Fall 2010

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Recommended Citation

Cartwright, K., E. Joseph, and E. Kaminsky, "Finding the Exact Maximum Impedance Resonant Frequency of a Practical Parallel Resonant Circuit without Calculus," Technology Interface Internat. J., vol.11, no. 1, Fall/Winter 2010, pp. 26-36.

FINDING THE EXACT MAXIMUM IMPEDANCE RESONANT FREQUENCY OF A PRACTICAL PARALLEL RESONANT CIRCUIT WITHOUT CALCULUS

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Abstract

A practical parallel resonant circuit has a resistor in series with an inductor, and that combination is in parallel with a capacitor. For such a circuit, it is well known that there are two possible definitions for the resonant frequency: (i) the resonant frequency f_p, which is the frequency at which the phase of the total impedance is zero, and (ii) the resonant frequency f_m, which is the frequency that achieves maximum magnitude of the total impedance. To find the latter traditionally requires calculus. However, in this paper, the authors show how f_m can be found exactly without using calculus. By modifying a formula that is given as an approximation to f_m in a popular technology textbook, an improvement in the accuracy of the approximation was achieved. Furthermore, a novel expression for the exact maximum impedance, as a function of $Q = \sqrt{L/C}\,/\,R.$ was derived. This has been approximated by previous authors as RQ^2 for $Q \geq 10$. However, in this report, the authors show that this approximation has a percentage error less than -2% for $Q \geq 5$, and less than -10% for $Q \geq 2$. Furthermore, it can be shown that the maximum impedance is also accurately approximated by $R\sqrt{Q^2\left(1+Q^2\right)}$, which has an excellent percentage error performance, even for $Q = 1$, with a percentage error of only -4% for this value, and less than -0.6% for $Q \geq 1.5$. Finally, the authors used PSpice simulations to verify their results.

Introduction

The parallel resonant circuit of Figure 1 is used in many technology texts. However, the parallel resonant circuit of Figure 2 is of more concern in practice. This is because it is virtually impossible to build a coil without resistance, which is represented by the resistor R in Figure 2. Hence, Figure 2 represents a practical parallel resonant circuit, whereas Figure 1 depicts the ideal case.

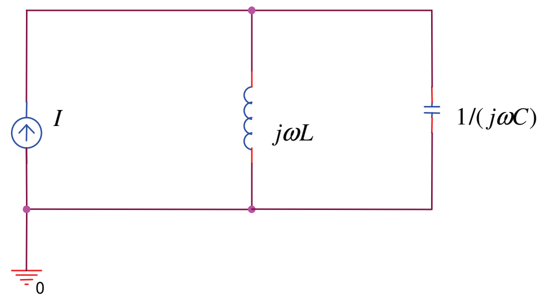

Figure 1. An ideal parallel resonant circuit

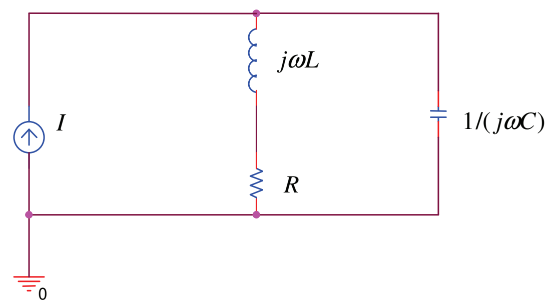

Figure 2. A practical parallel resonant circuit

Boylestad [1] is one technology author who considers Figure 2 in detail. He identifies two possible definitions for the resonant frequency:

(i) the resonant frequency f_p, which is the frequency at which the phase of the impedance of Figure 2 is zero, and

(ii) the resonant frequency f_m, which is the frequency for which the magnitude of the impedance is a maximum.

For completeness, $f_p = f_o\sqrt{1-\frac{1}{Q^2}}$, where $f_o = \frac{1}{2\pi\sqrt{LC}}$ was used in this report. Also, the derivation of f_p does not require calculus and is the same as that used by Boylestad [1]. On the other hand, the conventional method of finding f_m requires calculus [2]. However, the purpose of this paper is to show how this can be done without calculus. Interestingly, the authors found no engineering technology author who provided an exact equation for f_m. Boylestad [1] uses

$f_m = f_o\sqrt{1-\frac{1}{4Q^2}}$. However, as shown here, this equation is actually an approximation to the exact maximum impedance resonant frequency, f_m, which is derived here.

