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Blow-up of Solutions to the Generalized Inviscid Proudman-Johnson Equation

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Contents

List of Figures

Abstract

The generalized inviscid Proudman-Johnson equation serves as a model for n-dimensional incompressible Euler flow, gas dynamics, the orientation of waves in a massive director field of a nematic liquid crystal, and high-frequency waves in shallow waters. Furthermore, the equation also serves as a tool for studying the role that the natural fluid processes of convection and stretching play in the formation of spontaneous singularities, or of their absence.

In this work, we study blow-up, and blow-up properties, in solutions to the generalized inviscid Proudman-Johnson equation endowed with periodic or Dirichlet boundary conditions. More particularly, for $p \in [1, +\infty]$, regularity of solutions in an L^p setting will be measured via a direct approach which involves the derivation of representation formulae for solutions to the problem. For a parameter $\lambda \in \mathbb{R}$, several classes of initial data $u_0(x)$ are considered. These include the class of smooth functions with either zero or nonzero mean, a family of functions for which $u_0'(x)$ is piecewise constant, and a large class of initial data where u_0' is a bounded, at least continuous almost everywhere, function satisfying Hölder-type estimates near particular locations in the domain. Amongst other results, our analysis will indicate that for appropriate values of the parameter λ, the curvature of u_0 in a neighbourhood of these locations is responsible for an eventual breakdown of solutions, or their persistence for all time. Additionally, we will establish a nontrivial connection between the qualitative properties of L^∞ blow-up in u_x, and its L^p regularity for $p \in [1, +\infty)$. Finally, for smooth and non-smooth initial data, a special emphasis is made on the study of regularity of stagnation point-form solutions to the two ($\lambda = 1$) and three ($\lambda = 1/2$) dimensional incompressible Euler equations subject to periodic or Dirichlet boundary conditions.

Key words: Blow-up, generalized Proudman-Johnson equation, Euler equations.

Chapter 1

Introduction and Scope of the Dissertation

1.1 Introduction

In this work, we examine finite-time blow-up, or global existence in time, of solutions to the initial boundary value problem

$$\begin{cases} u_{xt} + uu_{xx} - \lambda u_x^2 = I(t), & t > 0, \\ u(x,0) = u_0(x), & x \in [0,1], \\ I(t) = -(1+\lambda)\int_0^1 u_x^2\,dx, \end{cases} \tag{1.1.1}$$

where $\lambda \in \mathbb{R}$ and solutions are subject to either the periodic boundary conditions

$$u(0,t) = u(1,t), \quad u_x(0,t) = u_x(1,t). \tag{1.1.2}$$

or the Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0. \tag{1.1.3}$$

For particular values of the parameter λ, equation (1.1.1)i), iii) is related to several important models from the field of fluid dynamics. Amongst the most prominent, one finds stagnation point-form solutions to the n-dimensional incompressible Euler equations, gas dynamics and orientation of waves in massive director nematic liquid crystals. Moreover, the equation appears as the short-wave (or high-frequency) limit to various models for shallow water waves. Finally, and from a more heuristic point of view, equation (1.1.1)i), iii) also serves as a tool for examining the competing effects that natural fluid processes such as convection and stretching have in the formation of spontaneous singularities, or of their absence. For these reasons, the initial boundary value problem for equation (1.1.1)i), iii) has been the subject of extensive research by the mathematical fluid dynamics community.

1.2 Literature Review

1.2.1 Physical Significance of the Equation

In 1962, Proudman and Johnson ([39]) studied the equation

$$u_{xt} + uu_{xx} - u_x^2 = \nu u_{xxx} + \frac{p_x}{y}, \qquad y \neq 0, \tag{1.2.1}$$

which they derived from the vertical component of the two dimensional incompressible Navier-Stokes system

$$\begin{aligned} &\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nu \Delta \boldsymbol{u} - \nabla p \\ &\nabla \cdot \boldsymbol{u} = 0 \end{aligned} \tag{1.2.2}$$

by considering stagnation point-form velocities

$$\boldsymbol{u}(x, y, t) = (u(x, t), -yu_x(x, t)) \tag{1.2.3}$$

in a semi-infinite domain $(x, y) \in [a, b] \times \mathbb{R}$ (a two dimensional channel). In (1.2.2), $\boldsymbol{u}$ represents the fluid velocity, p denotes the scalar pressure, $\nu \geq 0$ is the coefficient of kinematic viscosity and the term p_x/y is a function of time only. Next, after differentiating (1.2.1) with respect to x and inserting a parameter $a \in \mathbb{R}$, Okamoto and Zhu ([35]) introduced the generalized model

$$u_{xxt} + uu_{xxx} - au_x u_{xx} = \nu u_{xxxx}, \tag{1.2.4}$$

known as the generalized Proudman-Johnson equation. In this work, we will be concerned with the associated inviscid ($\nu = 0$) equation (1.1.1)i, iii), which may be obtained by integrating

$$u_{xxt} + uu_{xxx} + (1 - 2\lambda)u_x u_{xx} = 0, \qquad \lambda \in \mathbb{R} \tag{1.2.5}$$

in space and using either set of boundary conditions (1.1.2) or (1.1.3). As a result, we refer to (1.1.1)i), iii) as the generalized, inviscid Proudman-Johnson equation. We remark that the choice of the parameter λ, rather than $a = 2\lambda - 1$, is used, mostly, for notational convenience.

As a Model for n Dimensional Incompressible Euler Flow

From the above discussion, it follows that equation (1.1.1)i), iii) for $\lambda = 1$ is physically justified as the vertical component of the two dimensional incompressible Euler equations

$$\begin{aligned} &\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p \\ &\nabla \cdot \boldsymbol{u} = 0. \end{aligned} \tag{1.2.6}$$

However, in section 1.2.5 we will follow Saxton and Tiglay ([40]) to show that, for $\lambda = \frac{1}{n-1}$, $n \geq 2$, (1.1.1)i), iii) actually models stagnation point-form solutions

$$\boldsymbol{u}(x, \boldsymbol{x}', t) = (u(x,t), -\lambda \boldsymbol{x}' u_x(x,t)), \tag{1.2.7}$$

where $\boldsymbol{x}' = \{x_2, ..., x_n\}$, to the n dimensional incompressible Euler equations. Analogously, one may also use the cylindrical coordinate representation

$$u^r = -\lambda r u_x(x,t), \quad u^\theta \equiv 0, \quad u^x = u(x,t)$$

for $r = |\boldsymbol{x}'|$ ([43], [35], [23]).

As a Model for Gas Dynamics and Nematic Liquid Crystals

In addition to n-dimensional incompressible Euler flow, equation (1.1.1)i), iii) also occurs in several different contexts, either with or without the nonlocal term (1.1.1)iii).

- When $\lambda = -1$, (1.2.5) coincides with the inviscid Burgers' equation

  $$u_t + u u_x = 0,$$

  differentiated twice in space.

- If $\lambda = -1/2$, it reduces to the Hunter Saxton (HS) equation

  $$u_{xt} + u u_{xx} + \frac{1}{2} u_x^2 = -\frac{1}{2} \int_0^1 u_x^2 \, dx,$$

  which describes the orientation of waves in a massive director field for nematic liquid crystals ([28], [3], [15], [44]). For periodic functions, the HS equation also describes geodesics on the group $\mathcal{D}(\mathbb{S}) \backslash Rot(\mathbb{S})$ of orientation preserving diffeomorphisms on the unit circle $\mathbb{S} = \mathbb{R} \backslash \mathbb{Z}$, modulo the subgroup of rigid rotations with respect to the right-invariant metric ([32], [3], [41], [33])

  $$\langle f, g \rangle = \int_{\mathbb{S}} f_x g_x dx.$$

Moreover, we remark that in the local case $I(t) \equiv 0$, the equation appears as a special case of Calogero's equation

$$u_{xt} + uu_{xx} - \Phi(u_x) = 0$$

for $\Phi(z) = \lambda z^2$ ([4]).

The Role of Convection and Stretching in $3D$ Incompressible Euler Flow

From a more heuristic point of view, the introduction of the parameter λ can also be motivated as follows. Setting $\omega = u_{xx}$ into (1.2.5) yields

$$\omega_t + u\omega_x + (1 - 2\lambda)\omega u_x = 0. \tag{1.2.8}$$

If $\lambda = 1$, (1.2.8) becomes

$$\omega_t + u\omega_x - \omega u_x = 0, \tag{1.2.9}$$

which represents a one dimensional model ([16], c.f. also [13]) for

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}, \tag{1.2.10}$$

the vorticity equation of 3D incompressible Euler flow. The system (1.2.10) is obtained by taking the curl of (1.2.6)i) and defining

$$\frac{D}{Dt} \equiv \partial_t + (\boldsymbol{u} \cdot \nabla);$$

the so-called material or convective derivative. Further, (1.2.9) may also be obtained from (1.2.10) by considering a velocity field of the form

$$\boldsymbol{u}(x, y, t) = \{u(x, t), -yu_x(x, t), 0\}.$$

Now, blow-up is caused by nonlinear terms. Equation (1.2.9) has two of them, a "convection" term $u\omega_x$ and a "stretching" term ωu_x. It is well known that solutions to the 2D incompressible Euler equations which arise from smooth initial data with finite kinetic energy stay smooth for all time ([1]). This is as contrasted to the corresponding 3D problem for which the existence of smooth solutions remains inconclusive. The disparity between the 2D and 3D equations is generally attributed to the amplification of the vorticity that occurs exclusively in the 3D case due to the presence of the stretching term $(\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}$. Indeed, the absence of this term in the 2D equations implies a certain conservation of the vorticity which, in turn, guarantees the global existence. On the other hand, the corresponding convection term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega}$ has been alleged to play a neutral role in blow-up, however, Okamoto and Ohkitani ([36]) showed that it too can play a more prominent role. Specifically, their study of the generalized model (1.2.5) for $\lambda \in (1/2, 1)$ showed that solutions persist for all time, whereas finite-time blow-up occurs if the convection term is removed. In summary, their results suggest that the convection term plays a positive role in global existence. In this sense, the study of regularity in solutions to the generalized equation (1.1.1)i), iii) may lead to a better understanding of the roles played by the processes of convection and stretching in the formation of spontaneous singularities.

Remark 1.2.11. The existence of blow-up solutions to the 3D incompressible Euler, or Navier-Stokes, equations which arise from smooth initial data with finite kinetic energy is one of the most important problems in the fields of analysis and mathematical fluid dynamics. In fact, the 3D Navier-Stokes problem of existence and smoothness is considered one of the seven Millennium Prize problems by the Clay Mathematics Institute. The corresponding 3D Euler problem is, however, considered of far greater physical importance. Although these regularity questions lie outside the scope of this dissertation, due to the unbounded domain in stagnation point-form solutions[1], equation (1.1.1)i), iii) for $\lambda = 1/2, 1$ is (particularly) of great research interest given the complexity of the full Euler problem.

As a Model for High-Frequency Waves in Shallow Water

It is also worth noting that (1.1.1)i), iii) appears as the short-wave, or high-frequency, limit of the so-called b equation ([27], [22], [19], [18])

$$m_t + um_x + bmu_x = 0, \qquad m = u - \alpha^2 u_{xx} \tag{1.2.12}$$

for $(\alpha, b) \in \mathbb{R}^2$. Equation (1.2.12) is a dispersive wave equation which includes as special cases the Camassa-Holm (CH) equation if $\alpha = 1$ and $b = 2$ ([5]), and the Degasperis-procesi (DP) equation when $\alpha = 1$ and $b = 3$ ([17]). Both equations are bi-Hamiltonian (thus admit an infinite number of conservation laws), are completely integrable via the inverse scattering transform, and arise in the modeling of shallow water waves. The CH and DP equations

[1]See (1.2.7) above.

are appropriate for waves of medium amplitude and wave breaking phenomena can occur, that is, solutions stay continuous and bounded, while their slope may become infinite in finite-time.

The short-wave limit of (1.2.12) is achieved via the change of variables $t' = \epsilon t$, $x' = x/\epsilon$, $u(x,t) = \epsilon^2 u'(x',t')$, and then letting $\epsilon \to 0$ in the resulting equation. In this sense, the generalized inviscid Proudman-Johnson equation (1.1.1)i), iii) is the short-wave limit of (1.2.12) for $b = 1 - 2\lambda$ and $\alpha \neq 0$.

1.2.2 Some Terminology

Before giving a brief summary of earlier results and outlining the objectives of this Dissertation, we introduce some terminology ([20], [26]).

- For $p \in [1, +\infty]$ and $k \in \mathbb{N} \cup \{0\}$, $L^p(0,1)$ and $W^{k,p}(0,1)$ denote the standard Banach spaces. In addition, for a measurable function $f(x,t) : [0,1] \times [0,T) \to \mathbb{R}$ we use

$$\|f(\cdot,t)\|_{L^p(0,1)} = \|f(\cdot,t)\|_p\,, \qquad \|f(\cdot,t)\|_{W^{k,p}(0,1)} = \|f(\cdot,t)\|_{k,p}$$

as notation for the corresponding norms:

$$\|f(\cdot,t)\|_{L^p(0,1)} \equiv \begin{cases} \left(\int_0^1 |f(x,t)|^p\,dx\right)^{1/p}, & 1 \leq p < +\infty, \\ \operatorname{ess\,sup}_{x\in[0,1]} |f(x,t)|\,, & p = +\infty \end{cases} \tag{1.2.13}$$

and

$$\|f(\cdot,t)\|_{W^{k,p}(0,1)} \equiv \begin{cases} \left(\sum_{|\kappa|\leq k} \int_0^1 |D^\kappa f(x,t)|^p\,dx\right)^{1/p}, & 1 \leq p < +\infty, \\ \sum_{|\kappa|\leq k} \operatorname{ess\,sup}_{x\in[0,1]} |D^\kappa f(x,t)|\,, & p = +\infty \end{cases} \tag{1.2.14}$$

where $\kappa = (\kappa_1, \kappa_2, ..., \kappa_n)$ is a multiindex of order $|\kappa| = \sum_{i=1}^n \kappa_i$ and

$$D^\kappa g = \frac{\partial^{\kappa_1}}{\partial x_1^{\kappa_1}} \cdots \frac{\partial^{\kappa_n}}{\partial x_n^{\kappa_n}}\, g.$$

Also, we use the standard notation $H^k(0,1) = W^{k,2}(0,1)$, as well as H^{-k} for the dual of H_0^k.

- $PC(0,1)$ is the space of piecewise constant functions on $[0,1]$.

- $C^k(0,1)$ for $k \in \mathbb{N} \cup \{0\}$, denotes the space of continuous functions on $[0,1]$ with continuous derivatives up to order k, whereas C^∞ refers to the class of smooth functions. Furthermore, for $T > 0$, $C^k([0,T);C^j(0,1))$ and similar notations are defined straightforwardly.

- Let $x_0 \in \mathbb{R}$ and f be a function defined on a bounded set $\mathcal{D}$ containing x_0. If $0 < q < 1$, we say that f is Hölder continuous with exponent q at x_0 if the quantity

$$[f]_{q;x_0} := \sup_{x \in \mathcal{D}} \frac{|f(x) - f(x_0)|}{|x - x_0|^q} \tag{1.2.15}$$

is finite. We call $[f]_{q;x_0}$ the $q-$Hölder coefficient of f at x_0 with respect to $\mathcal{D}$. Clearly, if f is Hölder continuous at x_0, then f is continuous at x_0. When (1.2.15) is finite for $q = 1$, f is said to be Lipschitz continuous at x_0.

- A subscript '$\mathbb{R}$' is used to signify zero mean, i.e. $f(\cdot,t) \in L^2_{\mathbb{R}}(0,1)$ implies

$$f(\cdot,t) \in L^2(0,1), \qquad \int_0^1 f(x,t)\,dx = 0 \tag{1.2.16}$$

for as long as f is defined.

- For fixed spatial variable, $\dot{f}$ denotes differentiation with respect to time.

