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The complex Fresnel reflection coefficients r_p and r_s of p- and s-polarized light and their ratio $\rho = r_p/r_s$ at the pseudo-Brewster angle (PBA) ϕ_{pB} of a dielectric–conductor interface are evaluated for all possible values of the complex relative dielectric function $\varepsilon = |\varepsilon| \exp(-j\theta) = \varepsilon_r - j\varepsilon_i$, $\varepsilon_i > 0$ that share the same ϕ_{pB}. Complex-plane trajectories of r_p, r_s, and ρ at the PBA are presented at discrete values of ϕ_{pB} from 5° to 85° in equal steps of 5° as θ is increased from 0° to 180°. It is shown that for $\phi_{pB} > 70°$ (high-reflectance metals in the IR) r_p at the PBA is essentially pure negative imaginary and the reflection phase shift $\delta_p = \arg(r_p) \approx -90°$. In the domain of fractional optical constants (vacuum UV or light incidence from a high-refractive-index immersion medium) $0 < \phi_{pB} < 45°$ and r_p is pure real negative ($\delta_p = \pi$) when $\theta = \tan^{-1}(\sqrt{\cos(2\phi_{pB})})$, and the corresponding locus of ε in the complex plane is obtained. In the limit of $\varepsilon_i = 0$, $\varepsilon_r < 0$ (interface between a dielectric and plasmonic medium) the total reflection phase shifts δ_p, δ_s, $\Delta = \delta_p - \delta_s = \arg(\rho)$ are also determined as functions of ϕ_{pB}.

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1. INTRODUCTION

A salient feature of the reflection of collimated monochromatic p (TM)-polarized light at a planar interface between a transparent medium of incidence (dielectric) and an absorbing medium of refraction (conductor) is the appearance of a reflectance minimum at the pseudo-Brewster angle (PBA) ϕ_{pB}. If the medium of refraction is also transparent, the minimum reflectance is zero and ϕ_{pB} reverts back to the usual Brewster angle $\phi_B = \tan^{-1} n = \tan^{-1} \sqrt{\varepsilon_r}$. The PBA ϕ_{pB} is determined by the complex relative dielectric function $\varepsilon = \varepsilon_1/\varepsilon_0 = \varepsilon_r - j\varepsilon_i$, $\varepsilon_i > 0$, where ε_0 and ε_1 are the real and complex permittivities of the dielectric and conductor, respectively, by solving a cubic equation in $u = \sin^2 \phi_{pB}$ [1–5]. Measurement of ϕ_{pB} and of reflectance at that angle or at normal incidence enables the determination of complex ε [1,6–9]. It is also possible to determine ε of an optically thick absorbing film from two PBAs measured in transparent ambient and substrate media that sandwich the thick film [10]. Reflection at the PBA has also had other interesting applications [11,12].

For light reflection at any angle of incidence ϕ the complex-amplitude Fresnel reflection coefficients (see, e.g., [13]) of the p and s polarizations are given by

$$r_p = \frac{\varepsilon \cos \phi - (\varepsilon - \sin^2 \phi)^{1/2}}{\varepsilon \cos \phi + (\varepsilon - \sin^2 \phi)^{1/2}}, \tag{1}$$

$$r_s = \frac{\cos \phi - (\varepsilon - \sin^2 \phi)^{1/2}}{\cos \phi + (\varepsilon - \sin^2 \phi)^{1/2}}. \tag{2}$$

All possible values of complex $\varepsilon = (\varepsilon_r, \varepsilon_i)$ that share the same ϕ_{pB} are generated by using the following algorithm [8,14]:

$$\varepsilon_r = |\varepsilon| \cos \theta, \qquad \varepsilon_i = |\varepsilon| \sin \theta, \tag{3}$$

$$\begin{aligned} &|\varepsilon| = \ell \cos(\varsigma/3), \\ &\ell = 2u\left(1 - \frac{2}{3}u\right)^{1/2} / (1 - u), \\ &\varsigma = \cos^{-1}\left[-(1-u)\cos\theta \Big/ \left(1 - \frac{2}{3}u\right)^{3/2}\right], \\ &u = \sin^2 \phi_{pB}, \\ &0 \le \theta \le 180°. \end{aligned} \tag{4}$$

As θ is increased from 0° to 180°, the minimum reflectance $|r_p|_{\min}$ at a given ϕ_{pB} increases monotonically from 0 to 1 [15] and also as is evident in Fig. 1 of Section 2.