Furthermore, using the equation for the maximum impedance resonant frequency, a novel expression for the exact maximum impedance magnitude, as a function of Q can be derived. This has been previously approximated by other authors as RQ^2 for $Q \geq 10$. However, the authors show that this approximation is useable for smaller Q values: indeed, the aforementioned approximation has a percentage error less than -2% for $Q \geq 5$, and less than -10% for $Q \geq 2$. Additionally, it can be demonstrated that the maximum impedance is better approximated by $R\sqrt{Q^2\left(1+Q^2\right)}$. In fact, this approximation has an excellent percentage error performance: for $Q \geq 1.5$ it is less than -0.6% and for $Q = 1$ the percentage error is only -4%. Furthermore, the authors used PSpice simulations to confirm their results.

In the discussion above, the observant reader would realize that the definition of Q was neglected, which is the quality factor of the coil. In this paper, in order to get results consistent with Walton's [2], $Q = \sqrt{L/C}/R$. was used. However, it should be noted that other authors might use $Q_p = 2\pi f_p L/R$. Hence, the reader needs to exercise caution when reading the literature. Fortunately, using $f_p = f_o\sqrt{1-\frac{1}{Q^2}}$, it is easy to show that there is a simple relationship between the two, i.e., $Q = \sqrt{Q_p^2+1}$. Clearly, for large values of $Q_p, Q \approx Q_p$.

The authors would also like to point out that for most practical situations in electrical engineering or engineering technology, such as communication systems, $Q \geq 10$, as pointed out by Beasley and Miller [3]. Also, electrical engineering technology textbooks generally analyze the circuit of Figure 2 quite well for $Q \geq 10$. However, there are practical systems in electrical engineering technology for which $Q < 10$. For example, the ultra wideband FM demodulator [4] requires a practical parallel resonant circuit with $Q = 2.5$. Also, traffic detection loops have the equivalent circuit of Figure 2, where Q values can be as low as 5, according to Klein et al. [5]. Hence, it might be important that electrical engineering technologists understand the circuit of Figure 2 for low values of Q as well. In this paper, the authors show how the practical parallel resonant circuit of Figure 2 can be analyzed, even by the electrical engineering technology student who has not yet had the opportunity to study calculus.

Impedance of the Practical Parallel Resonant Circuit

In this section, an expression for the impedance of the circuit in Figure 2 is derived. In order to do this, many authors, including Boylestad [1], first convert Figure 2 to an equivalent parallel *RLC* circuit. However, in this paper, the authors follow Walton [2] and work directly with Figure 2. Indeed, the impedance of this circuit is given by

$$Z = \frac{\left(R + j\omega L\right)\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}} = \frac{R + j\omega L}{j\omega RC + j^2\omega^2 LC + 1} = \frac{R + j\omega L}{j\omega RC - \omega^2 LC + 1}. \tag{1}$$

From equation (1), the magnitude of the impedance is easily found to be

$$|Z| = \sqrt{\frac{R^2 + (\omega L)^2}{\left(\omega RC\right)^2 + \left(1 - LC\omega^2\right)^2}}. \tag{2}$$

In principle, it is possible to find the maximum impedance resonant frequency f_m from equation (2) using calculus. However, the math is less tedious if f_m is found from the square of equation (2), i.e., from

$$|Z|^2 = \frac{R^2 + (\omega L)^2}{\left(\omega RC\right)^2 + \left(1 - LC\omega^2\right)^2}. \tag{3}$$

Furthermore, as with Walton [2], it will be convenient to write equation (3) in terms of the square of the normalized frequency x, which is defined to be

$$x = \left(\frac{\omega}{\omega_0}\right)^2 = \left(\frac{\omega}{\frac{1}{\sqrt{LC}}}\right)^2 = LC\omega^2. \tag{4}$$

Substituting equation (4), $\omega^2 = x/LC$, into equation (3) yields

$$|Z|^2 = \frac{R^2 + \frac{Lx}{C}}{\frac{R^2Cx}{L} + (1-x)^2} = \frac{R^2\left(1+\frac{Lx}{R^2C}\right)}{\frac{R^2Cx}{L} + (1-x)^2}. \tag{5}$$