- The term "discrete blow-up" will apply to functions which diverge in time at a finite number of points in its spatial domain, namely, there exists $T \in (0,+\infty)$, $n \in \mathbb{N}$ and $x_k \in [0,1]$, $k = 1, 2, ..., n$ such that $f(x,t) : [0,1] \times [0,T) \to \mathbb{R}$ satisfies

$$\lim_{t \uparrow T} |f(x_k,t)| = +\infty \tag{1.2.17}$$

for each k. If (1.2.17) holds for every $x \in [0,1]$, the term 'everywhere' blow-up is used instead. Furthermore, if f diverges to either $+\infty$ or $-\infty$ only, we say the blow-up is "one-sided". If instead, blow-up occurs simultaneously to $+\infty$ and $-\infty$, we call it a "two-sided" blow-up.

Finally, we remark that all functions in this work are assumed to be real-valued.

1.2.3 Earlier Results

We begin this section by stating local-in-time existence Theorems for solutions to (1.1.1)-(1.1.2) or (1.1.3). Then, a review of earlier finite-time blow-up, or global existence in time, results is presented.

Local Existence

The following two Theorems concern the local-in-time existence of solutions to the periodic or Dirichlet problem for (1.1.1).

Theorem 1.2.18. *For any $u_0''(x) \in L^2(0,1)$ there exists $T > 0$ and a unique solution to (1.1.1)-(1.1.2) or (1.1.3) in the class*

$$u_{xx}(x,t) \in C^0([0,T]; L^2(0,1)) \cap C^1_w([0,T]; H^{-1}(0,1)),$$

where the subscript w implies weak topology. In addition, if $u_0''(x) \in H^m(0,1)$, with $m \in \mathbb{N}$, then $u_{xx}(x,t) \in C^0([0,T]; H^m(0,1))$.

We remark that Okamoto and Zhu proved Theorem 1.2.18 in ([35]) by using a result from Kato and Lai ([30]).

Now, in the context stagnation point-form solutions to the n dimensional incompressible Euler equations (see section 1.2.1), we have the following result by Saxton and Tiglay ([40]):

Theorem 1.2.19. *Suppose $u_0(x) \in C^1(0,1)$ and $\lambda = \frac{1}{n-1}$ for $n > 1$. Then, there exists a unique solution to (1.1.1), satisfying either (1.1.2) or (1.1.3), in the class*

$$u(x,t) \in C^0([0,T); C^1(0,1)) \cap C^1([0,T); C^0(0,1)).$$

Additionally, the solution depends continuously on the initial data.

Earlier Regularity Results

For the Dirichlet boundary condition (1.1.3), the earliest blow-up results in the nonlocal case

$$I(t) = -2\int_0^1 u_x^2\,dx, \qquad \lambda = 1$$

are due to Childress et al. ([9]), where the authors show that there are solutions which can blow-up in finite-time[2]. Specifically, they attribute the finite-time blow-up to the infinite

[2]Recall from section 1.2.1 that for $\lambda = 1$, (1.1.1)i), iii) models stagnation point-form solutions to the 2D incompressible Euler equations.

domain and unbounded initial vorticity $\boldsymbol{\omega}(x,y,0)$, where $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$. Indeed, for velocity fields (1.2.3), the vorticity's only nonzero component is given by $-yu_{xx}$. In proving that breakdown can occur, the authors employed both Lagrangian and Eulerian type methods to construct blow-up solutions. Their first Lagrangian method led to a "breakdown by example" type of proof where, starting from a choice of smooth initial data, a closed-form formula for u_x was derived and shown to diverge in finite-time at the boundary. In their second Lagrangian argument, more or less related to the first, the authors transformed (1.1.1)i), iii) into a Liouville-type equation, while, in their third method, they used the separation of variables $u(x,t) = \frac{F(x)}{t_*-t}$ into (1.1.1)i), iii) to derive an ordinary differential equation for $F(x)$, which they then showed had a nontrivial solution. In this last case, however, the ODE places restrictions on the regularity of the initial data.

The above results apply to Dirichlet boundary conditions (1.1.3). For spatially periodic solutions, the following holds:

- If $\lambda \in [-1/2, 0)$ and $u_0(x) \in W^{1,2}_{\mathbb{R}}(0,1)$, u_x remains bounded in the L^2 norm but blows up in the L^∞ norm ([38]).

- If $\lambda \in [-1, 0)$, $u_0(x) \in H^s_{\mathbb{R}}(0,1)$, $s \geq 3$ and u_0'' is not constant, then $\|u_x\|_\infty$ blows up ([42]). Similarly if $\lambda \in (-2,-1)$ as long as

$$\inf_{x\in[0,1]} \{u_0'(x)\} + \sup_{x\in[0,1]} \{u_0'(x)\} < 0. \tag{1.2.20}$$

- If $\lambda < -1/2$, $\|u_x\|_2$ blows up in finite-time as long as ([35])

$$\int_0^1 (u_0'(x))^3 dx < 0. \tag{1.2.21}$$

- If $\lambda = \infty$, there is blow-up iff the Lebesgue measure

$$\left| \left\{ x \in [0,1] \,:\, u_0(x) = \max_{y\in[0,1]} u_0(y) \right\} \right| \leq \frac{1}{2}.$$

- If $\lambda \in [0, 1/2)$ and $u_0''(x) \in L^{\frac{1}{1-2\lambda}}_{\mathbb{R}}(0,1)$, u exists globally in time. Similarly, for $\lambda = 1/2$ as long as $u_0(x) \in W^{2,\infty}_{\mathbb{R}}(0,1)$ ([38], [40]).

- If $\lambda \in [1/2, 1)$ and $u_0'''(x) \in L^{\frac{1}{2(1-\lambda)}}_{\mathbb{R}}(0,1), u$ exists globally in time ([38]).

We will return to these results in future sections. For the time being, the reader may refer to [9], [35], [40], [38], [42] and [12] for details.

1.2.4 Objectives and Outline of the Dissertation

In this section, some of the regularity questions that will serve as a guide for the development of this work are discussed. Then, a general outline of the Dissertation is provided.

Objectives

The main purpose of this work is to provide further insight on how solutions to (1.1.1)-(1.1.2) or (1.1.3) blow up for parameters $\lambda < 0$ as well as to study their regularity, under differing assumptions on the initial data $u_0(x)$, when $\lambda \geq 0$. For any $\lambda \in \mathbb{R}$, regularity will be examined using $L^p(0,1)$ Banach spaces for $p \in [1,+\infty]$. To do so, we employ a direct approach which involves the derivation of representation formulae for solutions along characteristics. Moreover, several classes of initial data will be considered. For the time being, we simply note that these include:

- Smooth initial data with zero or nonzero mean in $[0,1]$.
- Initial data with either u_0' or u_0'' in $PC_{\mathbb{R}}(0,1)$ or $PC(0,1)$, the family of piecewise constant functions.
- A class of initial data with arbitrary curvature near particular locations in the domain, and for which u_0' is bounded and, at least, continuous almost everywhere.

Next, we discuss the main regularity issues, and other aspects related to the problem, that will considered in this Dissertation.

- Regularity for parameters $\lambda > 1$; a case that has remained open until now for any set of boundary conditions.
- In the cases where spontaneous singularities form, we examine detailed features of $L^\infty(0,1)$ blow-up for any $\lambda \in \mathbb{R}$:

    1. Is it a discrete type of blow-up or an everywhere blow-up?
    2. Is the blow-up one-sided or two-sided?
    3. Relative to the data class, is there a threshold parameter value $\lambda_* \in \mathbb{R}$ separating solutions which blow-up in finite-time from those that persist globally in time?
    4. Parameters $\lambda > 0$ versus $\lambda < 0$.

5. Periodic versus Dirichlet boundary conditions.

- Further $L^p(0,1)$ regularity of u_x for $p \in [1, +\infty)$:

  1. Is there a correspondence between qualitative properties of L^∞ blow-up and L^p regularity of solutions?
  2. Study of the energy-related quantities $E(t) = \|u_x(\cdot, t)\|_2^2$ and $\dot{E}(t)$.

- A special emphasis will be given to finite-time blow-up, or global existence in time, in stagnation point-form solutions to the two ($\lambda = 1$) and three ($\lambda = 1/2$) dimensional incompressible Euler equations.

Organization

For convenience of the reader, in section 1.2.5 we follow an argument in [40] to show how equation (1.1.1)i), iii) can be derived from the n dimensional incompressible Euler equations for certain values of the parameter λ. Then, in Chapter 2, representation formulae for $u(x,t)$ and $u_x(x,t)$ along characteristics, as well as other important related quantities, are derived. This is done by rewriting (1.1.1)i), iii) as a second-order linear ODE in terms of the jacobian of the transformation and then using periodic or Dirichlet boundary conditions to solve the simpler, reformulated problem. With the formulae at hand, Chapter 3 is concerned with the study of L^p regularity of u_x for $p \in [1, +\infty]$, $\lambda \in \mathbb{R}$ and several classes of initial data. Particularly, the regularity issues discussed in section 1.2.4 above are addressed. Lastly, the reader may refer to Chapter 4 for specific examples.

1.2.5 Derivation of the Equation

In this section, we follow an argument used by Saxton and Tiglay ([40]) to derive equation (1.1.1)i), iii) from the n dimensional incompressible Euler equations.

Using the ansatz (1.2.7) on (1.2.6)i) yields the following system of n equations:

$$\begin{cases} u_t(x,t) + u(x,t)u_x(x,t) = -p_x(x, \boldsymbol{x'}, t), \\ u_{xt}(x,t) + u(x,t)u_{xx}(x,t) - \lambda u_x(x,t)^2 = \frac{1}{\lambda x_i} p_{x_i}(x, \boldsymbol{x'}, t) \to I_i(x,t), \end{cases} \tag{1.2.22}$$

where $x_i \neq 0$ for $i = 2, 3, ...n$, the functions I_i are yet to be determined and $\lambda = \frac{1}{n-1}$, since $\nabla \cdot \boldsymbol{u} = 0$.

Now, because u is independent of $\boldsymbol{x'} = (x_2, ..., x_3)$, we apply the operator $\nabla' \equiv (\partial_{x_2}, ..., \partial_{x_n})$ to equation (1.2.22)i) and find that

$$\nabla' p_x = 0.$$

As a result, taking ∂_x of (1.2.22)ii) implies that every I_i depends only on time. Suppose u satisfies either (1.1.2) or (1.1.3). Then, applying $\int_0^1 dx$ to (1.2.22)ii) and integrating by parts yields

$$I_i(t) = -(1+\lambda)\int_0^1 u_x(x,t)^2\, dx, \qquad i = 2, 3, ..., n.$$

Substituting the above into (1.2.22)ii) implies (1.1.1)i), iii).

Remark 1.2.23. Suppose the scalar pressure p is periodic in the variable x, i.e.

$$p(0, \boldsymbol{x'}, t) = p(1, \boldsymbol{x'}, t).$$

Then, integrating (1.2.22)i) in x and using either (1.1.2) or (1.1.3) implies that

$$\int_0^1 u(x,t)\, dx = \int_0^1 u_0(x)\, dx. \tag{1.2.24}$$

We remark that in the periodic setting (1.1.2), the 'conservation in mean' condition (1.2.24) is needed in section 2.2.2 to uniquely determine a representation formula for u along characteristics.

Chapter 2

The General Solution

For as long as solutions exist, define the characteristics, $\gamma(\alpha,t)$, as the solution to the initial value problem

$$\dot{\gamma}(\alpha,t)=u(\gamma(\alpha,t),t),\qquad \gamma(\alpha,0)=\alpha\in[0,1]. \tag{2.0.1}$$

Then

$$\dot{\gamma}_\alpha(\alpha,t)=u_x(\gamma(\alpha,t),t)\cdot\gamma_\alpha(\alpha,t). \tag{2.0.2}$$

For $\lambda\neq 0$, our first objective will be to derive a representation formula for $u_x(\gamma(\alpha,t),t)$ satisfying

$$\frac{d}{dt}(u_x(\gamma(\alpha,t),t))-\lambda u_x(\gamma(\alpha,t),t)^2=I(t), \tag{2.0.3}$$

which is simply (1.1.1)i) along characteristics. The case $\lambda=0$ is considered separately in appendix A.

2.1 The Representation Formula for $u_x(\gamma(\alpha,t),t)$

Using (1.1.1)i) and (2.0.2),

$$\begin{aligned}\ddot{\gamma}_\alpha&=(u_{xt}+uu_{xx})\circ\gamma\cdot\gamma_\alpha+(u_x\circ\gamma)\cdot\dot{\gamma}_\alpha\\&=(u_{xt}+uu_{xx})\circ\gamma\cdot\gamma_\alpha+u_x^2\circ\gamma\cdot\gamma_\alpha\\&=(\lambda+1)\left(u_x^2\circ\gamma-\int_0^1u_x^2dx\right)\cdot\gamma_\alpha\\&=(\lambda+1)\left((\gamma_\alpha^{-1}\cdot\dot{\gamma}_\alpha)^2-\int_0^1u_x^2dx\right)\cdot\gamma_\alpha\,.\end{aligned}\tag{2.1.1}$$

For $I(t)=-(\lambda+1)\int_0^1u_x^2\,dx$ and $\lambda\in\mathbb{R}\backslash\{0\}$, then

$$I(t)=\frac{\ddot{\gamma}_\alpha\cdot\gamma_\alpha-(\lambda+1)\cdot\dot{\gamma}_\alpha^2}{\gamma_\alpha^2}=-\frac{\gamma_\alpha^\lambda\cdot(\gamma_\alpha^{-\lambda})^{\ddot{}}}{\lambda} \tag{2.1.2}$$

and so

$$(\gamma_\alpha^{-\lambda})^{\cdot\cdot} + \lambda\gamma_\alpha^{-\lambda}\, I(t) = 0. \tag{2.1.3}$$

Setting

$$\omega(\alpha, t) = \gamma_\alpha(\alpha, t)^{-\lambda} \tag{2.1.4}$$

yields

$$\ddot{\omega}(\alpha, t) + \lambda I(t)\omega(\alpha, t) = 0, \tag{2.1.5}$$

an ordinary differential equation parametrized by α. Suppose we have two linearly independent solutions $\phi_1(t)$ and $\phi_2(t)$ to (2.1.5), satisfying $\phi_1(0) = \dot{\phi}_2(0) = 1$, $\dot{\phi}_1(0) = \phi_2(0) = 0$. Then by Abel's formula, $\mathrm{W}(\phi_1(t), \phi_2(t)) = 1$, $t \geq 0$, where $\mathrm{W}(g, h)$ denotes the wronskian of g and h. We look for solutions of (2.1.5), satisfying appropriate initial data, of the form

$$\omega(\alpha, t) = c_1(\alpha)\phi_1(t) + c_2(\alpha)\phi_2(t), \tag{2.1.6}$$

where reduction of order allows us to write $\phi_2(t)$ in terms of $\phi_1(t)$ as

$$\phi_2(t) = \phi_1(t) \int_0^t \frac{ds}{\phi_1^2(s)}.$$

Since $\gamma_\alpha(\alpha, 0) = 1$ and $\dot{\omega} = -\lambda\gamma_\alpha^{-(\lambda+1)}\dot{\gamma_\alpha}$, by (2.1.4), then $\omega(\alpha, 0) = 1$ and $\dot{\omega}(\alpha, 0) = -\lambda u_0'(\alpha)$, from which $c_1(\alpha)$ and $c_2(\alpha)$ are obtained. Combining these results reduces (2.1.6) to

$$\omega(\alpha, t) = \phi_1(t)\left(1 - \lambda\eta(t)u_0'(\alpha)\right), \qquad \eta(t) = \int_0^t \frac{ds}{\phi_1^2(s)}. \tag{2.1.7}$$

