f/\partial\psi\ ,\quad f_\Delta = \partial f/\partial\Delta\ . \tag{10}$$

Let a periodic sweep be applied to the analyzer A such that one or more of its parameters $\alpha_j$ ($j=1, 2, \ldots, k$) become oscillatory with time. Consequently, $f$ [Eq. (6)] and its derivatives $f_\psi$ and $f_\Delta$ [Eqs. (10)] become periodic functions of time, with (fundamental) frequency $\omega_a$, that can be expanded into their Fourier series:

$$f = f_0 + f_1\sin(\omega_a t + \theta_1) + f_2\sin(2\omega_a t + \theta_2) + \cdots\ , \tag{11a}$$

$$f_\psi = f_{\psi 0} + f_{\psi 1}\sin(\omega_a t + \theta_{\psi 1}) + f_{\psi 2}\sin(2\omega_a t + \theta_{\psi 2}) + \cdots\ , \tag{11b}$$

$$f_\Delta = f_{\Delta 0} + f_{\Delta 1}\sin(\omega_a t + \theta_{\Delta 1}) + f_{\Delta 2}\sin(2\omega_a t + \theta_{\Delta 2}) + \cdots\ . \tag{11c}$$

With the periodic sweep applied to the analyzer and with the beam modulation assumed zero, the detected signal (to be called the baseband signal) is given by

$$S_{bb} = \mathscr{I}_D\big|_{\substack{\omega_a\mathrm{on}\\ \omega_m\mathrm{off}}} = c\overline{I}[\,f_0 + f_1\sin(\omega_a t + \theta_1) + f_2\sin(2\omega_a t + \theta_2) + \cdots]\ , \tag{12}$$

as can be obtained by direct substitution from Eq. (11a) into Eq. (7). Beam modulation ($\delta I$, $\delta\psi$, and $\delta\Delta$) generates a small signal $\delta S_{mc}$ in the detector output that is superimposed on the baseband signal of Eq. (12):

$$\delta S_{mc} = \delta\mathscr{I}_D\big|_{\substack{\omega_a\mathrm{on}\\ \omega_m\mathrm{on}}} = \delta S_e\cos\omega_m t\ , \tag{13a}$$

$$\begin{aligned}\delta S_e = c\overline{I}\{&[\,f_0 + f_1\sin(\omega_a t + \theta_1) + f_2\sin(2\omega_a t + \theta_2) + \cdots](\widehat{\delta I/\overline{I}}) \\ &+ [\,f_{\psi 0} + f_{\psi 1}\sin(\omega_a t + \theta_{\psi 1}) + f_{\psi 2}\sin(2\omega_a t + \theta_{\psi 2}) + \cdots]\widehat{\delta\psi} \\ &+ [\,f_{\Delta 0} + f_{\Delta 1}\sin(\omega_a t + \theta_1) + f_{\Delta 2}\sin(2\omega_a t + \theta_{\Delta 2}) + \cdots]\widehat{\delta\Delta}\}\ .\end{aligned} \tag{13b}$$

Equations (13) are obtained by substituting $\delta I$, $\delta\psi$, and $\delta\Delta$ from Eq. (5), and $f$, $f_\psi$, and $f_\Delta$ from Eqs. (11), into Eq. (9). $\delta S_{mc}$, Eq. (13a), represents a carrier of frequency $\omega_m$ (the beam-modulation frequency) which is amplitude modulated by a periodic envelope signal $\delta S_e$, Eq. (13b), of (fundamental) frequency $\omega_a$ (the periodic-sweep frequency). The total detected signal when the modulated beam passes through the periodic analyzer is obtained by adding Eqs. (12) and (13a):

$$S_t = S_{bb} + \delta S_{mc}\ . \tag{14}$$

The frequency spectrum[1] of $S_t$ is shown schematically in Fig. 2. It consists of (1) the baseband spectrum of $S_{bb}$ [Eq. (12)] with frequencies 0, $\omega_a$, $2\omega_a, \ldots$, and (2) the modulated-carrier spectrum of $\delta S_{mc}$ [Eqs. (13)] with frequencies $\omega_m$, $\omega_m \pm \omega_a$, $\omega_m \pm 2\omega_a, \ldots$.