Recall that the quality factor of the coil is (by definition) $Q = \frac{\omega_0 L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$. Hence,

$$Q^2 = \left(\frac{\omega_0 L}{R}\right)^2 = \frac{L}{R^2C}. \tag{6}$$

Substituting equation (6) into equation (5) gives

$$|Z|^2 = \frac{R^2\left(1+Q^2x\right)}{\frac{x}{Q^2} + (1-x)^2}. \tag{7}$$

Multiplying both the numerator and denominator of equation (7) by Q^2 results in

$$|Z|^2 = \frac{R^2Q^2\left(1+Q^2x\right)}{x + Q^2(1-x)^2}. \tag{8}$$

Finally, dividing both sides of equation (8) by R^2 gives

$$\begin{aligned} \frac{|Z|^2}{R^2} = S &= \frac{Q^2\left(1+Q^2x\right)}{x+Q^2(1-x)^2} \\ &= \frac{Q^2\left(1+Q^2x\right)}{Q^2x^2 - \left(2Q^2-1\right)x + Q^2} \\ &= \frac{Q^2x+1}{x^2 - \left(2-1/Q^2\right)x + 1}. \end{aligned} \tag{9}$$

Note that in equation (9), S is defined as $\frac{|Z|^2}{R^2} = \left|\frac{Z}{R}\right|^2$, i.e., the square of the magnitude of the normalized impedance.

Maximum Impedance Resonant Frequency

Now that it has been established how the magnitude of the impedance of the practical parallel circuit is changing with frequency, one is in a position to find the frequency at which the maximum impedance magnitude occurs. The conventional way of doing this is with calculus; however, as stated earlier, the purpose of this paper is to show how this can be done without calculus, which will now be addressed in this section.

In order to find the maximum of equation (9) without calculus, by using the authors' method, equation (9) will have to be manipulated into the form $S = \frac{Ay}{By^2 + Cy + D}$, where A, B, C, D are constants, i.e., do not depend upon the square of the normalized frequency, x, whereas y is indeed a function of x.

To do this, let $y = x + \frac{1}{Q^2}$. Hence, equation (9) becomes

$$\begin{aligned} S &= \frac{Q^2\left(y - \frac{1}{Q^2}\right) + 1}{\left(y-\frac{1}{Q^2}\right)^2 - \left(2-\frac{1}{Q^2}\right)\left(y-\frac{1}{Q^2}\right) + 1} \\ &= \frac{Q^2y}{y^2 - \frac{2}{Q^2}y + \frac{1}{Q^4} - \left(2y - \frac{1}{Q^2}y - \frac{2}{Q^2} + \frac{1}{Q^4}\right) + 1} \\ &= \frac{Q^2y}{y^2 - \left(2+\frac{1}{Q^2}\right)y + 1 + \frac{2}{Q^2}}. \end{aligned} \tag{10}$$

Note that equation (10) is now in the required form, from which the maximum value of equation (10) is easily found without calculus, as will now be shown.

Equation (10) must first be rewritten as

$$\begin{aligned} S &= \frac{Q^2}{y + \frac{1+\frac{2}{Q^2}}{y} - \left(2+\frac{1}{Q^2}\right)} \\ &= \frac{Q^2}{\left(\sqrt{y} - \frac{\sqrt{1+\frac{2}{Q^2}}}{\sqrt{y}}\right)^2 + 2\sqrt{1+\frac{2}{Q^2}} - \left(2+\frac{1}{Q^2}\right)}. \end{aligned} \tag{11}$$

However, equation (11) can be written as

$$S = \frac{a}{b+c}, \tag{12}$$

where,

$$a = Q^2, b = \left(\sqrt{y} - \frac{\sqrt{1+\frac{2}{Q^2}}}{\sqrt{y}} \right)^2 \text{ and } c = 2\sqrt{1+\frac{2}{Q^2}} - \left(2+\frac{1}{Q^2}\right).$$

For equation (12), please note the following:

(i) a is a positive number that is not a function of frequency,

(ii) c is a positive number for $Q > 1/2$ (please see Appendix A for a proof of this) that is not a function of frequency, and

(iii) b is a non-negative number, i.e., $b \geq 0$, and is a function of frequency.