Now, (2.1.4) and (2.1.7)i) imply

$$\gamma_\alpha(\alpha, t) = \left(\phi_1(t)\mathcal{J}(\alpha, t)\right)^{-\frac{1}{\lambda}}, \tag{2.1.8}$$

where

$$\mathcal{J}(\alpha, t) = 1 - \lambda\eta(t)u_0'(\alpha), \qquad \mathcal{J}(\alpha, 0) = 1. \tag{2.1.9}$$

Suppose u satisfies the periodic boundary condition (1.1.2). Then, using the result on uniqueness and existence of solutions to ODE, (2.0.1) and periodicity of u requires

$$\gamma(\alpha + 1, t) - \gamma(\alpha, t) = 1 \tag{2.1.10}$$

for as long as u is defined. On the other hand, if u satisfies Dirichlet boundary conditions (1.1.3), then

$$\gamma(0,t) \equiv 0, \qquad \gamma(1,t) \equiv 1 \tag{2.1.11}$$

must hold instead. Either way, the jacobian γ_α has mean one in $[0,1]$. As a result, spatially integrating (2.1.8) yields

$$\phi_1(t) = \left(\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{\frac{1}{\lambda}}} \right)^{\lambda}, \tag{2.1.12}$$

and so, if we set

$$\mathcal{K}_i(\alpha,t) = \frac{1}{\mathcal{J}(\alpha,t)^{i+\frac{1}{\lambda}}}, \qquad \bar{\mathcal{K}}_i(t) = \int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{i+\frac{1}{\lambda}}} \tag{2.1.13}$$

for $i \in \mathbb{N} \cup \{0\}$, we can write γ_α in the form

$$\gamma_\alpha = \mathcal{K}_0 / \bar{\mathcal{K}}_0. \tag{2.1.14}$$

Therefore, using (2.0.2) and (2.1.14), we obtain

$$u_x(\gamma(\alpha,t),t) = \dot{\gamma}_\alpha(\alpha,t)/\gamma_\alpha(\alpha,t) = \left(\ln(\mathcal{K}_0/\bar{\mathcal{K}}_0)\right)^{\cdot}. \tag{2.1.15}$$

In addition, differentiating (2.1.7)ii) and using (2.1.12) and (2.1.13)ii), yields

$$\dot{\eta}(t) = \bar{\mathcal{K}}_0(t)^{-2\lambda}, \qquad \eta(0) = 0, \tag{2.1.16}$$

which upon integration gives

$$t(\eta) = \int_0^\eta \left(\int_0^1 \frac{d\alpha}{(1 - \lambda\mu u_0'(\alpha))^{\frac{1}{\lambda}}} \right)^{2\lambda} d\mu. \tag{2.1.17}$$

From (2.1.17), it follows that finite-time blow-up of $u_x(\gamma(\alpha,t),t)$ will depend, in part, upon the existence of a finite, positive limit

$$t_* \equiv \lim_{\eta \uparrow \eta_*} \int_0^\eta \left(\int_0^1 \frac{d\alpha}{(1 - \lambda\mu u_0'(\alpha))^{\frac{1}{\lambda}}} \right)^{2\lambda} d\mu \tag{2.1.18}$$

for $\eta_* \in \mathbb{R}^+$ to be defined. In an effort to simplify the arguments in future sections, we note that (2.1.15) can be rewritten in a slightly more useful form. The result is the representation formula

$$u_x(\gamma(\alpha,t),t) = \frac{1}{\lambda\eta(t)\bar{\mathcal{K}}_0(t)^{2\lambda}} \left(\frac{1}{\mathcal{J}(\alpha,t)} - \frac{\bar{\mathcal{K}}_1(t)}{\bar{\mathcal{K}}_0(t)} \right). \tag{2.1.19}$$

This is derived as follows. From (2.1.13) and (2.1.15),

$$u_x(\gamma(\alpha,t),t) = \frac{1}{\bar{\mathcal{K}}_0(t)^{2\lambda}}\left(\frac{u_0'(\alpha)}{\mathcal{J}(\alpha,t)} - \frac{1}{\bar{\mathcal{K}}_0(t)}\int_0^1 u_0'(\alpha)\mathcal{K}_1(\alpha,t)d\alpha\right). \tag{2.1.20}$$

However

$$\frac{u_0'(\alpha)}{\mathcal{J}(\alpha,t)} = \frac{1}{\lambda\eta(t)}\left(\frac{1}{\mathcal{J}(\alpha,t)} - 1\right), \tag{2.1.21}$$

by (2.1.9), and so

$$\int_0^1 u_0'(\alpha)\mathcal{K}_1(\alpha,t)d\alpha = \frac{\bar{\mathcal{K}}_1(t) - \bar{\mathcal{K}}_0(t)}{\lambda\eta(t)}. \tag{2.1.22}$$

Substituting (2.1.21) and (2.1.22) into (2.1.20) yields (2.1.19).

Now, assuming sufficient smoothness, we may use (2.1.14) and (2.1.19) to obtain ([40], [42])

$$u_{xx}(\gamma(\alpha,t),t) = u_0''(\alpha)(\gamma_\alpha(\alpha,t))^{2\lambda-1}. \tag{2.1.23}$$

Equation (2.1.23) implies that as long as a solution exists it will maintain its initial concavity profile. Also, since the exponent above changes sign through $\lambda = 1/2$, blow-up implies, relative to the value of λ, either vanishing or divergence of the jacobian. More explicitly, (2.1.14) and (2.1.23) yield

$$u_{xx}(\gamma(\alpha,t),t) = \frac{u_0''(\alpha)}{\mathcal{J}(\alpha,t)^{2-\frac{1}{\lambda}}}\left(\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{\frac{1}{\lambda}}}\right)^{1-2\lambda}. \tag{2.1.24}$$

Remark 2.1.25. Since $\eta(0) = 0$ and $\bar{\mathcal{K}}_i(0) = 1$, setting $t = 0$ into (2.1.19) yields an expression of the form $0/0$. The desired result, namely $u_0'(\alpha)$, follows by L'Hopital's rule.

Remark 2.1.26. Notice that either (2.1.20) or (2.1.19) imply that, for as long as solutions exist,

$$u_x(\gamma(\alpha_1,t),t) = u_x(\gamma(\alpha_2,t),t) \quad \Leftrightarrow \quad u_0'(\alpha_1) = u_0'(\alpha_2) \tag{2.1.27}$$

for $\lambda \in \mathbb{R}$ and $\alpha_1, \alpha_2 \in [0,1]$. This clearly agrees with the periodic boundary conditions (1.1.2), whereas, in the Dirichlet setting (1.1.3), (2.1.11) and (2.1.27) give

$$u_x(0,t) = u_x(1,t) \quad \Leftrightarrow \quad u_0'(0) = u_0'(1).$$

Remark 2.1.28. The representation formula (2.1.20) for $\lambda = 1$ (stagnation point-form solutions (1.2.3) to the 2D incompressible Euler equations) resembles a lower-dimensional analogue to the vertical component of an infinite energy, periodic class of solutions derived by

Constantin ([14]) for the corresponding 3D Euler problem. For $u(x,y,t) = (u_1(x,y,t), u_2(x,y,t))$ and $\nabla \cdot u = \partial_x u_1 + \partial_y u_2$, he considered the ansatz

$$\boldsymbol{u}(x,y,z,t) = (u_1(x,y,t), u_2(x,y,t), zv(x,y,t))$$

on an infinite 2D channel $(x,y,z) \in [0,L]^2 \times \mathbb{R}$. Using the above yields, as the vertical component of the 3D Euler system, the equation

$$\partial_t(\nabla \cdot u) + (u_1,u_2) \cdot \nabla(\nabla \cdot u) - (\nabla \cdot u)^2 = -\frac{2}{L^2}\int_0^L \int_0^L (\nabla \cdot u)^2 dxdy,$$

a higher dimensional analogue to (1.1.1)i), iii) (with $\lambda = 1$).

Next, we show that even though formula (2.1.19) holds for either periodic or Dirichlet boundary conditions, this is, generally, not the case for $u(\gamma(\alpha,t),t)$.

2.2 The Representation Formula for $u(\gamma(\alpha,t),t)$

By using the results from section 2.1, we now derive an expression for $u(\gamma(\alpha,t),t)$. In section 2.2.1, we look at the case of Dirichlet boundary conditions (1.1.3) for which a straight forward derivation follows. Then, in section 2.2.2 we examine the periodic setting (1.1.2). In this case, we find that additional information on u and/or the data is required to completely determine a representation formula.

2.2.1 Dirichlet Boundary conditions

Integrating (2.1.14) in α and using (2.1.11)i) and (2.1.13), we find that the characteristics, γ, are given by

$$\gamma(\alpha,t) = \bar{\mathcal{K}}_0(t)^{-1}\int_0^\alpha \mathcal{K}_0(y,t)\,dy. \tag{2.2.1}$$

Now, from (2.1.9) and (2.1.18) there is a time interval $[0,t_*)$, $0 < t_* \leq +\infty$ such that $\mathcal{J}(\alpha,t) > 0$ for all $\alpha \in [0,1]$.[3] Therefore, (2.0.1), (2.1.16) and (2.2.1) yield

$$u(\gamma(\alpha,t),t) = \bar{\mathcal{K}}_0(t)^{-2(1+\lambda)}\left(\bar{\mathcal{K}}_0(t)\int_0^\alpha u_0'(y)\mathcal{K}_1(y,t)dy - \int_0^\alpha \mathcal{K}_0(y,t)dy\int_0^1 u_0'(\alpha)\mathcal{K}_1(\alpha,t)d\alpha\right).$$

[3] See (3.0.6) for a formal definition of $\eta_* \in \mathbb{R}^+$.

for $t \in [0, t_*)$. The above formula may, in turn, be written in a slightly more useful form by using (2.1.21) and (2.1.22). The resulting expression is

$$u(\gamma(\alpha,t),t) = \frac{\bar{\mathcal{K}}_0(t)^{-2(1+\lambda)}}{\lambda\eta(t)}\left(\bar{\mathcal{K}}_0(t)\int_0^\alpha \mathcal{K}_1(y,t)dy - \bar{\mathcal{K}}_1(t)\int_0^\alpha \mathcal{K}_0(y,t)dy\right). \qquad (2.2.2)$$

2.2.2 Periodic Boundary conditions

Next, suppose u satisfies the periodic boundary conditions (1.1.2). Integrating (2.1.14) now leads to

$$\gamma(\alpha,t) = \gamma(0,t) + \bar{\mathcal{K}}_0(t)^{-1}\int_0^\alpha \mathcal{K}_0(y,t)\,dy. \qquad (2.2.3)$$

Then, (2.0.1) yields

$$u(\gamma(\alpha,t),t) = \dot{\gamma}(0,t) + \frac{\bar{\mathcal{K}}_0(t)^{-2(1+\lambda)}}{\lambda\eta(t)}\left(\bar{\mathcal{K}}_0(t)\int_0^\alpha \mathcal{K}_1(y,t)dy - \bar{\mathcal{K}}_1(t)\int_0^\alpha \mathcal{K}_0(y,t)dy\right), \quad (2.2.4)$$

where the time-dependent function $\dot{\gamma}(0,t)$ satisfies

$$\dot{\gamma}(0,t) = u(\gamma(0,t),t), \qquad \gamma(0,t) = \gamma(1,t) - 1$$

by (2.0.1) and (2.1.10). Below, we determine $\gamma(0,t)$ in two different ways. The first relies on assumptions on the data's symmetry, while the second pertains to the incompressible fluid case and uses the conservation in mean condition (1.2.24).

Odd Initial Data.

Under periodic boundary conditions, suppose the initial data $u_0(x)$ is odd about the midpoint $x = 1/2$. Then $u_0(0) = u_0(1) = 0$, by periodicity. Now, it is easy to see that (1.1.1)i), iii) is invariant under the transformation $u(x,t) = -u(-x,t)$. This implies that if $u_0(x)$ is odd, then $u(x,t)$ will remain odd for as long as it exists. As a result $u(0,t) = u(1,t) = 0$, and so (2.1.11) holds from uniqueness of solution to (2.0.1). Particularly, this last observation implies $\gamma(0,t) \equiv 0$, so that (2.2.4) reduces to (2.2.2). To summarize, if the initial data $u_0(x)$ is odd about the midpoint $x = 1/2$ and $\lambda \neq 0$, representation formulae for the characteristics and solutions $u(\gamma(\alpha,t),t)$ to (1.1.1)-(1.1.2) are given by (2.2.1) and (2.2.2), respectively.

n Dimensional Incompressible Euler Flow.

Recall from section 1.2.5 that for $\lambda = \frac{1}{n-1}$, $n \geq 2$, equation (1.1.1)i), iii) models stagnation point-form solutions (1.2.7) to the n dimensional incompressible Euler equations (1.2.6).

Furthermore, for a scalar pressure term that is periodic in the x variable, the conservation in mean condition (1.2.24) holds. Assume periodic boundary conditions. Since

$$\int_0^1 u_0(x)\,dx = \int_0^1 u(x,t)\,dx = \int_0^1 u(\gamma(\alpha,t),t)\gamma_\alpha(\alpha,t)\,d\alpha, \tag{2.2.5}$$

we multiply (2.2.4) by the mean-one function γ_α in (2.1.14), integrate in α, and use the identity

$$\int_0^1 \mathcal{K}_0(\alpha,t)\int_0^\alpha \mathcal{K}_0(y,t)dy\,d\alpha = \frac{1}{2}\int_0^1 \frac{d}{d\alpha}\left(\int_0^\alpha \mathcal{K}_0(y,t)dy\right)^2 d\alpha = \frac{\bar{\mathcal{K}}_0(t)^2}{2}$$

to obtain

$$\dot{\gamma}(0,t) = \int_0^1 u_0(\alpha)d\alpha + \frac{\bar{\mathcal{K}}_0(t)^{-2(1+\lambda)}}{\lambda\eta(t)}\left(\frac{\bar{\mathcal{K}}_0(t)\bar{\mathcal{K}}_1(t)}{2} - \int_0^1 \mathcal{K}_0(\alpha,t)\int_0^\alpha \mathcal{K}_1(y,t)dy\,d\alpha\right). \tag{2.2.6}$$

Substituting the above back into (2.2.4) yields

$$\begin{aligned} u(\gamma(\alpha,t),t) = &\int_0^1 u_0(\alpha)d\alpha + \frac{\bar{\mathcal{K}}_0(t)^{-2(1+\lambda)}}{\lambda\eta(t)}\left(\frac{\bar{\mathcal{K}}_0(t)\bar{\mathcal{K}}_1(t)}{2} + \bar{\mathcal{K}}_0(t)\int_0^\alpha \mathcal{K}_1(y,t)dy\right) \\ &- \frac{\bar{\mathcal{K}}_0(t)^{-2(1+\lambda)}}{\lambda\eta(t)}\left(\bar{\mathcal{K}}_1(t)\int_0^\alpha \mathcal{K}_0(y,t)dy + \int_0^1 \mathcal{K}_0(\alpha,t)\int_0^\alpha \mathcal{K}_1(y,t)dyd\alpha\right). \end{aligned} \tag{2.2.7}$$

Lastly, since $\gamma(0,0) = 0$, we may integrate (2.2.6) in time and use (2.2.3) to obtain an expression for the characteristics $\gamma(\alpha,t)$.

Remark 2.2.8. If the data is not odd or $\lambda \neq \frac{1}{n-1}$, $n \geq 2$, it is generally assumed that u has zero mean. In that case, expressions for γ and $u(\gamma(\alpha,t),t)$ can be obtained from the above formulas simply by setting $\int_0^1 u_0(\alpha)d\alpha = 0$.

Remark 2.2.9. For the remainder of this Dissertation, solutions to (1.1.1)-(1.1.2) are assumed to have zero mean.

Chapter 3

Global Estimates and Blow-up

In this chapter, we examine finite-time blow-up, or global existence in time, of solutions to the initial value problem (1.1.1) arising out of several classes of initial data and satisfying either periodic (1.1.2) or Dirichlet boundary conditions (1.1.3). Before discussing the classes of initial data to be considered, first, we make some definitions and introduce some of the tools and auxiliary results that will aid us in the study of blow-up.