In this paper, loci of all possible values of complex $r_p = |r_p| \exp(j\delta_p)$, $r_s = |r_s| \exp(j\delta_s)$, and $\rho = r_p/r_s = |\rho| \exp(j\Delta)$ at the PBA are determined at discrete values of ϕ_{pB} from 5° to 85° in equal steps of 5° and as $\theta = -\arg(\varepsilon)$ covers the full range $0° \le \theta \le 180°$. These results are presented in Sections 2, 3, and 4, respectively, and lead to interesting conclusions. In particular, questions related to phase shifts that accompany the reflection of p- and s-polarized light at the PBA (e.g., [12]) are settled. Section 5 summarizes the essential conclusions of this paper.

2. COMPLEX REFLECTION COEFFICIENT OF THE *p* POLARIZATION AT THE PBA

Figure 1 shows the loci of complex r_p as θ increases from 0° to 180° at constant values of ϕ_{pB} from 5° to 85° in equal steps of 5°. All constant-ϕ_{pB} contours begin at the origin O ($\theta = 0$) as a common point, that represents zero reflection at an ideal Brewster angle, and end on the 90° arc of the unit circle in the third quadrant (shown as a dotted line) that represents total reflection $|r_p| = 1$ at $\theta = 180°$ ($\varepsilon_i = 0, \varepsilon_r < 0$). A quick conclusion from Fig. 1 is that for $\phi_{pB} > 70°$ (high-reflectance metals) r_p at the PBA is essentially pure negative imaginary, and $\delta_p \approx -90°$.

In Fig. 1 the constant-ϕ_{pB} contours of r_p for $0 < \phi_{pB} < 45°$ spill over into a limited range of the second quadrant of the complex plane and each contour intersects the negative real axis. In Appendix A it is shown that θ at the point of intersection, where $\delta_p = \arg(r_p) = \pi$, is given by the remarkably simple formula

$$\theta(\delta_p = \pi) = \tan^{-1}\left(\sqrt{\cos(2\phi_{pB})}\right). \tag{5}$$

A graph of this function of Eq. (5) is shown in Fig. 2.

The locus of complex ε such that $\delta_p = \arg(r_p) = \pi$ at the PBA [as determined by Eqs. (3)–(5)] falls in the domain of fractional optical constants and is shown in Fig. 3. The end points (0, 0) and (1, 0) of this trajectory correspond to $\phi_{pB} = 0$ and 45°, respectively. At $\varepsilon = (0.6, 0.3)$, a point that falls exactly on the curve very near to its peak, $\phi_{pB} = 37.761°$.

For small PBAs, $\phi_{pB} \le 5°$, the upper limit on $|\varepsilon|$ is calculated from ℓ of Eq. (4), $|\varepsilon| = \ell \le 0.0153$, and represents the domain of so-called epsilon-near-zero (ENZ) materials [16].

Negative real values of ε at $\theta = 180°$ [14] are given by

$$\varepsilon = \varepsilon_r = -\frac{1}{2}\tan^2\phi_{pB}[1 + (9 - 8\sin^2\phi_{pB})^{1/2}] \tag{6}$$

and represent light reflection at an ideal dielectric–plasmonic medium interface. The corresponding total reflection phase shift δ_p as $\theta \to 180°$ (at the end point of each contour in Fig. 1)