Hence, to maximize equation (12) with respect to frequency only requires that b is properly chosen (a and c are independent of frequency). In fact, to maximize equation (12) with respect to frequency requires that the denominator of equation (12) be minimized, and in order to do the latter requires that $b = 0$ as c is a positive number. However,

$$b = \left(\sqrt{y} - \frac{\sqrt{1+\frac{2}{Q^2}}}{\sqrt{y}} \right)^2 = 0 \text{ requires that } y = \sqrt{1+\frac{2}{Q^2}} \text{ or}$$

$$x = \sqrt{1+\frac{2}{Q^2}} - \frac{1}{Q^2}.$$

Therefore,

$$\left(\frac{\omega_m}{\omega_0}\right)^2 = \left(\frac{f_m}{f_0}\right)^2 = \frac{-1}{Q^2} + \sqrt{\frac{2}{Q^2}+1}. \tag{13}$$

Solving equation (13) gives the maximum impedance resonant frequency as

$$f_m = f_o\sqrt{\frac{-1}{Q^2} + \sqrt{\frac{2}{Q^2}+1}}. \tag{14}$$

Fortunately, equation (14) is the same expression that is derived with calculus as given by Walton [2].

Furthermore, in order for equation (14) to be valid, one must have $\frac{-1}{Q^2} + \sqrt{\frac{2}{Q^2}+1} \geq 0$. Hence, $Q \geq \sqrt{\sqrt{2}-1} = .6436$, as shown in Appendix B. This is fortunate because in deriving equation (14) without calculus earlier, it was assumed that $Q > 0.5$. Therefore, this assumption has not imposed any limitation on the derivation. (By the way, for $Q \leq .6436$, $f_m = 0$, as can be easily verified by plotting equation (9)).

Approximations to the Maximum Impedance Resonant Frequency

A. Approximations to the Maximum Impedance Resonant Frequency, Equation (14)

For large values of Q, it is possible to simplify equation (14) because $\frac{2}{Q^2}$ is quite a bit smaller than 1 and

$$\sqrt{1+\frac{2}{Q^2}} \approx 1 + \frac{1}{Q^2} - \frac{1}{2Q^4}. \tag{15}$$

Equation (15) follows from the well-known fact that $\sqrt{1+a} \approx 1 + \frac{a}{2} - \frac{a^2}{8}$, if a is small. Indeed, the smaller a is, the more accurate the approximation becomes. Likewise, the larger Q is, the more accurate equation (15) becomes.

Substituting equation (15) into equation (13) gives an excellent approximation to the square of the maximum impedance resonant frequency, i.e.,

$$\left(\frac{\omega_m}{\omega_0}\right)^2 = 1 - \frac{1}{2Q^4}. \tag{16}$$

Hence, from equation (16), the maximum impedance resonant frequency is well approximated by

$$f_m = f_o\sqrt{1 - \frac{1}{2Q^4}}. \tag{17}$$

It is interesting that Boylestad [1], in his equation (20.32), equation (20.44), and Table 20.1, gives the maximum impedance resonant frequency as

$$f_m = f_o\sqrt{1 - \frac{1}{4Q^2}}. \tag{18}$$

It is not known how this equation was derived, as no derivation for this is given in his text; however, it is clearly an approximation to the exact value equation (14). It might be that this is simply a typographical error, with $4Q^2$ inadvertently being used in place of $2Q^4$.

In the subsection below, the authors show that equation (17) is a substantially more accurate approximation to equation (14) than equation (18) is for Q values that are normally of interest.

B. Accuracy of the Approximations to the Maximum Impedance Resonant Frequency

Now that two approximations to the maximum impedance resonant frequency have been established, the natural question is how accurate is each approximation. To answer this question, the percentage error of equations (17) and (18), plotted in Figure 3, is defined by

$$P_{error} = \left(\frac{\text{Approximation Equation (17) or Equation (18)}}{\text{Exact Equation (14)}} - 1 \right) 100\%. \tag{19}$$

From Figure 3, it is clear that the percentage error for equation (17) is substantially smaller than that of equation (18) for Q values that are normally of interest. Indeed, the higher the Q, the more accurate is the approximation for equations (17) and (18) for $Q \geq 1.41$.