As mentioned at the end of section 2.1, finite-time blow-up of (2.1.19) will depend, in part, upon the existence of a finite, positive limit

$$t_* \equiv \lim_{\eta \uparrow \eta_*} \int_0^\eta \left(\int_0^1 \frac{d\alpha}{(1 - \lambda\mu u_0'(\alpha))^{\frac{1}{\lambda}}} \right)^{2\lambda} d\mu \tag{3.0.1}$$

for $\eta_* > 0$ to be defined. Let us suppose a solution $u(x,t)$ exists on an interval $t \in [0, t_*)$ $0 < t_* \leq +\infty$. Define

$$M(t) \equiv \sup_{\alpha \in [0,1]} \{u_x(\gamma(\alpha,t),t)\}, \qquad M(0) = M_0, \tag{3.0.2}$$

and

$$m(t) \equiv \inf_{\alpha \in [0,1]} \{u_x(\gamma(\alpha,t),t)\}, \qquad m(0) = m_0. \tag{3.0.3}$$

Then, it follows from the representation formula (2.1.19) (see appendix C) that

$$M(t) = u_x(\gamma(\overline{\alpha}_i, t), t) \tag{3.0.4}$$

and

$$m(t) = u_x(\gamma(\underline{\alpha}_j, t), t), \tag{3.0.5}$$

where $\overline{\alpha}_i,\ i = 1,2,...,m$ and $\underline{\alpha}_j,\ j = 1,2,...,n$ denote the finite (or infinite) number of locations in $[0,1]$ where $u_0'(\alpha)$ attains its greatest and least values $M_0 > 0 > m_0$, respectively. Let

$$\eta_* = \begin{cases} \frac{1}{\lambda M_0}, & \lambda > 0, \\ \frac{1}{\lambda m_0}, & \lambda < 0, \end{cases} \tag{3.0.6}$$

then as $\eta \uparrow \eta_*$, the space-dependent term in (2.1.19) will diverge for certain choices of α and not at all for others. Specifically, for $\lambda > 0$, $\mathcal{J}(\alpha,t)^{-1}$ blows up earliest as $\eta \uparrow \eta_*$ at $\alpha = \overline{\alpha}_i$, since

$$\mathcal{J}(\overline{\alpha}_i,t)^{-1} = (1-\lambda\eta(t)M_0)^{-1} \to +\infty \quad \text{as} \quad \eta \uparrow \eta_* = \frac{1}{\lambda M_0}.$$

Similarly for $\lambda < 0$, $\mathcal{J}(\alpha,t)^{-1}$ diverges first at $\alpha = \underline{\alpha}_j$ and

$$\mathcal{J}(\underline{\alpha}_j,t)^{-1} = (1-\lambda\eta(t)m_0)^{-1} \to +\infty \quad \text{as} \quad \eta \uparrow \eta_* = \frac{1}{\lambda m_0}.$$

However, blow-up of (2.1.19) does not necessarily follow from this; we will need to estimate the behaviour of the time-dependent integrals

$$\bar{\mathcal{K}}_0(t) = \int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{\frac{1}{\lambda}}}, \qquad \bar{\mathcal{K}}_1(t) = \int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda}}}$$

as $\eta \uparrow \eta_*$. To this end, in some of the proofs we find convenient the use of the Gauss hypergeometric series[4] ([2], [21], [25])

$${}_2F_1\left[a,b;c;z\right] \equiv \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k\ k!} z^k, \qquad |z| < 1 \tag{3.0.7}$$

for $c \notin \mathbb{Z}^- \cup \{0\}$ and where $(x)_k$ denotes the Pochhammer symbol

$$(x)_k = \begin{cases} 1, & k = 0 \\ x(x+1)...(x+k-1), & k \in \mathbb{Z}^+. \end{cases} \tag{3.0.8}$$

Also, we will make use of the following results ([21], [25]):

Lemma 3.0.9. *Suppose $|arg(-z)| < \pi$ and $a,b,c,a-b \notin \mathbb{Z}$, then the analytic continuation for $|z| > 1$ of the series (3.0.7) is given by*

$$\begin{aligned} {}_2F_1[a,b;c;z] =& \frac{\Gamma(c)\Gamma(a-b)(-z)^{-b}{}_2F_1[b,1+b-c;1+b-a;z^{-1}]}{\Gamma(a)\Gamma(c-b)} \\ &+ \frac{\Gamma(c)\Gamma(b-a)(-z)^{-a}{}_2F_1[a,1+a-c;1+a-b;z^{-1}]}{\Gamma(b)\Gamma(c-a)} \end{aligned} \tag{3.0.10}$$

where $\Gamma(\cdot)$ denotes the standard gamma function.

[4] See appendix B for convergence results on (3.0.7).

Lemma 3.0.11. *Suppose* $b < 2$, $0 \leq |\beta - \beta_0| \leq 1$ *and* $\epsilon \geq C_0$ *for some* $C_0 > 0$. *Then*

$$\frac{1}{\epsilon^b}\frac{d}{d\beta}\left((\beta-\beta_0)\,{}_2F_1\left[\frac{1}{q}, b; 1+\frac{1}{q}; -\frac{C_0\,|\beta-\beta_0|^q}{\epsilon}\right]\right) = (\epsilon + C_0\,|\beta-\beta_0|^q)^{-b} \tag{3.0.12}$$

for all $q > 0$ *and* $b \neq 1/q$.

Proof. See appendix B. □

Our study of finite-time blow-up begins in section 3.1 where a family of smooth data with $u_0'''(\alpha) \neq 0$ in a neighbourhood of $\overline{\alpha}_i$ and/or $\underline{\alpha}_j$ is considered. Then, in section 3.2, u_0' in the class of piecewise constant functions is studied. Finally, section 3.3 is concerned with a large class of functions where $u_0(\alpha)$ is, at least, $C^1(0,1)$ *a.e.* and has arbitrary curvature near $\overline{\alpha}_i$ and/or $\underline{\alpha}_j$.

3.1 A Class of Smooth Initial Data

In this section, we study finite-time blow-up of solutions to (1.1.1)-(1.1.2) or (1.1.3) which arise from a class of smooth initial data $u_0(\alpha) \in C^\infty_{\mathbb{R}}(0,1)$ or $C^\infty(0,1)$. More particularly, for parameters $\lambda > 0$, we assume that the smooth, mean-zero function $u_0'(\alpha)$ attains its greatest value $M_0 > 0$ at, at most, finitely many locations $\overline{\alpha}_i \in [0,1]$ and that, near these locations, $u_0'''(\alpha) \neq 0$. Similarly, when $\lambda < 0$, we suppose that its least value, $m_0 < 0$, occurs at a discrete set of points $\underline{\alpha}_j \in [0,1]$ and $u_0'''(\alpha) \neq 0$ in a neighbourhood of every $\underline{\alpha}_j$. One possibility for admitting infinitely many $\overline{\alpha}_i$ and/or $\underline{\alpha}_j$ will be considered in section 3.2 for $u_0'(\alpha) \in PC_{\mathbb{R}}(0,1)$, the class of piecewise constant functions. Moreover, the cases where $u_0'''(\alpha)$, or higher derivatives, vanish near the locations in question is studied at the end of section 3.3.3. Below, is a summary of the results we will establish in this section. The case $\lambda = 0$ is treated separately in appendix A.

Theorem 3.1.1. *Consider the initial boundary value problem (1.1.1)-(1.1.2) or (1.1.3) for the generalized, inviscid, Proudman-Johnson equation. There exist smooth initial data such that:*

1. *For* $\lambda \in [0,1]$, *solutions exist globally in time. Particularly, these vanish as* $t \uparrow t_* = +\infty$ *for* $\lambda \in (0,1)$ *but converge to a nontrivial steady-state if* $\lambda = 1$.

2. *For* $\lambda \in (-\infty,-2] \cup (1,+\infty)$, *there exists a finite* $t_* > 0$ *such that both the maximum* $M(t)$ *and the minimum* $m(t)$ *diverge to* $+\infty$ *and respectively to* $-\infty$ *as* $t \uparrow t_*$. *Moreover, if* $\alpha \notin \{\overline{\alpha}_i, \underline{\alpha}_j\}$, $\lim_{t\uparrow t_*} |u_x(\gamma(\alpha,t),t)| = +\infty$ *(two-sided, everywhere blow-up).*

3. *For $\lambda \in (-2, 0)$, there is a finite $t_* > 0$ such that only the minimum diverges, $m(t) \to -\infty$, as $t \uparrow t_*$ (one-sided, discrete blow-up).*

4. *For $\lambda < 0$, suppose only Dirichlet boundary conditions (1.1.3) are considered and/or $u_0(\alpha)$ is odd about the midpoint $\alpha = 1/2$. Then, for every $\underline{\alpha}_j \in [0, 1]$ there exists a unique $\underline{x}_j \in [0, 1]$ given by*

$$\underline{x}_j = \frac{\int_0^{\underline{\alpha}_j} \left(1 + \frac{u_0'(\alpha)}{|m_0|}\right)^{\frac{1}{|\lambda|}} d\alpha}{\int_0^1 \left(1 + \frac{u_0'(\alpha)}{|m_0|}\right)^{\frac{1}{|\lambda|}} d\alpha} \tag{3.1.2}$$

such that $\lim_{t \uparrow t_*} u_x(\underline{x}_j, t) = -\infty$.

The next two results examine the behaviour, as $t \uparrow t_*$, of two quantities, the jacobian $\gamma_\alpha(\alpha, t)$ (see (2.1.14)), and the L^p norm

$$\|u_x(\cdot, t)\|_p = \left(\int_0^1 (u_x(\gamma(\alpha, t), t))^p \gamma_\alpha(\alpha, t)\, d\alpha\right)^{1/p}, \quad p \in [1, +\infty), \tag{3.1.3}$$

with particular emphasis given to the energy function

$$E(t) = \|u_x(\cdot, t)\|_2^2.$$

Remark 3.1.4. Corollary 3.1.5 and Theorem 3.1.7 below describe pointwise behaviour and L^p regularity of solutions as $t \uparrow t_*$ where, for $\lambda \in (-\infty, 0) \cup (1, +\infty)$, $t_* > 0$ refers to the *finite* L^∞ blow-up time for u_x in Theorem 3.1.1, otherwise the description is asymptotic, for $t \uparrow t_* = +\infty$.

Corollary 3.1.5. *Let $u(x, t)$ in Theorem 3.1.1 be a solution to the initial boundary value problem (1.1.1)-(1.1.2) or (1.1.3) defined for $t \in [0, t_*)$. Then*

$$\lim_{t \uparrow t_*} \gamma_\alpha(\alpha, t) = \begin{cases} +\infty, & \alpha = \overline{\alpha}_i, & \lambda > 0, \\ 0, & \alpha \neq \overline{\alpha}_i, & \lambda \in (0, 2], \\ C, & \alpha \neq \overline{\alpha}_i, & \lambda > 2, \\ 0, & \alpha = \underline{\alpha}_j, & \lambda < 0, \\ C, & \alpha \neq \underline{\alpha}_j, & \lambda < 0 \end{cases} \tag{3.1.6}$$

where the positive constants C depend on the choice of λ and α.

Table 3.1: Energy Estimates and L^p Regularity as $t \uparrow t_*$

λ	$E(t)$	$\dot{E}(t)$	u_x
$(-\infty, -2]$	$+\infty$	$+\infty$	$\notin L^p,\ p > 1$
$(-2, -2/3]$	$+\infty$	$+\infty$	$\in L^1,\ \notin L^2$
$(-2/3, -1/2)$	Bounded	$+\infty$	$\in L^2,\ \notin L^3$
$-1/2$	Constant	0	$\in L^2,\ \notin L^3$
$(-1/2, -2/5]$	Bounded	$-\infty$	$\in L^2,\ \notin L^3$
$\left(-\frac{2}{2p-1}, 0\right),\ p \geq 3$	Bounded	Bounded	$\in L^p$
$\left[-\frac{2}{p-1}, -\frac{2}{p}\right],\ p \geq 6$	Bounded	Bounded	$\notin L^p$
$[0, 1]$	Bounded	Bounded	$\in L^\infty$
$(1, +\infty)$	$+\infty$	$+\infty$	$\notin L^p,\ p > 1$

Theorem 3.1.7. *Let $u(x,t)$ in Theorem 3.1.1 be a solution to the initial boundary value problem (1.1.1)-(1.1.2) or (1.1.3) defined for $t \in [0, t_*)$. It holds*

1. *For $p \geq 1$ and $\lambda \in [0, 1]$, $\|u_x\|_p$ exists for all time.*

2. *For $p \geq 1$ and $\frac{2}{1-2p} < \lambda \leq 1$, $\lim_{t \uparrow t_*} \|u_x\|_p < +\infty$.*

3. *For $p \in (1, +\infty)$ and $\lambda \in (-\infty, -2/p] \cup (1, +\infty)$, $\lim_{t \uparrow t_*} \|u_x\|_p = +\infty$.*

4. *The energy $E(t) = \|u_x\|_2^2$ diverges as $t \uparrow t_*$ if $\lambda \in (-\infty, -2/3] \cup (1, +\infty)$ but remains finite for $t \in [0, t_*]$ when $\lambda \in (-2/3, 0)$. Moreover, $\dot{E}(t)$ blows up to $+\infty$ as $t \uparrow t_*$ if $\lambda \in (-\infty, -1/2) \cup (1, +\infty)$ and $\dot{E}(t) \equiv 0$ for $\lambda = -1/2$; whereas, $\lim_{t \uparrow t_*} \dot{E}(t) = -\infty$ when $\lambda \in (-1/2, -2/5]$ but remains bounded, for all $t \in [0, t_*]$, if $\lambda \in (-2/5, 0)$.*

See Table 3.1 for a summary of the results mentioned in Theorem 3.1.7. Theorem 3.1.1 and Corollary 3.1.5 are proved in sections 3.1.1, 3.1.2 and 3.1.3, whereas Theorem 3.1.7 is established in section 3.1.4.

3.1.1 Global estimates for $\lambda \in [0, 1]$ and blow-up for $\lambda > 1$

In this section, we establish finite-time blow-up of u_x in the L^∞ norm for $\lambda > 1$. In fact, we will find that blow-up is two-sided and occurs everywhere in the domain, an event we will refer to as "two-sided, everywhere blow-up." In contrast, for parameters $\lambda \in [0, 1]$, we show that solutions persist globally in time. More particularly, these vanish as $t \to +\infty$ for $\lambda \in (0,1)$ but converge to a non-trivial steady-state if $\lambda = 1$. Finally, the behaviour of the jacobian (2.1.14) is also studied. We refer to appendix A for the case $\lambda = 0$.

Theorem 3.1.8. *Consider the initial boundary value problem (1.1.1)-(1.1.2) or (1.1.3). There exist smooth initial data such that:*

1. *For $\lambda \in (0,1]$, solutions persist globally in time. Particularly, these vanish as $t \uparrow t_* = +\infty$ for $\lambda \in (0,1)$ but converge to a nontrivial steady-state if $\lambda = 1$.*

2. *For $\lambda > 1$, there exists a finite $t_* > 0$ such that both the maximum $M(t)$ and the minimum $m(t)$ diverge to $+\infty$ and respectively to $-\infty$ as $t \uparrow t_*$. Moreover, $\lim_{t \uparrow t_*} u_x(\gamma(\alpha,t),t) = -\infty$ for $\alpha \notin \{\overline{\alpha}_i, \underline{\alpha}_j\}$ (two-sided, everywhere blow-up).*

Finally, for t_* as above, the jacobian (2.1.14) satisfies

$$\lim_{t \uparrow t_*} \gamma_\alpha(\alpha,t) = \begin{cases} +\infty, & \alpha = \overline{\alpha}_i, & \lambda > 0, \\ 0, & \alpha \neq \overline{\alpha}_i, & \lambda \in (0,2], \\ C, & \alpha \neq \overline{\alpha}_i, & \lambda > 2 \end{cases} \tag{3.1.9}$$

where the positive constants C depend on the choice of λ and $\alpha \neq \overline{\alpha}_i$.