In the following we show how the average (quiescent) polarization ($\overline{\psi}$, $\overline{\Delta}$) and beam-modulation parameters ($\widehat{\delta I/\overline{I}}$, $\widehat{\delta\psi}$, $\widehat{\delta\Delta}$) can be determined from the spectral analysis of the baseband signal $S_{bb}$ and the modulated-carrier signal $\delta S_{mc}$, respectively. All of the characteristics of the periodic analyzer are considered known, including the functions $f$, $f_\psi$, $f_\Delta$ and their Fourier series.

### A. Determination of $\overline{\psi}$ and $\overline{\Delta}$ from $S_{bb}$

Let $s_\omega$ denote the amplitude of the Fourier component of frequency $\omega$ of the signal $S$, and let $\eta_\omega$ be the amplitude of that component normalized to $s_0$ (the dc component of $S$), i.e.,

$$\eta_\omega = s_\omega/s_0\ . \tag{15}$$

From measurements of the dc component $s_0^{bb}$ and the amplitudes of (the first) two nonzero harmonics[2] of $S_{bb}$, say $s_{p\omega_a}^{bb}$ and $s_{q\omega_a}^{bb}$ ($p$, $q$ integers $\geq 1$), we obtain $\eta_{p\omega_a}$, $\eta_{q\omega_a}$. From Eq. (12) we also have

$$\eta_{p\omega_a} = f_p/f_0\ ,\quad \eta_{q\omega_a} = f_q/f_0\ . \tag{16}$$

For a given periodic analyzer, the function $f$ and its

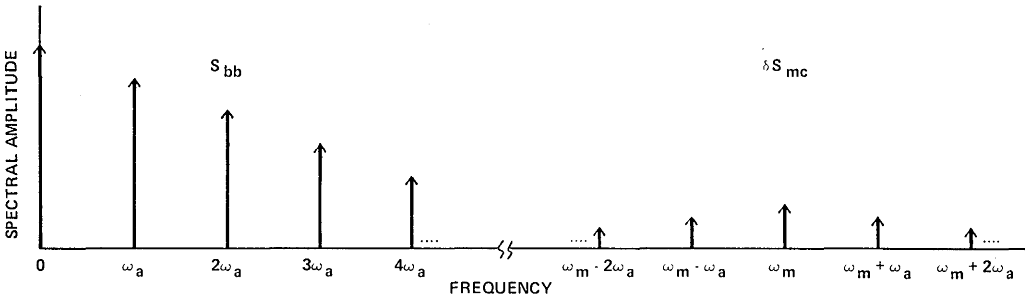


FIG. 2. Frequency spectrum of the baseband ($S_{bb}$) and modulated-carrier ($\delta S_{mc}$) components of the total detected signal $S_t$. $\omega_m$ is the light-modulation frequency and $\omega_a$ is the frequency of the periodic analyzer.

Fourier components $f_n$ are known (or can be determined). $\bar\psi$ and $\bar\Delta$ are arguments of the functions $f_n$, and Eqs. (16) provide two equations that can be solved[3] for $\bar\psi$ and $\bar\Delta$.

## B. Determination of $\widehat{\delta I/\bar I}$, $\widehat{\delta\psi}$, and $\widehat{\delta\Delta}$ from $\delta S_{mc}$