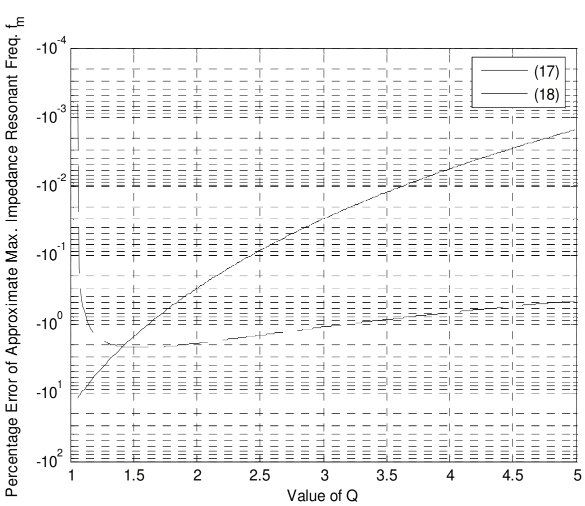

Figure 3. Percentage error of the approximations for equations (17) and (18) to the true maximum impedance resonant frequency given by equation (14)

Interestingly, equation (18) is more accurate than equation (17) for a limited range and actually improves as Q is lowered over that range, which is upper bounded by $Q = 1.41$. The percentage error of equation (18) is actually zero when equation (14) equals equation (18), or $Q = \frac{3}{2\sqrt{2}} \approx 1.0607$.

Finding the Impedance Magnitude at the Three Resonant Frequencies

Now that equations for the three resonant frequencies f_p, f_m and f_o have been established, they can be used along with equation (9) to find the actual impedance magnitudes at these frequencies. Note that f_o is being referred to as a resonant frequency, as it is the resonant frequency of Figure 2 when $R = 0$; indeed, for this case, $f_p = f_m = f_o$.

A. Impedance Magnitude at $\omega_o = \frac{1}{\sqrt{LC}}$.

When the frequency of the source is $\omega = \omega_o = 1/\sqrt{LC}$, $x = 1$; hence, from equation (9), the magnitude of the impedance is given by

$$|Z|_o = R\sqrt{Q^2\left(1+Q^2\right)}. \tag{20}$$

For large $Q, 1+Q^2 \approx Q^2$. Hence, equation (20) becomes

$$|Z|_o \approx RQ^2. \tag{21}$$

B. Impedance Magnitude at the Zero-Phase Resonant Frequency, $\omega_p = \omega_o\sqrt{1-\frac{1}{Q^2}}$.

When the source frequency is the zero-phase resonant frequency, i.e., $\omega_p = \omega_o\sqrt{1-\frac{1}{Q^2}}$, $x = 1 - 1/Q^2$, and the magnitude of the impedance is again found from Equation (9) to be

$$\begin{aligned} |Z|_p &= R\sqrt{\frac{Q^2\left(1+Q^2\left(1-\frac{1}{Q^2}\right)\right)}{\left(1-\frac{1}{Q^2}+Q^2\left(1-\left(1-\frac{1}{Q^2}\right)\right)^2\right)}} \\ &= R\sqrt{\frac{Q^4}{1-\frac{1}{Q^2}+Q^2\frac{1}{Q^4}}} \\ &= RQ^2. \end{aligned} \tag{22}$$

Comparing equation (22) to equation (20) shows that the magnitude of the impedance at the zero-phase resonant frequency is always smaller than that at ω_o, i.e., $|Z|_p < |Z|_o$. However, for large values of Q, these are very close in value, i.e., $|Z|_p \approx |Z|_o$.

It should also be noted that many authors, including Boylestad [1], use equation (22) as the approximation for the maximum impedance magnitude when $Q \geq 10.$ However, equation (20) is bigger than equation (22) for all values of Q. Hence, equation (20) might be a better approximation to the exact maximum impedance magnitude, especially for low Q values. Indeed, this is the case as verified below.