Proof. For simplicity, assume $M_0 > 0$ is attained at a single location[5] $\overline{\alpha} \in (0,1)$. We consider the case where, near $\overline{\alpha}$, $u_0'(\alpha)$ has non-vanishing second order derivative, so that, locally

$$u_0'(\alpha) \sim M_0 + C_1(\alpha - \overline{\alpha})^2$$

for $0 \leq |\alpha - \overline{\alpha}| \leq s$, $0 < s \leq 1$ and $C_1 = u_0'''(\overline{\alpha})/2 < 0$. Then, for $\epsilon > 0$

$$\epsilon - u_0'(\alpha) + M_0 \sim \epsilon - C_1(\alpha - \overline{\alpha})^2. \tag{3.1.10}$$

Global existence for $\lambda \in (0,1]$

By (3.1.10) above and the change of variables $\alpha = \sqrt{\frac{\epsilon}{|C_1|}} \tan\theta + \overline{\alpha}$, we have that

$$\int_{\overline{\alpha}-s}^{\overline{\alpha}+s} \frac{d\alpha}{(\epsilon - C_1(\alpha - \overline{\alpha})^2)^{\frac{1}{\lambda}}} \sim \frac{\epsilon^{\frac{1}{2}-\frac{1}{\lambda}}}{\sqrt{-C_1}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(\theta))^{2\left(\frac{1}{\lambda}-1\right)} d\theta \tag{3.1.11}$$

[5] The case of a finite number of $\overline{\alpha}_i \in [0,1]$ follows similarly.

for $\epsilon > 0$ small and $\lambda \in (0, 1]$. But from properties of the gamma function (see for instance [24]), the identity

$$\int_0^1 t^{p-1}(1-t)^{q-1}dt = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} \tag{3.1.12}$$

holds for all $p, q > 0$. Therefore, setting $p = \frac{1}{2}$, $q = \frac{1}{\lambda} - \frac{1}{2}$ and $t = \sin^2\theta$ into (3.1.12) gives

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(\theta))^{2\left(\frac{1}{\lambda}-1\right)} d\theta = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{\lambda} - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{\lambda}\right)},$$

which we use, along with (3.1.10) and (3.1.11), to obtain

$$\int_0^1 \frac{d\alpha}{(\epsilon - u_0'(\alpha) + M_0)^{\frac{1}{\lambda}}} \sim \frac{\Gamma\left(\frac{1}{\lambda} - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{\lambda}\right)} \sqrt{-\frac{\pi}{C_1}}\, \epsilon^{\frac{1}{2} - \frac{1}{\lambda}}. \tag{3.1.13}$$

Consequently, setting $\epsilon = \frac{1}{\lambda\eta} - M_0$ into (3.1.13) yields

$$\bar{\mathcal{K}}_0(t) \sim C_3 \mathcal{J}(\overline{\alpha}, t)^{\frac{1}{2} - \frac{1}{\lambda}} \tag{3.1.14}$$

for $\eta_* - \eta > 0$ small, $\mathcal{J}(\overline{\alpha}, t) = 1 - \lambda\eta(t)M_0$, $\eta_* = \frac{1}{\lambda M_0}$ and positive constants C_3 given by

$$C_3 = \frac{\Gamma\left(\frac{1}{\lambda} - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{\lambda}\right)} \sqrt{-\frac{\pi M_0}{C_1}}. \tag{3.1.15}$$

Similarly,

$$\int_{\overline{\alpha}-s}^{\overline{\alpha}+s} \frac{d\alpha}{(\epsilon - C_1(\alpha - \overline{\alpha})^2)^{1+\frac{1}{\lambda}}} \sim \frac{\Gamma\left(\frac{1}{2} + \frac{1}{\lambda}\right)}{\Gamma\left(1 + \frac{1}{\lambda}\right)} \sqrt{-\frac{\pi}{C_1}}\, \epsilon^{-\left(\frac{1}{2} + \frac{1}{\lambda}\right)} \tag{3.1.16}$$

so that

$$\bar{\mathcal{K}}_1(t) \sim \frac{C_4}{\mathcal{J}(\overline{\alpha}, t)^{\frac{1}{2} + \frac{1}{\lambda}}} \tag{3.1.17}$$

for $\lambda \in (0, 1]$ and positive constants C_4 determined by

$$C_4 = \frac{\Gamma\left(\frac{1}{2} + \frac{1}{\lambda}\right)}{\Gamma\left(1 + \frac{1}{\lambda}\right)} \sqrt{-\frac{\pi M_0}{C_1}}. \tag{3.1.18}$$

Using (3.1.14) and (3.1.17) with (2.1.19) implies

$$u_x(\gamma(\alpha, t), t) \sim \frac{C}{\mathcal{J}(\overline{\alpha}, t)^{\lambda - 1}} \left(\frac{\mathcal{J}(\overline{\alpha}, t)}{\mathcal{J}(\alpha, t)} - \frac{C_4}{C_3} \right) \tag{3.1.19}$$

for $\eta_* - \eta > 0$ small. But ([24])

$$\Gamma(y+1) = y\,\Gamma(y), \qquad y \in \mathbb{R}^+,$$

so that

$$\frac{C_4}{C_3} = \frac{\Gamma\left(\frac{1}{\lambda}\right)\,\Gamma\left(\frac{1}{\lambda}-\frac{1}{2}+1\right)}{\Gamma\left(\frac{1}{\lambda}+1\right)\,\Gamma\left(\frac{1}{\lambda}-\frac{1}{2}\right)} = 1-\frac{\lambda}{2} \in [1/2,1) \tag{3.1.20}$$

for $\lambda \in (0,1]$. Then, by (3.1.19), (3.0.4)i) and the definition of M_0

$$\begin{aligned} M(t) &\to 0^+, \qquad \alpha = \overline{\alpha}, \\ u_x(\gamma(\alpha,t),t) &\to 0^-, \qquad \alpha \neq \overline{\alpha} \end{aligned} \tag{3.1.21}$$

as $\eta \uparrow \eta_*$ for all $\lambda \in (0,1)$. For the threshold parameter $\lambda_* = 1$, we keep track of the positive constant C prior to (3.1.19) and find that, for $\alpha = \overline{\alpha}$,

$$M(t) \to -\frac{u_0'''(\overline{\alpha})}{(2\pi)^2} > 0 \tag{3.1.22}$$

as $\eta \uparrow \frac{1}{M_0}$, whereas

$$u_x(\gamma(\alpha,t),t) \to \frac{u_0'''(\overline{\alpha})}{(2\pi)^2} < 0 \tag{3.1.23}$$

for $\alpha \neq \overline{\alpha}$. Finally, from (2.1.16)

$$dt = \bar{\mathcal{K}}_0(t)^{2\lambda} d\eta, \tag{3.1.24}$$

then (3.1.14) implies

$$t_* - t \sim C \int_{\eta}^{\eta_*} (1-\lambda\mu M_0)^{\lambda-2} d\mu. \tag{3.1.25}$$

As a result, $t_* = +\infty$ for all $\lambda \in (0,1]$. See §4.1.1 for examples.

Two-sided, everywhere blow-up for $\lambda \in (1,+\infty)$

For $\lambda \in (1,+\infty)\backslash\{2\}$, set $b = \frac{1}{\lambda}$ and $q = 2$ in Lemma 3.0.11 to obtain

$$\int_{\overline{\alpha}-s}^{\overline{\alpha}+s} \frac{d\alpha}{(\epsilon - C_1(\alpha-\overline{\alpha})^2)^{\frac{1}{\lambda}}} = 2s\epsilon^{-\frac{1}{\lambda}}\,{}_2F_1\left[\frac{1}{2},\frac{1}{\lambda};\frac{3}{2};\frac{s^2C_1}{\epsilon}\right] \tag{3.1.26}$$

where the above series is defined by (3.0.7) as long as $\epsilon \geq -C_1 \geq -s^2C_1 > 0$, namely $-1 \leq \frac{s^2C_1}{\epsilon} < 0$. However, we are ultimately interested in the behaviour of (3.1.26) for $\epsilon > 0$ arbitrarily small, so that, eventually $\frac{s^2C_1}{\epsilon} < -1$. To achieve this transition of the series argument across -1 in a well-defined, continuous fashion, we use Lemma 3.0.9 which provides us with the analytic continuation of the series in (3.1.26) from argument values inside the unit circle, particularly $-1 \leq \frac{s^2C_1}{\epsilon} < 0$, to those found outside, and thus for $\frac{s^2C_1}{\epsilon} < -1$. Consequently, for ϵ small enough so that $-s^2C_1 > \epsilon > 0$, proposition 3.0.9 implies

$$2s\epsilon^{-\frac{1}{\lambda}}\,{}_2F_1\left[\frac{1}{2},\frac{1}{\lambda};\frac{3}{2};\frac{s^2C_1}{\epsilon}\right] = C\,\Gamma\left(\frac{1}{\lambda}-\frac{1}{2}\right)\epsilon^{\frac{1}{2}-\frac{1}{\lambda}} + \frac{C}{\lambda-2} + \psi(\epsilon) \tag{3.1.27}$$

for $\psi(\epsilon) = o(1)$ as $\epsilon \to 0$ and positive constant C which may depend on λ and can be obtained explicitly from (3.0.10). Then, substituting $\epsilon = \frac{1}{\lambda\eta} - M_0$ into (3.1.27) and using (3.1.10) along with (3.1.26), yields

$$\bar{\mathcal{K}}_0(t) \sim \begin{cases} C_3\mathcal{J}(\overline{\alpha},t)^{\frac{1}{2}-\frac{1}{\lambda}}, & \lambda \in (1,2), \\ C, & \lambda \in (2,+\infty) \end{cases} \tag{3.1.28}$$

for $\eta_* - \eta > 0$ small and positive constants C_3 given by (3.1.15) for $\lambda \in (1,2)$. Similarly, by following an identical argument, with $b = 1 + \frac{1}{\lambda}$ instead, we find that estimate (3.1.17), derived initially for $\lambda \in (0,1]$, holds for $\lambda \in (1,+\infty)$ as well. First suppose $\lambda \in (1,2)$, then (2.1.19), (3.1.17) and (3.1.28)i) imply estimate (3.1.19). However, by (3.1.20) we now have

$$\frac{C_4}{C_3} = 1 - \frac{\lambda}{2} \in (0,1/2)$$

for $\lambda \in (1,2)$. As a result, setting $\alpha = \overline{\alpha}$ in (3.1.19), we obtain

$$M(t) \sim \frac{C}{\mathcal{J}(\overline{\alpha},t)^{\lambda-1}} \to +\infty \tag{3.1.29}$$

as $\eta \uparrow \eta_*$. On the other hand, if $\alpha \neq \overline{\alpha}$, the definition of M_0 gives

$$u_x(\gamma(\alpha,t),t) \sim -\frac{C}{\mathcal{J}(\overline{\alpha},t)^{\lambda-1}} \to -\infty. \tag{3.1.30}$$

The existence of a finite $t_* > 0$ follows from (3.1.24) and (3.1.28)i), which imply

$$t_* - t \sim C(\eta_* - \eta)^{\lambda-1}. \tag{3.1.31}$$

For $\lambda \in (2,+\infty)$, we use (2.1.19), (3.1.17) and (3.1.28)ii) to get

$$u_x(\gamma(\alpha,t),t) \sim \frac{C}{\mathcal{J}(\overline{\alpha},t)}\left(\frac{\mathcal{J}(\overline{\alpha},t)}{\mathcal{J}(\alpha,t)} - C\mathcal{J}(\overline{\alpha},t)^{\frac{1}{2}-\frac{1}{\lambda}}\right). \tag{3.1.32}$$

Then, setting $\alpha = \overline{\alpha}$ in (3.1.32), we obtain

$$M(t) \sim \frac{C}{\mathcal{J}(\overline{\alpha},t)} \to +\infty \tag{3.1.33}$$

as $\eta \uparrow \eta_*$. Similarly, for $\alpha \neq \overline{\alpha}$,

$$u_x(\gamma(\alpha,t),t) \sim -\frac{C}{\mathcal{J}(\overline{\alpha},t)^{\frac{1}{2}+\frac{1}{\lambda}}} \to -\infty. \tag{3.1.34}$$

A finite blow-up time $t_* > 0$ follows from (3.1.24) and (3.1.28)ii), which yield

$$t_* - t \sim C(\eta_* - \eta).$$

For the case $\lambda = 2$ and $\eta_* - \eta = \frac{1}{2M_0} - \eta > 0$ small, we have

$$\bar{\mathcal{K}}_0(t) \sim -C\ln\left(\mathcal{J}(\overline{\alpha},t)\right), \qquad \bar{\mathcal{K}}_1(t) \sim \frac{C}{\mathcal{J}(\overline{\alpha},t)}. \tag{3.1.35}$$

Two-sided blow-up for $\lambda = 2$ then follows from (2.1.19), (3.1.24) and (3.1.35). Finally, the behaviour of the jacobian in (3.1.9) is deduced from (2.1.14) and the estimates (3.1.14), (3.1.28) and (3.1.35). See section 4.1.1 for examples. □

Remark 3.1.36. As discussed in section 1.2.3, several methods were used in [9] to show that there are stagnation point-form blow-up solutions to the 2D incompressible Euler equations ($\lambda = 1$) under Dirichlet boundary conditions. We remark that these do not conflict with our global result in part 1 of Theorem 3.1.8 as long as the data is smooth and, under certain circumstances, its local behaviour near the endpoints $\alpha = \{0, 1\}$ allows for a smooth, periodic extension of u_0' to all $\alpha \in \mathbb{R}$. We will return to this issue in section 3.3. Also in that section, we will show that if u_0' behaves linearly, instead of quadratically, near $\overline{\alpha}_i$ then finite-time blow-up occurs for all $\lambda > 1/2$, whereas, global existence in time follows if $\lambda \in [0, 1/2]$. In particular, this will provide us with blow-up criteria for the 2D Euler case and allow for a better understanding of the role played by the corresponding set of boundary conditions in the breakdown of solutions that arise from smooth data.

3.1.2 Blow-up for $\lambda < -1$

Theorem 3.1.37 below proves the existence of smooth data and a finite $t_* > 0$ such that u_x undergoes a two-sided, everywhere blow-up for $\lambda \leq -2$. If instead $\lambda \in (-2, -1)$, we show that only the minimum diverges, $m(t) \to -\infty$, at a finite number of locations in the domain. We will refer to this last type of blow-up as "one-sided, discrete blow-up".

Theorem 3.1.37. *Consider the initial boundary value problem (1.1.1)-(1.1.2) or (1.1.3). There exist smooth initial data such that:*

1. *For $\lambda \leq -2$, there is a finite $t_* > 0$ such that both the maximum $M(t)$ and the minimum $m(t)$ diverge to $+\infty$ and respectively to $-\infty$ as $t \uparrow t_*$. Also, $\lim_{t\uparrow t_*} u_x(\gamma(\alpha, t), t) = +\infty$ for $\alpha \notin \{\overline{\alpha}_i, \underline{\alpha}_j\}$ (two-sided, everywhere blow-up).*

2. *For $\lambda \in (-2, -1)$, there exists a finite $t_* > 0$ such that only the minimum diverges, $m(t) \to -\infty$, as $t \uparrow t_*$ (one-sided, discrete blow-up).*

3. *For $\lambda < -1$, suppose only Dirichlet boundary conditions are considered and/or u_0 is odd about the midpoint. Then, for every $\underline{\alpha}_j \in [0, 1]$ there exists a unique $\underline{x}_j \in [0, 1]$ given by (3.1.2) such that $\lim_{t\uparrow t_*} u_x(\underline{x}_j, t) = -\infty$.*

Finally, for $\lambda < -1$ and $t_* > 0$ as above, the jacobian (2.1.14) satisfies

$$\lim_{t\uparrow t_*} \gamma_\alpha(\alpha, t) = \begin{cases} 0, & \alpha = \underline{\alpha}_j, \\ C, & \alpha \neq \underline{\alpha}_j \end{cases} \tag{3.1.38}$$

where the positive constants C depend on the choice of λ and $\alpha \neq \underline{\alpha}_j$.