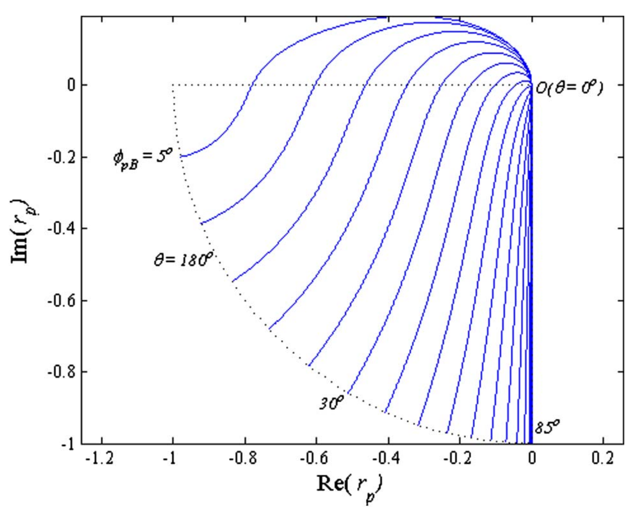

Fig. 1. Complex-plane trajectories of r_p at discrete values of the PBA ϕ_{pB} from 5° to 85° in equal steps of 5° as $\theta = -\arg(\varepsilon)$ covers the full range $0° \le \theta \le 180°$.

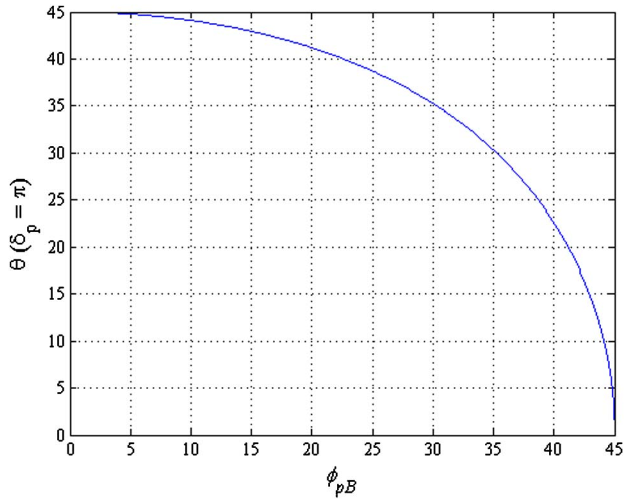

Fig. 2. Graph of the function of Eq. (5). Both ϕ_{pB} and θ are in degrees.

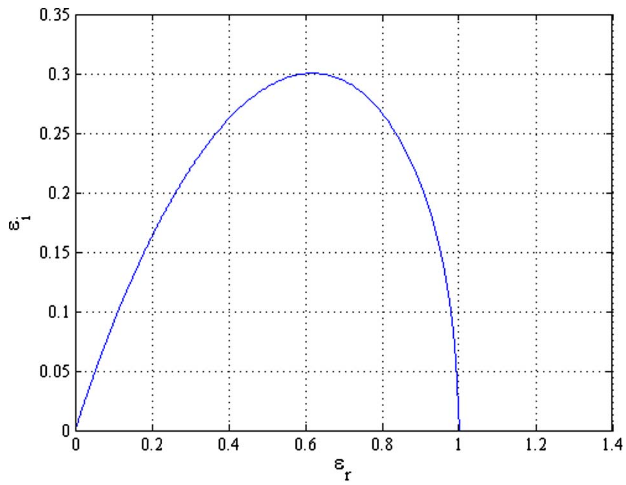

Fig. 3. Locus of all possible values of complex ε such that $\delta_p = \arg(r_p) = \pi$ at the PBA.

is obtained from Eqs. (1) and (6) and is plotted as a function of ϕ_{pB} in Fig. 4. In Fig. 4 δ_p increases monotonically from −180° to −90° as ϕ_{pB} increases from 0° to 90°. The initial rise of δ_p with respect to ϕ_{pB} is linear for $\phi_{pB} < 20°$ and then transitions to saturation at $\phi_{pB} > 70°$, in accord with Fig. 1.

In Fig. 5 δ_p is plotted as a function of θ for ϕ_{pB} from 10° to 40° in equal steps of 10°. Vertical transitions from +180° to −180° are located at θ values that agree with Eq. (5).

Another family of δ_p-versus-θ curves for ϕ_{pB} from 45° to 85° in equal steps of 5° is shown in Fig. 6. For $\phi_{pB} > 45°$ the δ_p-versus-θ curve first exhibits a minimum then reaches saturation as $\theta \to 180°$. The saturated value of δ_p is a function of ϕ_{pB} and is shown in Fig. 4.