C. Maximum Impedance Magnitude, $|Z|_{\max}$: Impedance at the Maximum Impedance Resonant Frequency, $\omega_m = \omega_o\sqrt{-\frac{1}{Q^2}+\sqrt{1+\frac{2}{Q^2}}}$

From equation (12), the maximum impedance magnitude, $S_{\max}$, occurs for $b = 0;$ hence,

$$S_{\max} = \frac{a}{c} = \frac{Q^2}{2\sqrt{1+\frac{2}{Q^2}}-2-\frac{1}{Q^2}} = \frac{Q^2}{2\frac{1}{Q}\sqrt{Q^2+2}-2-\frac{1}{Q^2}} = \frac{Q^4}{2Q\sqrt{Q^2+2}-2Q^2-1}. \tag{23}$$

Using $S_{\max} = |Z|_{\max}^2 / R^2$, equation (23) becomes

$$|Z|_{\max} = RQ^2\sqrt{\frac{1}{2Q\sqrt{Q^2+2}-2Q^2-1}}. \tag{24}$$

To the authors' knowledge, the expression in equation (24) has never appeared in the open literature before. Interestingly, for large values of Q, $\sqrt{Q^2+2} = Q\sqrt{1+\frac{2}{Q^2}} \approx Q\left(1+\frac{1}{Q^2}\right)$, and so the denominator under the square root sign of equation (24) becomes unity. Thus, for large values of Q, $|Z|_{\max} \approx RQ^2$; hence, $|Z|_p \approx |Z|_o \approx |Z|_{\max}$.

D. Accuracy of the Approximations to the Maximum Impedance Magnitude

Now that the approximations of equations (20) and (22) have been established to the exact maximum impedance, given by equation (24), the accuracy of each should be tested. This can be done by computing the percentage error, given by

$$P_{error} = \left(\frac{\text{Approximation Equation (20) or Equation (22)}}{\text{Exact Equation (24)}}\text{-1}\right)100\%. \tag{25}$$

These percentage errors are plotted in Figure 4.

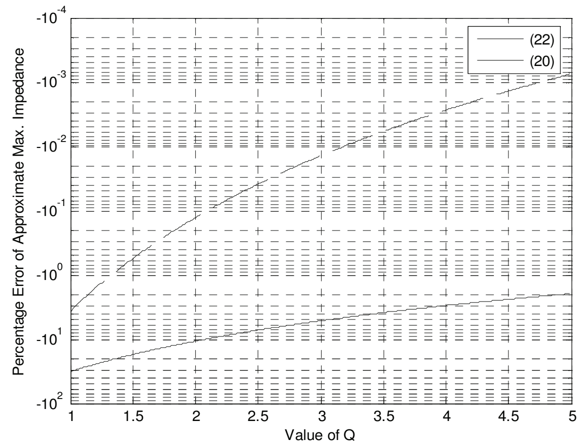

Figure 4. Percentage error of the maximum impedance magnitude estimates. Clearly, equations (20) and (22) always underestimate the true maximum impedance magnitude. Furthermore, equation (20) is a better approximation than equation (22) for all *Q* values of interest

As mentioned earlier, equation (22) is used to approximate the maximum magnitude of the impedance for $Q \geq 10.$ However, Figure 4 shows that equation (22) has a percentage error of less than -2% for $Q \geq 5,$ and less than -10% for $Q \geq 2.$ Hence, equation (22) can be used for very small Q values with a tolerable percentage error. Furthermore, equation (22) has the advantage of being the simplest approximation.

On the other hand, equation (20) has excellent percentage error performance, even for $Q = 1,$ with a percentage error of only -4%, whereas for $Q \geq 1.5,$ the percentage error is less than -0.6%.

Verification through PSpice Simulations

In this section, PSpice is used to verify the authors' derived equations. For all simulations, the authors used $C = 10\mu\text{F}$ and $L = 25.33\mu\text{H}$; (see Figure 2); hence, $f_o = 10k\text{Hz}$. Due to space limitations, $R = 2\pi f_o L / Q = \sqrt{L/C}/Q$, was chosen for only two Q values, as listed in Table 1.

Table 1. *R* values needed to achieve desired *Q* values for the circuit of Figure 2

Desired *Q* Value	Corresponding *R* Value (Ω)
1.5	1.06103
3	0.53051

For all of the simulation plots of this section, the amplitude of the voltage across the practical parallel circuit is plotted, i.e., the voltage across the capacitor, divided by the magnitude of the current source, which is 1A, the magnitude of the impedance seen by the current source.