Proof. For $\lambda < -1$ and

$$\eta_* = \frac{1}{\lambda m_0},$$

smoothness of u_0' implies that

$$\bar{\mathcal{K}}_0(t) = \int_0^1 \mathcal{J}(\alpha, t)^{\frac{1}{|\lambda|}} d\alpha, \qquad \bar{\mathcal{K}}_0(0) = 1$$

remains finite, and positive, for all $\eta \in [0, \eta_*]$. Indeed, suppose there is an earliest $t_1 > 0$ such that $\eta_1 = \eta(t_1) > 0$ and

$$\bar{\mathcal{K}}_0(t_1) = \int_0^1 (1 - \lambda\eta_1 u_0'(\alpha))^{\frac{1}{|\lambda|}} d\alpha = 0.$$

Since

$$\int_0^1 \left(1 + \frac{u_0'(\alpha)}{|m_0|}\right)^{\frac{1}{|\lambda|}} d\alpha > 0,$$

then $\eta_1 \neq \eta_*$. Also, by periodicity of the data, there are $[0, 1] \ni \alpha' \neq \underline{\alpha}_j$ where

$$(1 - \lambda\eta_1 u_0'(\alpha'))^{\frac{1}{|\lambda|}} = 1.$$

As a result, $\bar{\mathcal{K}}_0(t_1) = 0$ implies the existence of at least one $[0,1] \ni \alpha'' \neq \underline{\alpha}_j$ where $u_0'(\alpha'') = \frac{1}{\lambda\eta_1}$. But $u_0'(\alpha) \geq m_0$ and $\eta_* = \frac{1}{\lambda m_0}$, then

$$\eta_* < \eta_1. \tag{3.1.39}$$

In fact, (3.1.39) and $m_0 \leq u_0'(\alpha) \leq M_0$ yield

$$0 < \int_0^1 \left(1 + \frac{u_0'(\alpha)}{|m_0|}\right)^{\frac{1}{|\lambda|}} d\alpha \leq \bar{\mathcal{K}}_0(t) \leq 1 \tag{3.1.40}$$

for all $\eta \in [0, \eta_*]$. Next, for $\lambda < -1$, we need to examine the behaviour of

$$\bar{\mathcal{K}}_1(t) = \int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda}}}$$

as $\eta \uparrow \eta_*$. We will do so by following an argument analogous to the one used in the derivation of (3.1.28) in the previous section. For simplicity, we will assume that m_0 occurs at a single location $\underline{\alpha} \in (0,1)$.[6] Also, we consider smooth functions $u_0'(\alpha)$ with non-vanishing second order derivative near $\underline{\alpha}$, so that, locally

$$u_0'(\alpha) \sim m_0 + C_2(\alpha - \underline{\alpha})^2$$

for $0 \leq |\alpha - \underline{\alpha}| \leq r$, $0 < r \leq 1$ and $C_2 = u_0'''(\underline{\alpha})/2 > 0$. Then, for arbitrary $\epsilon > 0$,

$$\epsilon + u_0'(\alpha) - m_0 \sim \epsilon + C_2(\alpha - \underline{\alpha})^2. \tag{3.1.41}$$

Given $\lambda < -1$, set $b = 1 + \frac{1}{\lambda}$ and $q = 2$ into (3.0.12) of Lemma 3.0.11, to find

$$\int_{\underline{\alpha}-r}^{\underline{\alpha}+r} \frac{d\alpha}{(\epsilon + C_2(\alpha - \underline{\alpha})^2)^{1+\frac{1}{\lambda}}} = \frac{2r}{\epsilon^{1+\frac{1}{\lambda}}}\, {}_2F_1\left[\frac{1}{2}, 1 + \frac{1}{\lambda}; \frac{3}{2}; -\frac{r^2 C_2}{\epsilon}\right] \tag{3.1.42}$$

for $\epsilon \geq C_2 \geq r^2 C_2 > 0$ and $\lambda \in (-\infty, -1)\backslash\{-2\}$.[7] Now, if we let $\epsilon > 0$ become small enough, so that eventually $-\frac{r^2 C_2}{\epsilon} < -1$, we may use Lemma 3.0.9 to obtain a continuous, well-defined transition of the series argument, $-\frac{r^2 C_2}{\epsilon}$, across -1. We find

$$\frac{2r}{\epsilon^{1+\frac{1}{\lambda}}}\, {}_2F_1\left[\frac{1}{2}, 1 + \frac{1}{\lambda}; \frac{3}{2}; -\frac{r^2 C_2}{\epsilon}\right] = \frac{C}{\lambda+2} + \frac{C\,\Gamma\left(\frac{1}{2} + \frac{1}{\lambda}\right)}{\epsilon^{\frac{1}{2}+\frac{1}{\lambda}}} + \xi(\epsilon) \tag{3.1.43}$$

for $\xi(\epsilon) = o(1)$ as $\epsilon \to 0$ and positive constants C which may depend on the choice of λ and can be obtained explicitly from (3.0.10). Accordingly, we use (3.1.41), (3.1.42) and (3.1.43) to obtain

$$\int_{\underline{\alpha}-r}^{\underline{\alpha}+r} \frac{d\alpha}{(\epsilon + u_0'(\alpha) - m_0)^{1+\frac{1}{\lambda}}} \sim \frac{C}{\lambda+2} + \frac{C\,\Gamma\left(\frac{1}{2} + \frac{1}{\lambda}\right)}{\epsilon^{\frac{1}{2}+\frac{1}{\lambda}}} \tag{3.1.44}$$

[6] The case of finitely many $\underline{\alpha}_j \in [0,1]$ follows similarly.
[7] The case $\lambda = -2$ is treated separately.

for small $\epsilon > 0$. Finally, setting $\epsilon = m_0 - \frac{1}{\lambda\eta}$ implies

$$\bar{\mathcal{K}}_1(t) \sim \begin{cases} C, & \lambda \in (-2,-1), \\ C_5 \mathcal{J}(\underline{\alpha},t)^{-\left(\frac{1}{2}+\frac{1}{\lambda}\right)}, & \lambda < -2 \end{cases} \tag{3.1.45}$$

for $\eta_* - \eta > 0$ small, $\eta_* = \frac{1}{\lambda m_0}$, $\mathcal{J}(\underline{\alpha},t) = 1 - \lambda\eta(t)m_0$ and

$$C_5 = \frac{\Gamma\left(\frac{1}{2}+\frac{1}{\lambda}\right)}{\Gamma\left(1+\frac{1}{\lambda}\right)} \sqrt{-\frac{\pi m_0}{C_2}} > 0, \qquad \lambda < -2. \tag{3.1.46}$$

Setting $\alpha = \underline{\alpha}$ in (2.1.19) and using (3.0.5), (3.1.40) and (3.1.45), we find that

$$m(t) \sim -\frac{C}{\mathcal{J}(\underline{\alpha},t)} \to -\infty \tag{3.1.47}$$

as $\eta \uparrow \eta_*$ for all $\lambda \in (-\infty,-1)\backslash\{-2\}$, a one-sided, discrete blow-up.

On the other hand, (2.1.19), (3.1.40), (3.1.45), and the definition of m_0, imply that for $\alpha \neq \underline{\alpha}$,

$$\begin{cases} |u_x(\gamma(\alpha,t),t)| < +\infty, & \lambda \in (-2,-1), \\ u_x(\gamma(\alpha,t),t) \sim C\mathcal{J}(\underline{\alpha},t)^{-\left(\frac{1}{2}+\frac{1}{\lambda}\right)} \to +\infty, & \lambda < -2 \end{cases} \tag{3.1.48}$$

as $\eta \uparrow \eta_*$. A one-sided, discrete blow-up for $\lambda \in (-2,-1)$ follows from (3.1.47) and (3.1.48)i), whereas a two-sided, everywhere blow-up for $\lambda < -2$ results from (3.1.47) and (3.1.48)ii). The existence of a finite blow-up time $t_* > 0$ and formula (3.1.2) follow from (2.1.16) and (2.2.1), respectively, along with (3.1.40) as $\eta \uparrow \eta_*$. Particularly, we have the lower bound

$$\eta_* \leq t_*. \tag{3.1.49}$$

The case $\lambda = -2$ can be treated directly. We find

$$\bar{\mathcal{K}}_1(t) \sim -C \ln\left(\mathcal{J}(\underline{\alpha},t)\right) \tag{3.1.50}$$

for $\eta_* - \eta > 0$ small. A two-sided, everywhere blow-up then follows as above. Finally, (3.1.38) is deduced from (2.1.14) and (3.1.40). See section 4.1.1 for examples. □

3.1.3 One-sided, discrete blow-up for $\lambda \in [-1,0)$

Theorem 3.1.51 below will extend the one-sided, discrete blow-up found in Theorem 3.1.37 for parameters $\lambda \in (-2,-1)$ to all $\lambda \in (-2,0)$. It is also valid for arbitrary smooth initial data.

Theorem 3.1.51. *Consider the initial boundary value problem (1.1.1)-(1.1.2) or (1.1.3) for arbitrary smooth initial data. If $\lambda \in [-1, 0)$, there exists a finite $t_* > 0$ such that only the minimum diverges, $m(t) \to -\infty$, as $t \uparrow t_*$ (one-sided, discrete blow-up). Also, if only the Dirichlet setting (1.1.3) is considered and/or u_0 is odd about the midpoint, then formula (3.1.2) gives the corresponding blow-up locations in the Eulerian variable $x \in [0, 1]$. Finally, the jacobian (2.1.14) satisfies*

$$\lim_{t\uparrow t_*} \gamma_\alpha(\alpha, t) = \begin{cases} 0, & \alpha = \underline{\alpha}_j, \\ C, & \alpha \neq \underline{\alpha}_j \end{cases} \tag{3.1.52}$$

where the positive constants C depend on the choice of λ and $\alpha \neq \underline{\alpha}_j$.

Proof. Since u_0' is smooth and $\lambda \in [-1, 0)$, both integrals $\bar{\mathcal{K}}_i(t)$, $i = 0, 1$ remain finite (and positive) for all $\eta \in [0, \eta_*)$, $\eta_* = \frac{1}{\lambda m_0}$. Also, $\bar{\mathcal{K}}_0(t)$ does not vanish as $\eta \uparrow \eta_*$. In fact

$$1 \leq \bar{\mathcal{K}}_0(t) \leq \left(1 + \frac{M_0}{|m_0|}\right)^{\frac{1}{|\lambda|}} \tag{3.1.53}$$

for all $\eta \in [0, \eta_*]$. Indeed, notice that $\dot{\bar{\mathcal{K}}}_0(0) = 0$ and

$$\ddot{\bar{\mathcal{K}}}_0(t) = \left((1+\lambda)\int_0^1 \frac{u_0'(\alpha)^2 d\alpha}{\mathcal{J}(\alpha, t)^{2+\frac{1}{\lambda}}} - 2\lambda \left(\int_0^1 \frac{u_0'(\alpha)\, d\alpha}{\mathcal{J}(\alpha, t)^{1+\frac{1}{\lambda}}}\right)^2\right) \bar{\mathcal{K}}_0(t)^{-4\lambda} > 0$$

for $\lambda \in [-1, 0)$ and $\eta \in (0, \eta_*)$. This implies

$$\dot{\bar{\mathcal{K}}}_0(t) = \bar{\mathcal{K}}_0(t)^{-2\lambda} \int_0^1 \frac{u_0'(\alpha) d\alpha}{\mathcal{J}(\alpha, t)^{1+\frac{1}{\lambda}}} > 0. \tag{3.1.54}$$

Then, (3.1.54), $\bar{\mathcal{K}}_0(0) = 1$ and $m_0 \leq u_0'(\alpha) \leq M_0$ yield (3.1.53). Similarly, one can show that

$$1 \leq \bar{\mathcal{K}}_1(t) \leq \left(\frac{|m_0|}{M_0 + |m_0|}\right)^{1+\frac{1}{\lambda}}. \tag{3.1.55}$$

Consequently, (2.1.19), (3.0.5), (3.1.53), and (3.1.55) imply that

$$m(t) \to -\infty$$

as $\eta \uparrow \eta_*$. On the other hand, by (3.1.53), (3.1.55) and the definition of m_0, we find that $u_x(\gamma(\alpha, t), t)$ remains bounded for all $\alpha \neq \underline{\alpha}_j$ as $\eta \uparrow \eta_*$. The existence of a finite blow-up time

$t_* > 0$ and formula (3.1.2) follow from (2.1.16) and (2.2.1), respectively, along with (3.1.53). Although t_* can be computed explicitly from (2.1.18), (3.1.53) provides the simple estimate[8]

$$\eta_* \left(\frac{m_0}{m_0 - M_0}\right)^2 \leq t_* \leq \eta_*. \tag{3.1.56}$$

Also, since the maximum $M(t)$ remains finite as $t \uparrow t_*$, setting $\alpha = \overline{\alpha}$ in (2.1.19) and using (3.0.4) and (2.0.3) gives $\dot{M}(t) < \lambda(M(t))^2 < 0$, which implies

$$0 < M(t) \leq M_0$$

for all $t \in [0, t_*]$ and $\lambda \in [-1, 0)$. Finally, (3.1.52) follows directly from (2.1.14), (3.1.53) and the definition of m_0. See section 4.1.1 for examples. □

This concludes the proof of Theorem 3.1.1 and Corollary 3.1.5.

3.1.4 Further L^p Regularity

In this section, we prove Theorem 3.1.7. Particularly, we will find that the two-sided, everywhere blow-up (or one-sided, discrete blow-up) from Theorem 3.1.1, can be associated with stronger (or weaker) L^p regularity.

Before proving the Theorem, we use (2.1.14) and (2.1.19) to derive basic upper and lower bounds for the $L^p(0,1)$ norm

$$\|u_x(\cdot,t)\|_p = \left(\int_0^1 (u_x(\gamma(\alpha,t),t))^p \gamma_\alpha(\alpha,t)\, d\alpha\right)^{1/p}, \quad p \in [1, +\infty), \tag{3.1.57}$$

as well as write down explicit formulas for the energy function $E(t) = \|u_x(\cdot,t)\|_2^2$, its time derivative $\dot{E}(t)$, and estimate the blow-up rates of relevant time-dependent integrals.

First of all, let $t_* > 0$ be as in Theorem 3.1.1, namely, for parameters $\lambda \in (-\infty, 0) \cup (1, +\infty)$, $t_* > 0$ denotes the *finite*, L^∞ blow-up time for u_x, otherwise $t_* = +\infty$. From (2.1.14) and (2.1.19),

$$|u_x(\gamma(\alpha,t),t)|^p\, \gamma_\alpha(\alpha,t) = \frac{|f(\alpha,t)|^p}{|\lambda\eta(t)|^p\, \bar{\mathcal{K}}_0(t)^{1+2\lambda p}} \tag{3.1.58}$$

for $t \in [0, t_*)$, $p \in [1, +\infty)$, $\lambda \neq 0$ and

$$f(\alpha,t) = \frac{1}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda p}}} - \frac{\bar{\mathcal{K}}_1(t)}{\bar{\mathcal{K}}_0(t)\mathcal{J}(\alpha,t)^{\frac{1}{\lambda p}}}.$$

[8]Which we may contrast to (3.1.49). Notice that (2.1.18) implies that the two cases coincide ($t_* = \eta_*$) in the case of Burgers' equation $\lambda = -1$.