3. COMPLEX REFLECTION COEFFICIENT OF THE *s* POLARIZATION AT THE PBA

Figure 7 shows the loci of complex r_s as θ increases from 0° to 180° at discrete values of ϕ_{pB} from 5° to 85° in equal steps of 5°. All curves start on the real axis at $\theta = 0$, $r_s = \cos(2\phi_{pB})$, which is the s amplitude reflectance at the Brewster angle of a

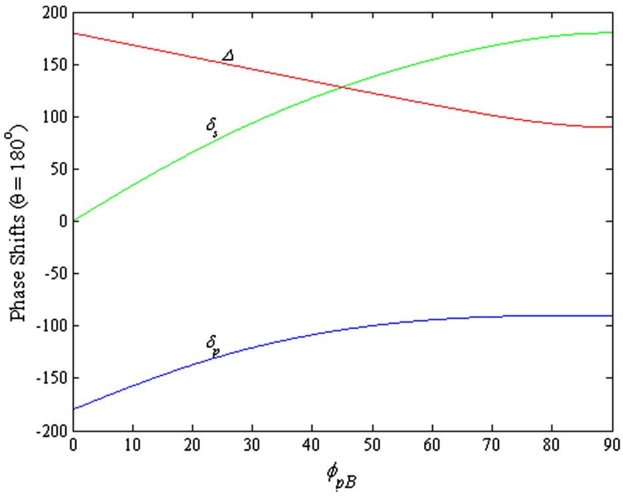

Fig. 4. Total reflection phase shifts δ_p, δ_s, and $\Delta = \delta_p - \delta_s + 360°$ at the interface between a dielectric and plasmonic medium in the limit as $\theta \to 180°$ ($\varepsilon_i = 0, \varepsilon_r < 0$) are plotted as a functions of ϕ_{pB}. All angles are in degrees.

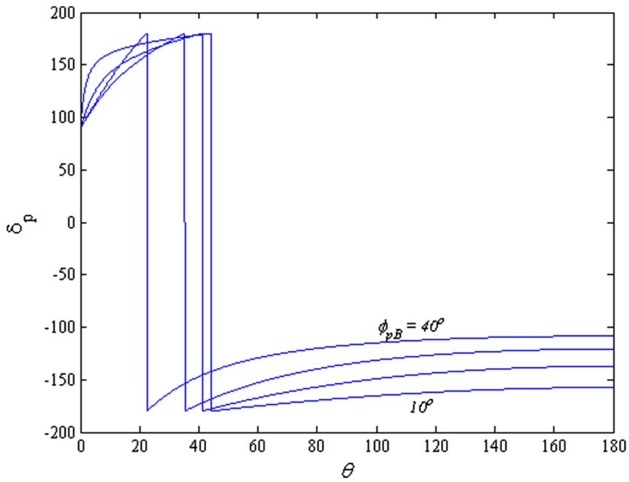

Fig. 5. Family of δ_p versus θ curves for ϕ_{pB} from 10° to 40° in equal steps of 10°. Both θ and δ_p are in degrees.

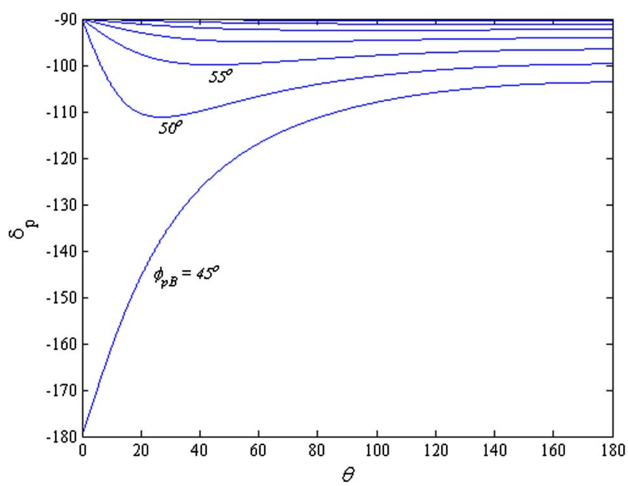

Fig. 6. Family of δ_p versus θ curves for ϕ_{pB} from 45° to 85° in equal steps of 5°. Both θ and δ_p are in degrees.