### *1. Special case*

We deal first with the special case of a periodic analyzer for which the function $f$ and its derivatives $f_\psi$ and $f_\Delta$ are all in phase. Under such conditions, the Fourier components of $f$, $f_\psi$, and $f_\Delta$ that have the same frequency are also in phase. Consequently,

$$\theta_{\psi n} = \theta_{\Delta n} = \theta_n \quad \text{for all } n\,, \tag{17}$$

which allows Eqs. (13) to be rewritten as

$$\begin{aligned}\delta S_{mc} &= c\bar I[f_0(\widehat{\delta I/\bar I}) + f_{\psi 0}\widehat{\delta\psi} + f_{\Delta 0}\widehat{\delta\Delta}]\cos\omega_m t \\ &\quad + c\bar I\left\{\sum_{n=1}^{\infty}[f_n(\widehat{\delta I/\bar I}) + f_{\psi n}\widehat{\delta\psi} + f_{\Delta n}\widehat{\delta\Delta}]\sin(n\omega_a t + \theta_n)\right\} \\ &\quad \times\cos\omega_m t\,. \end{aligned}\tag{18}$$

From Eq. (18), the amplitudes of the carrier and $n$th-order sidebands of $\delta S_{mc}$ are given by

$$\begin{aligned} s^{mc}_{\omega_m} &= c\bar I[f_0(\widehat{\delta I/\bar I}) + f_{\psi 0}\widehat{\delta\psi} + f_{\Delta 0}\widehat{\delta\Delta}]\,, \\ s^{mc}_{\omega_m \pm n\omega_a} &= \tfrac{1}{2}c\bar I[f_n(\widehat{\delta I/\bar I}) + f_{\psi n}\widehat{\delta\psi} + f_{\Delta n}\widehat{\delta\Delta}]\,. \end{aligned}\tag{19}$$

From measurements of the dc component $s_0^t$ of $S_t$ (or $S_{bb}$), and the amplitudes of the carrier $s^{mc}_{\omega_m}$ and (the first) two nonzero sidebands of $\delta S_{mc}$, say $s^{mc}_{\omega_m\pm p\omega_a}$, $s^{mc}_{\omega_m\pm q\omega_a}$ ($p$, $q$ integers $\geq 1$), we determine the normalized amplitudes $\eta_{\omega_m}$, $\eta_{\omega_m\pm p\omega_a}$, $\eta_{\omega_m\pm q\omega_a}$ [using the definition of Eq. (15)]. By use of Eqs. (19), and with $s_0^t = c\bar I f_0$ [Eq. (12)], we get

$$\begin{aligned} \eta_{\omega_m} &= (\widehat{\delta I/\bar I}) + (f_{\psi 0}/f_0)\widehat{\delta\psi} + (f_{\Delta 0}/f_0)\widehat{\delta\Delta}\,, \\ 2\eta_{\omega_m\pm p\omega_a} &= (f_p/f_0)(\widehat{\delta I/\bar I}) + (f_{\psi p}/f_0)\widehat{\delta\psi} + (f_{\Delta p}/f_0)\widehat{\delta\Delta}\,, \\ 2\eta_{\omega_m\pm q\omega_a} &= (f_q/f_0)(\widehat{\delta I/\bar I}) + (f_{\psi q}/f_0)\widehat{\delta\psi} + (f_{\Delta q}/f_0)\widehat{\delta\Delta}\,. \end{aligned}\tag{20}$$

These three equations can be solved for the three unknowns[4] $\widehat{\delta I/\bar I}$, $\widehat{\delta\psi}$, and $\widehat{\delta\Delta}$, which represent the desired modulation parameters. In the left-hand side of Eqs. (20), the $\eta$'s are quantities to be measured; in the right-hand side, the coefficients of $\widehat{\delta I/\bar I}$, $\widehat{\delta\psi}$, and $\widehat{\delta\Delta}$ are calculated for the given periodic analyzer.

### *2. General case*

The periodic functions $f$, $f_\psi$, and $f_\Delta$ are not in phase so that Eqs. (17) are not satisfied. From Eqs. (13) it is evident that measurements of the amplitudes of the carrier and two different-order sidebands of $\delta S_{mc}$, normalized to the dc component of $S_t$, are sufficient to determine $\widehat{\delta I/\bar I}$, $\widehat{\delta\psi}$, and $\widehat{\delta\Delta}$. However, the equations in this case become nonlinear (quadratic), making the simultaneous solution for the modulation parameters difficult.