Furthermore, the PSpice simulation data was exported to Matlab for actual plotting; it was found that this would allow for more readable graphs. Another advantage is that Matlab can then be used to search for the maximum impedance point, with great precision. In fact, this is how the PSpice simulated values given in Tables 2(a) to 3(b), were found.

A. Results for *Q*=1.5

Figure 5 shows a PSpice simulation plot of the magnitude of the impedance of the practical parallel circuit for *Q*=1.5.

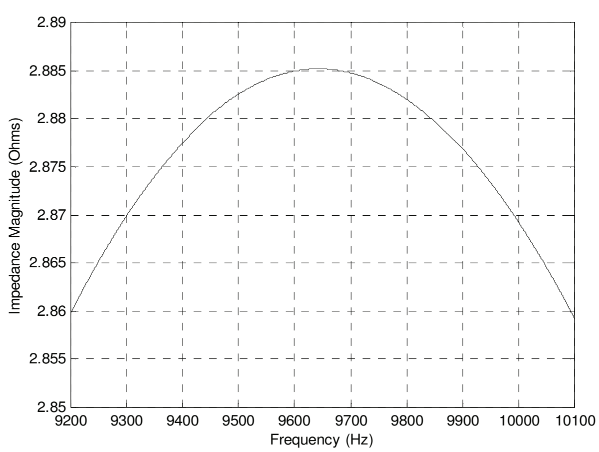

Figure 5. Magnitude of the impedance of the practical parallel circuit of Figure 2 for *Q*=1.5

For convenience, the simulated results are presented and compared to the theoretical results in Tables 2(a) and 2(b).

Table 2(a). Simulated results compared with theoretical results for the maximum impedance resonant frequency when *Q*=1.5

Maximum Impedance Resonant Frequency	Value	Percentage error compared with exact theoretical Equation (14)
PSpice simulated value	9643.4 Hz	0.00104%
Theoretical value from Equation (14)	9643.3 Hz	NA
Approximate value with Equation (18)	9428.1 Hz	−2.232%
Approximate value with Equation (17)	9493.3 Hz	−1.555%

Table 2(b). Simulated results compared with theoretical results for the maximum impedance magnitude when *Q*=1.5

Maximum Impedance Magnitude	Value	Percentage error compared with exact theoretical Equation (24)
PSpice simulated value	2.8852 Ω	0%
Theoretical value from Equation (24)	2.8852 Ω	NA
Approximate value with Equation (22)	2.3873 Ω	−17.26%
Approximate value with Equation (20)	2.8692 Ω	−.5546%

B. Results for *Q*=3

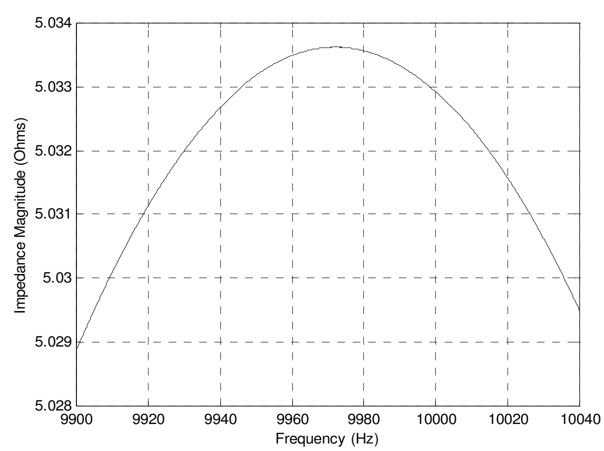

Figure 6. Magnitude of the impedance of the practical parallel circuit of Figure 2 for *Q* = 3

Table 3(a). Simulated results compared with theoretical results for the maximum impedance resonant frequency when *Q*=3

Maximum Impedance Resonant Frequency	**Value**	**Percentage error compared with exact theoretical Equation (14)**
PSpice simulated value	9972.1 Hz	0 %
Theoretical value from Equation (14)	9972.1 Hz	NA
Approximate valuewith Equation (18)	9860.1 Hz	−1.123%
Approximate value with Equation (17)	9969.1 Hz	−0.030%

Table 3(b). Simulated results compared with theoretical results for the maximum impedance magnitude when *Q*=3

Maximum Impedance Magnitude	**Value**	**Percentage error compared with exact theoretical Equation (24)**
PSpice simulated value	5.0336 Ω	0%
Theoretical value from Equation (24)	5.0336 Ω	NA
Approximate value with Equation (22)	4.7746 Ω	−5.145%
Approximate value with Equation (20)	5.0329 Ω	−0.0139%

Figure 6 shows a PSpice simulation plot of the magnitude of the impedance of the practical parallel circuit for *Q*=3. Again, the simulated results are presented and compared to the theoretical results in Tables 3(a) and 3(b).