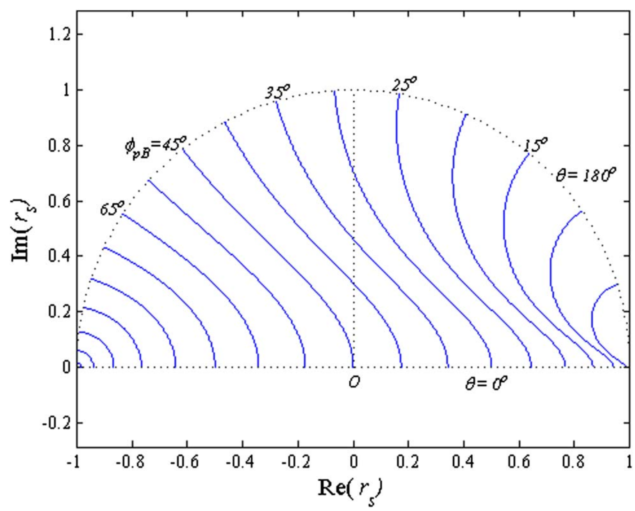

Fig. 7. Complex-plane contours of r_s at discrete values of the PBA ϕ_{pB} from 5° to 85° in equal steps of 5° as $\theta = -\arg(\varepsilon)$ covers the full range from 0° to 180°.

dielectric–dielectric interface [17], and terminate on the upper half of the unit circle (dotted line) that represents total reflection $r_s = \exp(j\delta_s)$ at $\theta = 180°$ ($\varepsilon_i = 0, \varepsilon_r < 0$). The associated total reflection phase shift δ_s along the dotted semicircle is a function of ϕ_{pB} as shown in Fig. 4.

Although we are locked on the PBA, all possible values of complex r_s (within the upper half of the unit circle) are generated at that angle. This is not the case of complex r_p at the PBA (Fig. 1) which is squeezed mostly in the third quadrant of the unit circle. Recall that the unconstrained domain of r_p for light reflection at all dielectric–conductor interfaces is on and inside the full unit circle [17].

4. RATIO OF COMPLEX REFLECTION COEFFICIENTS OF THE *p* AND *s* POLARIZATIONS AT THE PBA

The ratio of complex p and s reflection coefficients, also known as the ellipsometric function $\rho = \tan\psi \exp(j\Delta)$ [13], is obtained from Eqs. (1) and (2) as

$$\rho = r_p/r_s = \frac{\sin\phi \tan\phi - (\varepsilon - \sin^2\phi)^{1/2}}{\sin\phi \tan\phi + (\varepsilon - \sin^2\phi)^{1/2}}. \tag{7}$$

Figure 8 shows loci of complex ρ as θ increases from 0° to 180° at constant values of ϕ_{pB} from 5° to 85° in equal steps of 5°. All contours begin at the origin O (as a common point that represents the ideal Brewster-angle condition of $r_p = 0$ at $\theta = 0$), then fan out and terminate on the 90° arc of the unit circle in the second quadrant of the complex plane (dotted line), so that $\rho = \exp(j\Delta)$ at $\theta = 180°$ ($\varepsilon_i = 0, \varepsilon_r < 0$). The differential reflection phase shift $\Delta = \delta_p - \delta_s + 360°$ at $\theta = 180°$ decreases monotonically from 180° to 90° as ϕ_{pB} increases from 0° to 90° as shown in Fig. 4.

5. SUMMARY

The Fresnel complex reflection coefficients r_p, r_s and their ratio $\rho = r_p/r_s$ are evaluated at the PBA ϕ_{pB} of a dielectric–conductor interface for all possible values of the complex relative dielectric function $\varepsilon = |\varepsilon|\exp(-j\theta) = \varepsilon_r - j\varepsilon_i$, $\varepsilon_i > 0$.