An alternative procedure for measuring the modulation parameters is to employ envelope detection to obtain $\delta S_e$, Eq. (13b). The dc component and the (first) nonzero $p$ harmonic of $\delta S_e$ are given by

$$\begin{aligned} s_0^e &= c\bar I[f_0(\widehat{\delta I/\bar I}) + f_{\psi 0}\widehat{\delta\psi} + f_{\Delta 0}\widehat{\delta\Delta}]\,, \\ s^e_{p\omega_a} &= c\bar I[f_p(\widehat{\delta I/\bar I})\sin(p\omega_a t + \theta_p) + f_{\psi p}\sin(p\omega_a t + \theta_{\psi a})\widehat{\delta\psi} \\ &\quad + f_{\Delta p}\sin(p\omega_a t + \theta_{\Delta p})\widehat{\delta\Delta}]\,, \end{aligned}\tag{21}$$

where the time dependence has been retained in the latter expression of $s^e_{p\omega_a}$. From measurements of $s_0^t$, $s_0^e$, and the amplitudes $s^{ec}_{p\omega_a}$, $s^{es}_{p\omega_a}$ of the cosine and sine components of $s^e_{p\omega_a}$, we obtain the normalized amplitudes $\eta_0^e$, $\eta^{ec}_{p\omega_a}$ and $\eta^{es}_{p\omega_a}$ [see Eq. (15)]. By use of Eqs. (21), and with $s_0^t = c\bar I f_0$ [Eq. (12)], we get

$$\begin{aligned} \eta_0^e &= (\widehat{\delta I/\bar I}) + (f_{\psi 0}/f_0)\widehat{\delta\psi} + (f_{\Delta 0}/f_0)\widehat{\delta\Delta}\,, \\ \eta^{ec}_{p\omega_a} &= (f_p\sin\theta_p/f_0)(\widehat{\delta I/\bar I}) + (f_{\psi p}\sin\theta_{\psi p}/f_0)\widehat{\delta\psi} \\ &\quad + (f_{\Delta p}\sin\theta_{\Delta p}/f_0)\widehat{\delta\Delta}\,, \\ \eta^{es}_{p\omega_a} &= (f_p\cos\theta_p/f_0)(\widehat{\delta I/\bar I}) + (f_{\psi p}\cos\theta_{\psi p}/f_0)\widehat{\delta\psi} \\ &\quad + (f_{\Delta p}\cos\theta_{\Delta p}/f_0)\widehat{\delta\Delta}\,. \end{aligned}\tag{22}$$

Equations (22) can be solved for the three modulation parameters[4] $\widehat{\delta I/\bar I}$, $\widehat{\delta\psi}$, and $\widehat{\delta\Delta}$. The left-hand side of Eqs. (22) represent quantities to be measured, whereas the coefficients of $\widehat{\delta I/\bar I}$, $\widehat{\delta\psi}$, and $\widehat{\delta\Delta}$ in the right-hand side represent quantities that can be calculated for a given periodic analyzer. In the above detection scheme, the envelope signal $\delta S_e$ is to be phase locked with the periodic sweep applied to the analyzer.

Figure 3 is a block diagram of the general scheme that we propose for the detection of the average polar-