Integrating (3.1.58) in α and using periodicity, or the Dirichlet boundary conditions, then gives

$$\|u_x(\cdot,t)\|_p^p = \frac{1}{|\lambda\eta(t)|^p\,\bar{\mathcal{K}}_0(t)^{1+2\lambda p}}\int_0^1 |f(\alpha,t)|^p\,d\alpha. \tag{3.1.59}$$

In particular, setting $p=2$ yields, after simplification, the following formula for the energy $E(t)$:

$$E(t) = \left(\lambda\eta(t)\bar{\mathcal{K}}_0(t)^{1+2\lambda}\right)^{-2}\left(\bar{\mathcal{K}}_0(t)\bar{\mathcal{K}}_2(t) - \bar{\mathcal{K}}_1(t)^2\right). \tag{3.1.60}$$

Furthermore, multiplying (1.1.1)i) by u_x, integrating by parts and using either (1.1.2) or (1.1.3), along with (2.1.14) and (2.1.19), gives

$$\begin{aligned}\dot{E}(t) &= (1+2\lambda)\int_0^1 (u_x(x,t))^3 dx\\ &= (1+2\lambda)\int_0^1 (u_x(\gamma(\alpha,t),t))^3\gamma_\alpha(\alpha,t)\,d\alpha\\ &= \frac{1+2\lambda}{(\lambda\eta(t))^3}\left[\frac{\bar{\mathcal{K}}_3(t)}{\bar{\mathcal{K}}_1(t)} - \frac{3\bar{\mathcal{K}}_2(t)}{\bar{\mathcal{K}}_0(t)} + 2\left(\frac{\bar{\mathcal{K}}_1(t)}{\bar{\mathcal{K}}_0(t)}\right)^2\right]\frac{\bar{\mathcal{K}}_1(t)}{\bar{\mathcal{K}}_0(t)^{6\lambda+1}}.\end{aligned} \tag{3.1.61}$$

Now, $\bar{\mathcal{K}}_i(t)$ and $\mathcal{J}(\alpha,t)$ stay positive and bounded for all $\alpha\in[0,1]$ and $\eta\in[0,\eta_*)$ (i.e. for $t\in[0,t_*)$), as a result

$$|f(\alpha,t)|^p \le 2^{p-1}\left(\frac{1}{\mathcal{J}(\alpha,t)^{p+\frac{1}{\lambda}}} + \frac{\bar{\mathcal{K}}_1(t)^p}{\bar{\mathcal{K}}_0(t)^p\,\mathcal{J}(\alpha,t)^{\frac{1}{\lambda}}}\right), \tag{3.1.62}$$

where we used the simple inequality (see appendix D)

$$(a+b)^p \le 2^{p-1}(a^p+b^p)$$

valid for $p\ge 1$ and non-negative numbers a and b. Then, we integrate (3.1.62) in space and use (3.1.58) to obtain the upper bound

$$\|u_x(\cdot,t)\|_p^p \le \frac{2^{p-1}}{|\lambda\eta(t)|^p\,\bar{\mathcal{K}}_0(t)^{1+2\lambda p}}\left(\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{p+\frac{1}{\lambda}}} + \frac{\bar{\mathcal{K}}_1(t)^p}{\bar{\mathcal{K}}_0(t)^{p-1}}\right) \tag{3.1.63}$$

for $t\in[0,t_*)$, $p\in[1,+\infty)$ and $\lambda\ne 0$.

For a lower bound, notice that by Jensen's inequality (see appendix D, [20]),

$$\int_0^1 |f(\alpha,t)|^p\,d\alpha \ge \left|\int_0^1 f(\alpha,t)d\alpha\right|^p$$

for $p \in [1, +\infty)$. Using the above on (3.1.59), we find

$$\|u_x(\cdot,t)\|_p \geq \frac{1}{|\lambda\eta(t)|\,\bar{\mathcal{K}}_0(t)^{2\lambda+\frac{1}{p}}}\left|\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda p}}} - \frac{\bar{\mathcal{K}}_1(t)}{\bar{\mathcal{K}}_0(t)}\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{\frac{1}{\lambda p}}}\right|. \tag{3.1.64}$$

Although the right-hand side of (3.1.64) is identically zero for $p = 1$, it does allow for the study of L^p regularity of solutions when $p \in (1, +\infty)$.[9]

Next, we need to determine any blow-up rates for the appropriate integrals in (3.1.60)-(3.1.64). By following the argument in Theorems 3.1.8 and 3.1.37, we go through the derivation of estimates for the term

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda p}}}$$

with $\lambda > 1$ and $p \geq 1$, whereas those for

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{\frac{1}{\lambda p}}}, \qquad \int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{p+\frac{1}{\lambda}}}$$

follow similarly and will be simply stated here.

For simplicity, assume u_0' attains its maximum value $M_0 > 0$ at a single location $\overline{\alpha} \in (0,1)$. As before, we consider the case where, near $\overline{\alpha}$, u_0' has non-vanishing second order derivative. Accordingly, there is $0 < s \leq 1$ small enough such that, by a simple Taylor expansion,

$$u_0'(\alpha) \sim M_0 + C_1(\alpha - \overline{\alpha})^2$$

for $0 \leq |\alpha - \overline{\alpha}| \leq s$ and $C_1 = u_0'''(\overline{\alpha})/2 < 0$. Then

$$\epsilon - u_0'(\alpha) + M_0 \sim \epsilon - C_1(\alpha - \overline{\alpha})^2$$

for $\epsilon > 0$. Given $\lambda > 1$ and $p \geq 1$, we let $b = 1 + \frac{1}{\lambda p}$ and $q = 2$ in Lemma 3.0.11 to obtain

$$\int_{\overline{\alpha}-s}^{\overline{\alpha}+s} \frac{d\alpha}{(\epsilon - u_0'(\alpha) + M_0)^b} \sim \int_{\overline{\alpha}-s}^{\overline{\alpha}+s} \frac{d\alpha}{(\epsilon - C_1(\alpha - \overline{\alpha})^2)^b} = \frac{2s}{\epsilon^b}\,{}_2F_1\left[\frac{1}{2}, b; \frac{3}{2}; \frac{C_1 s^2}{\epsilon}\right] \tag{3.1.65}$$

for $\epsilon \geq -C_1 \geq -C_1 s^2 > 0$. Now, letting $\epsilon > 0$ become small enough, so that eventually $\frac{C_1 s^2}{\epsilon} < -1$, Lemma 3.0.9 implies

$$\frac{2s}{\epsilon^b}\,{}_2F_1\left[\frac{1}{2}, b; \frac{3}{2}; \frac{C_1 s^2}{\epsilon}\right] = \frac{2s}{(1-2b)(-s^2 C_1)^b} + \frac{\Gamma\left(b - \frac{1}{2}\right)}{\Gamma(b)}\sqrt{-\frac{\pi}{C_1}}\,\epsilon^{\frac{1}{2}-b} + \zeta(\epsilon)$$

[9] Also, for $p \in (1, +\infty)$, (3.1.64) makes sense as $t \downarrow 0$ due to the periodicity of u_0, or its vanishing at the endpoints.

for $\lambda \neq 2/p$, and $\zeta(\epsilon) = o(1)$ as $\epsilon \to 0$. Using the above on (3.1.65) yields

$$\int_0^1 \frac{d\alpha}{(\epsilon - u_0'(\alpha) + M_0)^b} \sim \frac{\Gamma(b-1/2)}{\Gamma(b)}\sqrt{-\frac{\pi}{C_1}}\,\epsilon^{\frac{1}{2}-b} \tag{3.1.66}$$

for $\epsilon > 0$ small. Then, setting $\epsilon = \frac{1}{\lambda\eta} - M_0$ into (3.1.66) gives

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda p}}} \sim C\mathcal{J}(\overline{\alpha},t)^{-(\frac{1}{2}+\frac{1}{\lambda p})} \tag{3.1.67}$$

for $\eta_* - \eta > 0$ small, $\eta_* = \frac{1}{\lambda M_0}$, $\lambda > 1$, and $p \geq 1$.[10] For the other cases and remaining integrals, we follow a similar argument to find

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda p}}} \sim \frac{C}{\mathcal{J}(\underline{\alpha},t)^{\frac{1}{2}+\frac{1}{\lambda p}}}, \qquad \lambda < -\frac{2}{p}, \quad p \geq 1, \tag{3.1.68}$$

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{\frac{1}{\lambda p}}} \sim \begin{cases} C, & \lambda > \frac{2}{p}, \ \ p \geq 1 \ \ \text{or} \ \ \lambda < 0, \\ C\mathcal{J}(\overline{\alpha},t)^{\frac{1}{2}-\frac{1}{\lambda p}}, & 1 < \lambda < \frac{2}{p}, \quad 1 < p < 2 \end{cases} \tag{3.1.69}$$

and

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{p+\frac{1}{\lambda}}} \sim C, \qquad \frac{2}{1-2p} < \lambda < 0, \ \ p \geq 1 \tag{3.1.70}$$

where the positive constants C may depend on the choices for λ and p.

Recall from Theorem 3.1.1 (see also appendix A) that

$$\lim_{t\to+\infty} \|u_x\|_\infty < +\infty, \qquad \lambda \in [0,1]. \tag{3.1.71}$$

In contrast, we also showed that there is a finite $t_* > 0$ such that

$$\lim_{t\uparrow t_*} \|u_x\|_\infty = +\infty, \qquad \lambda \in \mathbb{R}\backslash[0,1]. \tag{3.1.72}$$

In the case of (3.1.72), Theorem 3.1.73 below further examines the L^p regularity of u_x as t approaches the finite L^∞ blow-up time t_*.

[10] For $\lambda = 2/p$, we have that $b = 3/2$ and (3.1.67) reduces to (3.1.35)ii).

Theorem 3.1.73. *Consider the initial boundary value problem (1.1.1)-(1.1.2) or (1.1.3) and let $t_* > 0$ denote the finite L^∞ blow-up time for u_x in Theorem 3.1.1. There exist smooth initial data such that:*

1. *For $p \in (1, +\infty)$ and $\lambda \in (-\infty, -2/p] \cup (1, +\infty)$, $\lim_{t\uparrow t_*} \|u_x\|_p = +\infty$.*
2. *For $p \in [1, +\infty)$ and $\frac{2}{1-2p} < \lambda < 0$, $\lim_{t\uparrow t_*} \|u_x\|_p < +\infty$.*
3. *The energy $E(t) = \|u_x\|_2^2$ diverges as $t \uparrow t_*$ if $\lambda \in (-\infty, -2/3] \cup (1, +\infty)$ but remains finite for $t \in [0, t_*]$ when $\lambda \in (-2/3, 0)$. Moreover, $\dot{E}(t)$ blows up to $+\infty$ as $t \uparrow t_*$ if $\lambda \in (-\infty, -1/2) \cup (1, +\infty)$ and $\dot{E}(t) \equiv 0$ for $\lambda = -1/2$; whereas, $\lim_{t\uparrow t_*} \dot{E}(t) = -\infty$ when $\lambda \in (-1/2, -2/5]$ but remains bounded for all $t \in [0, t_*]$ if $\lambda \in (-2/5, 0)$.*

Proof. Let C denote a positive constant which may depend on λ and p.

Case λ, $p \in (1, +\infty)$

First, consider the lower bound (3.1.64) for $p \in (1, 2)$ and $\lambda \in (1, 2/p)$. Then, $\lambda \in (1, 2)$ so that (3.1.28)i), (3.1.17), (3.1.67) and (3.1.69)ii) imply

$$\|u_x\|_p^p \geq \frac{\left|\int_0^1 f(\alpha, t)d\alpha\right|^p}{|\lambda\eta(t)|^p \bar{\mathcal{K}}_0(t)^{1+2\lambda p}} \sim C\mathcal{J}(\overline{\alpha}, t)^{\sigma(\lambda,p)}$$

for $\eta_* - \eta > 0$ small and $\sigma(\lambda, p) = \frac{3p}{2} - \frac{1}{2} - \lambda p$. By the above restrictions on λ and p, we have that $\sigma(\lambda, p) < 0$ for

$$\frac{1}{2}\left(3 - \frac{1}{p}\right) < \lambda < \frac{2}{p}, \qquad p \in (1, 5/3).$$

Then, by choosing $p - 1 > 0$ arbitrarily small,

$$\lim_{t\uparrow t_*} \|u_x(\cdot, t)\|_p = +\infty$$

for $\lambda \in (1, 2)$. Next, let $\lambda > 2$ and $p \in (1, +\infty)$. This means $\lambda > \frac{2}{p}$, and so, (3.1.28)ii), (3.1.17), (3.1.67) and (3.1.69)i) now yield

$$\|u_x\|_p \geq \frac{\left|\int_0^1 f(\alpha, t)d\alpha\right|}{|\lambda\eta(t)| \bar{\mathcal{K}}_0(t)^{2\lambda+\frac{1}{p}}} \sim \frac{C}{\mathcal{J}(\overline{\alpha}, t)^{\frac{1}{2}+\frac{1}{\lambda}}} \to +\infty \tag{3.1.74}$$

as $t \uparrow t_*$. This proves *1* of the Theorem for $\lambda > 1$.[11]

[11] If $\lambda = 2$, $\lambda > \frac{2}{p}$ for $p > 1$ and result follows from (3.1.35), (3.1.64), (3.1.67) and (3.1.69)i).

Case $\lambda < 0$ and $p \in [1, +\infty)$

For $\lambda < 0$, we keep in mind the estimates (3.1.40), (3.1.45)i), (3.1.53) and (3.1.55), which describe the behaviour of $\bar{\mathcal{K}}_i(t)$, $i = 0, 1$ as $\eta \uparrow \eta_*$.

Consider the upper bound (3.1.63) for $p \in [1, +\infty)$ and $\frac{2}{1-2p} < \lambda < 0$. Then $\lambda \in (-2, 0)$ so that (3.1.70), along with the aforementioned estimates, imply

$$\|u_x(\cdot,t)\|_p^p \leq \frac{2^{p-1}}{|\lambda\eta(t)|^p\, \bar{\mathcal{K}}_0(t)^{1+2\lambda p}} \left(\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{p+\frac{1}{\lambda}}} + \frac{\bar{\mathcal{K}}_1(t)^p}{\bar{\mathcal{K}}_0(t)^{p-1}} \right) \to C$$

as $t \uparrow t_*$. By the above, we conclude that

$$\lim_{t\uparrow t_*} \|u_x(\cdot,t)\|_p < +\infty$$

for $\frac{2}{1-2p} < \lambda < 0$ and $p \in [1, +\infty)$. Now, consider the lower bound (3.1.64) with $p \in (1, +\infty)$ and $-2 < \lambda < -\frac{2}{p} < \frac{2}{1-2p}$. Then, by (3.1.68), (3.1.69)i) and corresponding estimates on $\bar{\mathcal{K}}_i(t)$, $i = 0, 1$, we find that

$$\|u_x(\cdot,t)\|_p \geq \frac{\left|\int_0^1 f(\alpha,t)d\alpha\right|}{|\lambda\eta(t)|\, \bar{\mathcal{K}}_0(t)^{2\lambda+\frac{1}{p}}} \sim C\mathcal{J}(\underline{\alpha},t)^{-\left(\frac{1}{2}+\frac{1}{\lambda}\right)} \tag{3.1.75}$$

for $\eta_* - \eta > 0$ small. Therefore,

$$\lim_{t\uparrow t_*} \|u_x(\cdot,t)\|_p = +\infty \tag{3.1.76}$$

for $p \in (1, +\infty)$ and $\lambda \in (-2, -2/p]$.[12] Finally, let $\lambda < -2$ and $p \in (1, +\infty)$. Then $\lambda < -\frac{2}{p}$ and it is easy to check that (3.1.75), with different constants C, also holds. As a result, (3.1.76) follows for $p > 1$ and $\lambda \leq -2$.[13] This concludes the proof of statements *1* and *2*. For statement *3*, notice that when $p = 2$, *1* and *2*, as well as Theorem 3.1.1 imply that, as $t \uparrow t_*$, both $E(t) = \|u_x\|_2^2$ and $\dot{E}(t)$ diverge to $+\infty$ for $\lambda \in (-\infty, -1] \cup (1, +\infty)$ while $E(t)$ remains finite if $\lambda \in (-2/3, 1]$. Therefore we still have to establish the behaviour of $E(t)$ when $\lambda \in (-1, -2/3]$ and $\dot{E}(t)$ for $\lambda \in (-1, 0)\backslash\{-1/2\}$. However, from (3.1.53), (3.1.55) and (3.1.60), we see that, as $t \uparrow t_*$, any blow-up in $E(t)$ for $\lambda \in (-1, -2/3]$ must come from the $\bar{\mathcal{K}}_2(t)$ term. Using Lemmas 3.0.9 and 3.0.11, we estimate[14]

$$\bar{\mathcal{K}}_2(t) \sim \begin{cases} C\mathcal{J}(\underline{\alpha},t)^{-\left(\frac{3}{2}+\frac{1}{\lambda}\right)}, & \lambda \in (-1, -2/3), \\ -C\log\left(\mathcal{J}(\underline{\alpha},t)\right), & \lambda = -2/3, \\ C, & \lambda \in (-2/3, 0) \end{cases} \tag{3.1.77}$$

[12]For the case $\lambda = -\frac{2}{p}$ with $p \in (1, +\infty)$, we simply use (3.1.50) instead of (3.1.68).