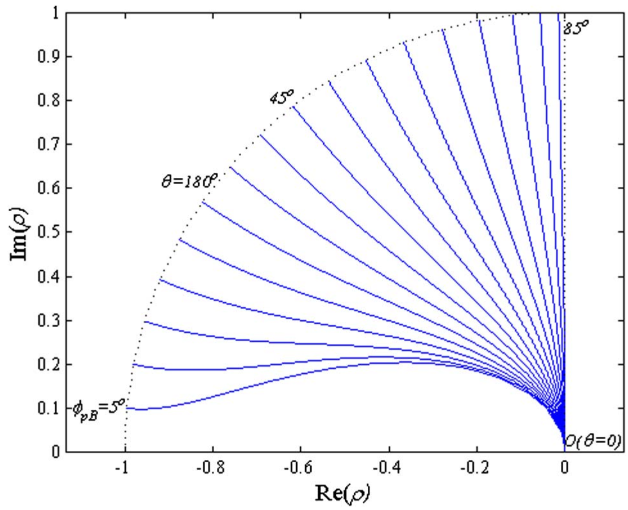

Fig. 8. Complex-plane trajectories of the ratio $\rho = r_p/r_s$ at discrete values of the PBA ϕ_{pB} from 5° to 85° in equal steps of 5° as $\theta = -\arg(\varepsilon)$ covers the full range $0° \le \theta \le 180°$.

Complex-plane loci of r_p, r_s, and ρ at the PBA are obtained at discrete values of ϕ_{pB} from 5° to 85° in equal steps of 5° and as θ increases from 0° to 180°; these are presented in Figs. 1, 7, and 8, respectively. The reflection phase shift δ_p of the p polarization at the PBA is plotted as function of θ in Figs. 5 and 6 for two different sets of ϕ_{pB}. For $\phi_{pB} > 70°$ (e.g., high-reflectance metals in the IR), r_p at the PBA is essentially pure negative imaginary and $\delta_p = \arg(r_p) \approx -90°$. In the domain of fractional optical constants (vacuum UV or light incidence from a high-refractive-index immersion medium) $0° < \phi_{pB} < 45°$, and r_p is pure real negative ($\delta_p = \pi$) at $\theta = \tan^{-1}(\sqrt{\cos(2\phi_{pB})})$. The associated locus of complex ε is shown in Fig. 3. Finally, the total reflection phase shifts δ_p, δ_s, $\Delta = \arg(\rho)$ at an ideal dielectric–plasmonic medium interface ($\varepsilon_i = 0, \varepsilon_r < 0$), are shown as functions of ϕ_{pB} in Fig. 4.

APPENDIX A

By setting $\varepsilon_r = x$ and $\varepsilon_i = y$, the Cartesian equation of a constant-ϕ_{pB} contour (a cardioid [8]) takes the form [10]

$$y^2 = a + (a^2 - bx)^{1/2} - x^2, \tag{A1}$$

$$a = u^2(1.5 - u)/(1 - u)^2, b = u^3/(1 - u)^2, u = \sin^2\phi_{pB}. \tag{A2}$$

The locus of complex ε such that $\delta_p = \arg(r_p) = \pi$ at a given angle of incidence $\phi = \sin^{-1}\sqrt{u}$ is a circle [18]

$$y^2 = 2ux - x^2. \tag{A3}$$

Equations (A1) and (A3) are satisfied simultaneously if their right-hand sides are equal; this gives

$$(a^2 - bx)^{1/2} = 2ux - a. \tag{A4}$$

By squaring both sides of Eq. (A4) we obtain

$$4u^2x^2 = (4au - b)x. \tag{A5}$$

Equation (A5) is obviously satisfied when $x = 0$, and from Eq. (A3) one gets $y = 0$ and $\varepsilon = 0$. The more significant solution of Eq. (A5) is

$$x = (4au - b)/(4u^2). \tag{A6}$$

Substitution of a and b from Eq. (A2) in Eq. (A6) leads to the simple result

$$x = u/(1 - u). \tag{A7}$$

The associated value of y is then obtained from Eq. (A3) as

$$y = u\sqrt{1 - 2u}/(1 - u). \tag{A8}$$

The angle $\theta = \arg(\varepsilon)$ is determined from Eqs. (A7) and (A8) by

$$\tan\theta = y/x = \sqrt{1 - 2u}. \tag{A9}$$

Finally, substitution of $u = \sin^2\phi_{pB}$ in Eq. (A9) gives

$$\theta(\delta_p = \pi) = \tan^{-1}\left(\sqrt{\cos(2\phi_{pB})}\right). \tag{A10}$$

This completes the proof of Eq. (5).

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