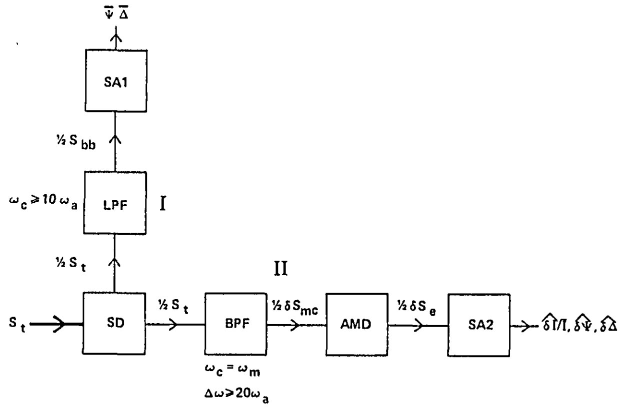


FIG. 3. A block diagram of the electronic signal-processing units needed for frequency-mixing detection of polarization-modulated light. The output $S_t$ of the photodetector (Fig. 1) is divided by the signal divider SD into two equal signals. One signal passes through channel I that consists of the low-pass filter LPF (of cutoff frequency $\omega_c \geq 10\omega_a$, where $\omega_a$ is the periodic-analyzer frequency) and the spectral analyzer SA1. The output of channel I determines the average polarization of the beam $\bar{\psi}$, $\bar{\Delta}$. The second signal passes through channel II that consists of the band-pass filter BPF (of center frequency $\omega_c = \omega_m$, the light-modulation frequency, and bandwidth $\Delta\omega \geq 20\omega_a$), the amplitude-modulation (or envelope) detector AMD, and the signal analyzer SA2. The output of channel II determines the intensity- $(\widehat{\delta I/I})$ and polarization-modulation $(\widehat{\delta\psi}, \widehat{\delta\Delta})$ parameters of the beam. $S_{bb}$, $\delta S_{mc}$ and $\delta S_e$ are the baseband, modulated-carrier, and envelope signals, respectively.

ization $(\bar{\psi}, \bar{\Delta})$ and modulation parameters $(\widehat{\delta I/I}, \widehat{\delta\psi}$, and $\widehat{\delta\Delta})$ of the light beam.

## III. A SIMPLE EXAMPLE

The theory of Sec. II applies to any type of periodic analyzer. For the purpose of demonstration, we take as a specific example a linear analyzer that is rotated at constant angular speed $\frac{1}{2}\omega_a$. If $A$ is the azimuth of the transmission axis of the analyzer, measured from the $x$ axis of the reference Cartesian coordinate system (see Fig. 1), the function $f$ of Eq. (6) assumes the form

$$f = (1+\cos 2A) + \tan^2\psi(1-\cos 2A) + 2\tan\psi\cos\Delta\sin 2A\ , \quad (23)$$

where a constant multiplier (equal to $\frac{1}{8}$ for an ideal linear analyzer) has been dropped.[5] In Eq. (23), $f$ contains only one parameter, $\alpha_1 = A$. By partial differentiation of Eq. (23), we obtain

$$f_\psi = 2\tan\psi\sec^2\psi(1-\cos 2A) + 2\sec^2\psi\cos\Delta\sin 2A\ ,$$
$$f_\Delta = -2\tan\psi\sin\Delta\sin 2A\ . \quad (24)$$

Because the analyzer is rotating, we substitute

$$A = \tfrac{1}{2}\omega_a t\ , \quad (25)$$

into Eqs. (23) and (24); this gives

$$f = \sec^2\psi + (1-\tan^2\psi)\cos\omega_a t + 2\tan\psi\cos\Delta\sin\omega_a t\ , \quad (26a)$$

$$f_\psi = 2\tan\psi\sec^2\psi - 2\tan\psi\sec^2\psi\cos\omega_a t + 2\sec^2\psi\cos\Delta\sin\omega_a t\ , \quad (26b)$$

$$f_\Delta = -2\tan\psi\sin\Delta\sin\omega_a t\ . \quad (26c)$$

As can be seen from Eqs. (26), in the case of a rotating analyzer each of the periodic functions $f$, $f_\psi$, and $f_\Delta$ consists only of a constant term (equal to zero for $f_\Delta$) plus a single spectral component of frequency $\omega_a$.