C. Discussion

Based upon the tables above and other simulations not reported here due to space limitations, the following observations can be made concerning the maximum impedance frequency:

1. The theoretical value given by equation (14) and the PSpice simulations show excellent agreement.
2. The approximate theoretical value given by equation (17) is more accurate than the value from equation (18) for $Q \geq 1.41$. Both of these approximations improve with increasing Q values.

Furthermore, the following observations can be made concerning the maximum impedance:

1. The theoretical value given by equation (24) and the PSpice simulations show excellent agreement.
2. It is quite apparent that the maximum impedance magnitude is in fact well approximated by the impedance magnitude given by equation (20), which of course is the impedance magnitude at the resonant frequency, f_o. Additionally, this approximation improves with increasing Q values.

Conclusion

In this paper, the maximum impedance resonant frequency was derived without calculus. In so doing, the authors also modified a formula that is given for this in a popular technology textbook, thereby increasing its accuracy. Furthermore, the authors also found a novel expression for the exact maximum impedance. This has been approximated by previous authors as RQ^2 for $Q \geq 10$. However, it was shown here that this approximation has a percentage error less than -2% for $Q \geq 5$, and less than -10% for $Q \geq 2$. It was further shown that the maximum impedance is also well approximated by $R\sqrt{Q^2\left(1+Q^2\right)}$, which has excellent percentage error performance, even for $Q = 1$, with a percentage error of only -4% for this value, and less than -0.6% for $Q \geq 1.5$. Finally, the authors used PSpice simulations to verify their results.

Appendix A

In this appendix, the authors show that

$$2\sqrt{1+\frac{2}{Q^2}} - \left(2+\frac{1}{Q^2}\right) > 0. \tag{A1}$$

Rearranging equation (A1) produces

$$2\sqrt{1+\frac{2}{Q^2}} > 2+\frac{1}{Q^2} \;\Rightarrow\; \sqrt{1+\frac{2}{Q^2}} > 1+\frac{1}{2Q^2}. \tag{A2}$$

Further rearrangement of equation (A2) gives

$$\begin{aligned} &1+\frac{2}{Q^2} > \left(1+\frac{1}{2Q^2}\right)^2 \Rightarrow 1+\frac{2}{Q^2} > 1+\frac{1}{Q^2}+\frac{1}{4Q^4} \\ &\Rightarrow \frac{1}{Q^2} > \frac{1}{4Q^4} \Rightarrow 1 > \frac{1}{4Q^2} \\ &\Rightarrow Q^2 > \frac{1}{4} \Rightarrow Q > \frac{1}{2}. \end{aligned} \tag{A3}$$

Appendix B

In this appendix, the authors show that

$$\frac{-1}{Q^2}+\sqrt{\frac{2}{Q^2}+1}\geq 0 \text{ for } Q\geq\sqrt{\sqrt{2}-1}=.6436.$$

Beginning with

$$\sqrt{\frac{2}{Q^2}+1}\geq\frac{1}{Q^2}\Rightarrow\frac{2}{Q^2}+1\geq\frac{1}{Q^4} \qquad \text{(B1)}$$
$$\Rightarrow 2Q^2+Q^4\geq 1\Rightarrow Q^4+2Q^2-1\geq 0.$$

Solving equation (B1) gives

$$Q^2\geq -1\pm\frac{\sqrt{4+4}}{2}=-1\pm\frac{2\sqrt{2}}{2}. \qquad \text{(B2)}$$

However, Q^2 must be non-negative, so $Q^2\geq -1+\sqrt{2}$.

Hence, $Q\geq\sqrt{\sqrt{2}-1}=.6436,$ as stated previously.

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