The average polarization $(\bar{\psi}, \bar{\Delta})$ of the modulated beam is determined not in terms of the normalized amplitudes of two different harmonics as suggested by Eqs. (16), but rather by the normalized amplitudes of the cosine and sine components of the same spectral component of frequency $\omega_a$. Instead of Eqs. (16), we now have

$$\eta^c_{\omega_a} = f_1\sin\theta_1/f_0\ , \quad \eta^s_{\omega_a} = f_1\cos\theta_1/f_0\ . \quad (27)$$

From Eq. (26a), we get

$$f_0 = \sec^2\bar{\psi}\ , \quad f_1\sin\theta_1 = (1-\tan^2\bar{\psi})\ , \quad f_1\cos\theta_1 = 2\tan\bar{\psi}\cos\bar{\Delta}\ , \quad (28)$$

which can be substituted into Eqs. (27) to give,

$$\eta^c_{\omega_a} = (1-\tan^2\bar{\psi})/\sec^2\bar{\psi} = \cos 2\bar{\psi}\ , \quad (29a)$$

$$\eta^s_{\omega_a} = 2\tan\bar{\psi}\cos\bar{\Delta}/\sec^2\bar{\psi} = \sin 2\bar{\psi}\cos\bar{\Delta}\ . \quad (29b)$$

Equation (29a) readily gives $\bar{\psi}$; $\bar{\Delta}$ is obtained subsequently from Eq. (29b).[6]

Because $f$, $f_\psi$, and $f_\Delta$ are not in phase, Eqs. (26), the modulation parameters $\widehat{\delta I/I}$, $\widehat{\delta\psi}$, and $\widehat{\delta\Delta}$ are obtained by use of Eqs. (22). In Eqs. (22), we now have $p=1$ and the coefficients on the right-hand side are identified by Eqs. (28) and by

$$f_{\psi 0} = 2\tan\bar{\psi}\sec^2\bar{\psi}\ , \quad f_{\psi 1}\sin\theta_{\psi 1} = -2\tan\bar{\psi}\sec^2\bar{\psi}\ ,$$
$$f_{\psi 1}\cos\theta_{\psi 1} = 2\sec^2\bar{\psi}\cos\bar{\Delta}\ , \quad (30)$$
$$f_{\Delta 0} = 0\ , \quad f_{\Delta 1}\sin\theta_{\Delta 1} = 0\ , \quad f_{\Delta 1}\cos\theta_{\Delta 1} = -2\tan\bar{\psi}\sin\bar{\Delta}\ ,$$

which follow from Eqs. (26b) and (26c). Therefore, Eqs. (22) now read

$$\eta^e_0 = (\widehat{\delta I/I}) + (2\tan\bar{\psi})\widehat{\delta\psi}\ , \quad (31a)$$

$$\eta^{ec}_{\omega_a} = (\cos 2\bar{\psi})(\widehat{\delta I/I}) + (-2\tan\bar{\psi})\widehat{\delta\psi}\ , \quad (31b)$$

$$\eta^{es}_{\omega_a} = (\sin 2\bar{\psi}\cos\bar{\Delta})(\widehat{\delta I/I}) + (2\cos\bar{\Delta})\widehat{\delta\psi} + (-2\tan\bar{\psi}\sin\bar{\Delta})\widehat{\delta\Delta}\ . \quad (31c)$$

Equations (31) are readily solved for the modulation parameters

$$\frac{\widehat{\delta I}}{I} = \frac{\eta^e_0 + \eta^{ec}_{\omega_a}}{1+\cos 2\bar{\psi}}\ , \quad (32a)$$

$$\widehat{\delta\psi} = \frac{\eta^e_0 - (\widehat{\delta I/I})}{2\tan 2\bar{\psi}}\ , \quad (32b)$$

$$\widehat{\delta\Delta} = \frac{-\eta^{es}_{\omega_a} + (\sin 2\bar{\psi}\cos\bar{\Delta})(\widehat{\delta I/I}) + (2\cos\bar{\Delta})\widehat{\delta\psi}}{2\tan\bar{\psi}\sin\bar{\Delta}}\ , \quad (32c)$$

where $\bar{\psi}$ and $\bar{\Delta}$ are now known from measurements of the average polarization, Eqs. (29). The example of frequency-mixing detection of polarization-modulated signals by a rotating analyzer is now complete.

## IV. APPLICATIONS

In optical communication systems information may be impressed on the light beam as polarization (and intensity) modulation.[7] FMD, as described in this paper, provides a means of demodulation for the purpose of information retrieval. Although we have assumed that the modulation is sinusoidal, extension to arbitrary modulation waveforms is straightforward by decomposing the waveform into its sinusoidal Fourier components and applying the principle of superposition in the case of small-level modulation.

The results of this paper have a direct bearing on ellipsometry. The discussion of Sec. IIA represents a unified treatment for the measurement of unmodulated polarization states of light by means of periodic analyzers. The case of the rotating analyzer[8–10] was considered in Sec. III as a simple example. The same procedure can be applied to the oscillating-analyzer,[11] rotating-analyzer/fixed-analyzer,[12] rotating-compensator/fixed-analyzer,[13] rotating-compensator/rotating-analyzer,[14] and the oscillating-phase compensator/fixed-analyzer[15] ellipsometers.

FMD makes possible the automation of modulated ellipsometry[16] (ME) and modulated generalized ellipsometry[17] (MGE). In ME and MGE, a beam of light of constant polarization is reflected from or transmitted through an optical sample that is subjected to a modulating stimulus, such as temperature, stress, electric, or magnetic field. Modulation of the sample causes modulation of the intensity and polarization of the beam that can be measured by FMD. Consider, for instance, an isotropic surface that reflects linearly polarized light of 45° azimuth from the plane of incidence. If a sinusoidal perturbation is applied to the surface, changes will occur in its reflectance and ellipsometric parameters; consequently, the reflected light will be both intensity and polarization modulated. Because the incident light is linearly polarized at 45° azimuth, it can be readily seen that the polarization modulation of the reflected light $\widehat{\delta\psi}$ and $\widehat{\delta\Delta}$ can be identified with the changes of the ellipsometric parameters of the surface. The intensity modulation $\widehat{\delta I}/\overline{I}$ gives $\widehat{\delta R}/\overline{R}$, where $\overline{R}\,[=\frac{1}{2}(\overline{R}_p+\overline{R}_s)]$ is the reflectance for unpolarized light. Thus FMD using any periodic analyzer as described in Sec. II, e.g., using the rotating analyzer as described in Sec. III, can be applied to automate modulated ellipsometry.

Another related application of FMD is the automation of AIDER[18] (angle-of-incidence-derivative ellipsometry and reflectormetry). In this case, a light beam is obliquely reflected from an angularly vibrating surface and the state of polarization of the reflected beam is therefore modulated. Such modulation can be measured by FMD using a periodic analyzer (e.g., a rotating analyzer); hence the angle-of-incidence derivatives of the reflectance and ellipsometric parameters of the surface can be determined.

Finally, we should mention that the principle of FMD is applicable to the measurement of polarization modulation of other electromagnetic waves, even though we have referred to light waves in particular throughout this paper.

[1] To prevent overlapping between the spectra of $S_{bb}$ and $\delta S_{mc}$, we select the frequency of the periodic analyzer $\omega_a$ to be much smaller than the beam-modulation frequency $\omega_m$ (e.g., $\omega_m > 10\omega_a$), and restrict $\omega_m/\omega_a$ not to equal the ratio of two integers.

[2] Alternatively, we may measure the amplitudes of the cosine and sine components of one nonzero harmonic of $S_{bb}$. This is the case of the example considered in Sec. III.

[3] Dependent on the type of periodic analyzer that we choose, Eqs. (16) may or may not have an explicit, or a unique, solution for $\overline{\psi}$ and $\overline{\Delta}$.

[4] This requires, of course, that the three equations be linearly independent. This is satisfied in general, unless the periodic analyzer, the chosen harmonics $(p,\ q)$, and/or the quiescent polarization $(\overline{\psi},\ \overline{\Delta})$ happen to be such that two (or all three) equations become linearly dependent.

[5] Such a constant can be absorbed in the multipler $c$ that appears in Eq. (7).

[6] This is in agreement with results to be found in Refs. 8–10.


[7] See, for example, W. K. Pratt, *Laser Communication Systems* (Wiley, New York, 1969).

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