[13]If $\lambda = -2$, $\lambda < -\frac{2}{p}$ for $p > 1$. Result follows as above with (3.1.50) instead of (3.1.68).

[14]Under the usual assumption $u_0'''(\underline{\alpha}) \neq 0$.

for $\eta_* - \eta > 0$ small. Then, (3.1.53), (3.1.55), and (3.1.60) imply that, as $t \uparrow t_*$, both $E(t)$ and $\dot{E}(t)$ blow-up to $+\infty$ for $\lambda \in (-1, -2/3]$.

Now, from (3.1.61)i),

$$\left|\dot{E}(t)\right| \leq |1 + 2\lambda| \, \|u_x\|_3^3 \tag{3.1.78}$$

so that (3.1.53), (3.1.55) and (3.1.61)iii) imply that

$$\lim_{t \uparrow t_*} \left|\dot{E}(t)\right| < +\infty$$

for $\lambda \in [-1/3, 1]$.[15] Moreover, by part *2*

$$\lim_{t \uparrow t_*} \|u_x\|_3 < +\infty$$

for $\lambda \in (-2/5, 0)$. Then, (3.1.78) implies that $\dot{E}(t)$ also remains finite for $\lambda \in (-2/5, -1/3)$.

Lastly, estimating $\bar{\mathcal{K}}_3(t)$ yields

$$\bar{\mathcal{K}}_3(t) \sim \begin{cases} C\mathcal{J}(\underline{\alpha}, t)^{-\left(\frac{5}{2} + \frac{1}{\lambda}\right)}, & \lambda \in (-2/3, -2/5), \\ -C \log\left(\mathcal{J}(\underline{\alpha}, t)\right), & \lambda = -2/5. \end{cases} \tag{3.1.79}$$

As a result, (3.1.53), (3.1.55), (3.1.77)iii) and (3.1.61)iii) imply that

$$\lim_{t \uparrow t_*} \dot{E}(t) = +\infty$$

for $\lambda \in (-2/3, -1/2)$ but

$$\lim_{t \uparrow t_*} \dot{E}(t) = -\infty$$

when $\lambda \in (-1/2, -2/5]$. This concludes the proof of the Theorem. We refer the reader to table 3.1 in section 3.1 for a summary of the above results. □

Remark 3.1.80. Theorem 3.1.73 implies that for every $p > 1$, L^p blow-up occurs for u_x if $\lambda \in \mathbb{R}\backslash(-2, 1]$, whereas for $\lambda \in (-2, 0)$, u_x remains in L^1 but blows up in particular, smaller L^p spaces. This suggests a weaker type of blow-up for the latter which certainly agrees with our L^∞ results where a "stronger", two-sided, everywhere blow-up takes place for $\lambda \in \mathbb{R}\backslash(-2, 1]$, but a "weaker", one-sided, discrete blow-up occurs when $\lambda \in (-2, 0)$.

Finally, in addition to the energy results notice that Theorem 3.1.73 and inequality (3.1.78) yield a complete description of the L^3 regularity for u_x:

$$\lim_{t \uparrow t_*} \|u_x(\cdot, t)\|_3 = \begin{cases} +\infty, & \lambda \in \mathbb{R}\backslash(-2/5, 1], \\ C, & \lambda \in (-2/5, 1] \end{cases} \tag{3.1.81}$$

[15] The result for $\lambda \in [0, 1]$ follows from Theorem 3.1.1 and appendix A

where the positive constant C depends on the choice of $\lambda \in (-2/5, 1]$.

Remark 3.1.82. For $V(t) = \int_0^1 u_x^3 dx$, the authors in [35] derived a finite upper bound

$$T^* = \left(\frac{3}{|1+2\lambda|\, E(0)}\right)^{\frac{1}{2}} \tag{3.1.83}$$

for the blow-up time of $E(t)$ for $\lambda < -1/2$ and

$$V(0) < 0, \qquad \frac{|1+2\lambda|}{2} V(0)^2 \geq \frac{2}{3} E(0)^3. \tag{3.1.84}$$

If (3.1.84)i) holds but we reverse (3.1.84)ii), then they proved that $\dot{E}(t)$ blows up instead. Now, from *3* in Theorem 3.1.73 we have that, in particular for $\lambda \in (-2/3, -1/2)$, $E(t)$ remains bounded for all $t \in [0, t_*]$ but $\dot{E}(t) \to +\infty$ as $t \uparrow t_*$. Here, $t_* > 0$ denotes the finite L^∞ blow-up time for u_x (see Theorem 3.1.51) and satisfies (3.1.56). Therefore, further discussion is required to clarify the apparent discrepancy between the two results for $\lambda \in (-2/3, -1/2)$ and u_0' satisfying both conditions in (3.1.84). Our claim is that for these values of λ,

$$t_* < T^*. \tag{3.1.85}$$

Specifically, $E(t)$ remains finite for all $t \in [0, t_*] \subset [0, T^*]$, while $\dot{E}(t) \to +\infty$ as $t \uparrow t_*$. From (3.1.61)i) and (3.1.84)ii), we have that

$$\frac{\dot{E}(0)^2}{2\,|1+2\lambda|} \geq \frac{2}{3} E(0)^3,$$

or equivalently

$$\frac{1}{(|1+2\lambda|\, E(0))^3} \geq \frac{4}{3(|1+2\lambda|\, \dot{E}(0))^2}.$$

As a result, (3.1.83) yields

$$T^* \geq \left(\frac{6}{|1+2\lambda|\, \dot{E}(0)}\right)^{\frac{1}{3}} \tag{3.1.86}$$

where we used $\dot{E}(0) > 0$; a consequence of (3.1.61)i), (3.1.84)i) and $\lambda \in (-2/3, -1/2)$. Now, for instance, suppose $0 < M_0 \leq |m_0|$.[16] Then

$$-V(0) = \left| \int_0^1 u_0'(x)^3 dx \right| \leq \max_{x \in [0,1]} |u_0'(x)|^3 = |m_0|^3, \tag{3.1.87}$$

[16] A natural case to consider given (3.1.84)i).

which we use on (3.1.61)i) to obtain $0 < \dot{E}(0) \leq |1+2\lambda|\,|m_0|^3$, or

$$\frac{6}{|1+2\lambda|\,\dot{E}(0)} \geq \frac{6}{|1+2\lambda|^2\,|m_0|^3}. \tag{3.1.88}$$

Consequently, (3.1.56), (3.1.86) and (3.1.88) yield

$$T^* \geq \left(\frac{6}{|1+2\lambda|^2\,|m_0|^3}\right)^{\frac{1}{3}} > \frac{1}{|1+2\lambda|^{\frac{2}{3}}\,|m_0|} > \frac{1}{|\lambda|\,|m_0|} = \eta_* \geq t_* \tag{3.1.89}$$

for $\lambda \in (-2/3, -1/2)$. If $\lambda \leq -2/3$, both results concerning L^2 blow-up of u_x coincide. Furthermore, in [10] the authors derived a finite upper bound

$$T_* = \frac{3}{(1+3\lambda)} V(0)^{-\frac{1}{3}}$$

for the blow-up time of $V(t)$ to negative infinity valid as long as $V(0) < 0$ and $\lambda < -1/3$. Clearly, T_* also serves as an upper bound for the breakdown of $\|u_x\|_3$ for $\lambda < -1/3$, or $\dot{E}(t) = (1+2\lambda)V((t)$ if $\lambda \in (-\infty, -1/3)\backslash\{-1/2\}$. However, (3.1.81) and *1* in Theorem 3.1.73 prove the existence of a finite $t_* > 0$ such that, particularly for $\lambda \in (-2/5, -1/3]$, $\|u_x\|_3$ remains finite for $t \in [0, t_*]$ while $\lim_{t\uparrow t_*} \|u_x\|_6 = +\infty$. This in turn implies the local boundedness of $\dot{E}(t)$ for $t \in [0, t_*]$ and $\lambda \in (-2/5, -1/3]$. Similar to the previous case, we claim that $t_* < T_*$. Here, once again, we consider the case $0 < M_0 \leq |m_0|$. Accordingly, (3.1.56) and (3.1.87) imply that

$$T_* = \frac{3}{(1+3\lambda)V(0)^{\frac{1}{3}}} \geq \frac{3}{|1+3\lambda|\,|m_0|} > \frac{1}{|\lambda|\,|m_0|} = \eta_* \geq t_*.$$

For the remaining values $\lambda \leq -2/5$, both our results and those established in [10] regarding blow-up of $V(t)$ agree. A simple example is given by

$$u_0'(x) = \sin(2\pi x) + \cos(4\pi x)$$

for which $V(0) = -3/4$, $E(0) = 1$, $m_0 = -2$ and $M_0 \sim 1.125$. Then, for $\lambda = -3/5 \in (-2/3, -1/2)$, we have that

$$T^* = \sqrt{15} > \eta_* = 5/6 \geq t_* \geq 0.34,$$

whereas, if $\lambda = -7/20 \in (-2/5, -1/3)$,

$$T_* = 20(6)^{2/3} > 10/7 = \eta_* \geq t_* \geq 0.59.$$

Remark 3.1.90. Global weak solutions to (1.1.1)i) having $I(t) = 0$ and $\lambda = -1/2$ have been studied by several authors, ([29], [3], [33]). Such solutions have also been constructed for $\lambda \in [-1/2, 0)$ ([10], c.f. also [11]) by extending an argument used in [3]. Notice that Theorems 3.1.1 and 3.1.73 imply the existence of smooth data and a finite $t_* > 0$ such that strong solutions to (1.1.1)-(1.1.2) with $\lambda \in (-2/3, 0)$ satisfy $\lim_{t\uparrow t_*} \|u_x\|_\infty = +\infty$ but $\lim_{t\uparrow t_*} E(t) < +\infty$. As a result, it is possible that the representation formulae derived in chapter 2 can lead to similar construction of global, weak solutions for $\lambda \in (-2/3, 0)$.

3.2 $n-$phase Piecewise Constant $u_0'(x)$

Up to this point, we have considered smooth data u_0' which attained its extreme values $M_0 > 0 > m_0$ at finitely many points $\overline{\alpha}_i$ and $\underline{\alpha}_j \in [0, 1]$, respectively, with u_0' having, relative to the sign of λ, quadratic local behaviour near these locations. In this section, we consider a class of functions which violates these assumptions, namely $u_0'(\alpha) \in PC_{\mathbb{R}}(0, 1)$, the class of mean-zero piecewise constant functions. Specifically, we will be concerned with the L^p regularity of solutions for $p \geq 1$.

Let $\chi_i(\alpha)$, $i = 1, ..., n$ denote the characteristic function for the intervals $\Omega_i = (\alpha_{i-1}, \alpha_i) \subset [0, 1]$ with $\alpha_0 = 0$, $\alpha_n = 1$ and $\Omega_j \cap \Omega_k = \varnothing$, $j \neq k$, i.e.

$$\chi_i(\alpha) = \begin{cases} 1, & \alpha \in \Omega_i, \\ 0, & \alpha \notin \Omega_i. \end{cases} \tag{3.2.1}$$

Then, for $h_i \in \mathbb{R}$, let $PC_{\mathbb{R}}(0, 1)$ denote the space of mean-zero, simple functions:

$$\left\{ g(\alpha) \in C^0(0, 1) \; a.e. \;\middle|\; g(\alpha) = \sum_{i=1}^{n} h_i \chi_i(\alpha) \text{ and } \sum_{i=1}^{n} h_i \mu(\Omega_i) = 0 \right\} \tag{3.2.2}$$

where $\mu(\Omega_i) = \alpha_i - \alpha_{i-1}$, the Lebesgue measure of Ω_i. Observe that for $u_0'(\alpha) \in PC_{\mathbb{R}}(0, 1)$ and $\lambda \neq 0$, (2.1.13), (3.2.1) and (3.2.2) imply that

$$\bar{\mathcal{K}}_i(t) = \sum_{j=1}^{n} (1 - \lambda \eta(t) h_j)^{-i-\frac{1}{\lambda}} \mu(\Omega_j). \tag{3.2.3}$$

We prove the following Theorem:

Theorem 3.2.4. *Consider the initial boundary value problem (1.1.1)-(1.1.2) or (1.1.3) for* $u_0'(\alpha) \in PC_{\mathbb{R}}(0,1)$. *Let* $T > 0$ *and assume solutions are defined for all* $t \in [0,T]$. *Then, the representation formula (2.1.19) implies that no global* $W^{1,\infty}(0,1)$ *solution can exist if* $T \geq t_*$, *where* $t_* = +\infty$ *for* $\lambda \geq 0$ *and* $0 < t_* < +\infty$ *otherwise. In addition,* $\lim_{t\uparrow t_*} \|u_x(\cdot,t)\|_1 = +\infty$ *if* $\lambda < -1$, *while*

$$\lim_{t\uparrow t_*} \|u_x(\cdot,t)\|_p = \begin{cases} C, & -\frac{1}{p} \leq \lambda < 0, \\ +\infty, & -1 \leq \lambda < -\frac{1}{p} \end{cases}$$

for $p \geq 1$ *and positive constants* C *that depend on the choice of* λ *and* p.

Proof. Let C denote a generic constant which may depend on λ and p. Since

$$u_0'(\alpha) = \sum_{i=1}^{n} h_i \chi_i(\alpha), \tag{3.2.5}$$

for $h_i \in \mathbb{R}$ as in (3.2.2), then (2.1.14) and (3.2.3) give

$$\gamma_\alpha(\alpha,t)^{-\lambda} = (1 - \lambda\eta(t)\sum_{i=1}^{n} h_i\chi_i(\alpha)) \left(\sum_{i=1}^{n}(1-\lambda\eta(t)h_i)^{-\frac{1}{\lambda}}\mu(\Omega_i)\right)^{\lambda} \tag{3.2.6}$$

for $\eta \in [0,\eta_*)$, η_* as defined in (3.0.6) and

$$\begin{cases} M_0 = \max_i h_i > 0, \\ m_0 = \min_i h_i < 0. \end{cases} \tag{3.2.7}$$

Let $\mathcal{I}_{max}$ and $\mathcal{I}_{min}$ denote the sets of indexes for the intervals $\overline{\Omega}_i$ and $\underline{\Omega}_i$, respectively, defined by

$$\overline{\Omega}_i \equiv \{\overline{\alpha} \in [0,1] \,|\, u_0'(\overline{\alpha}) = M_0\}, \qquad \underline{\Omega}_i \equiv \{\underline{\alpha} \in [0,1] \,|\, u_0'(\underline{\alpha}) = m_0\}. \tag{3.2.8}$$

Global estimates for $\lambda > 0$

Let $\lambda > 0$ and $\eta_* = \frac{1}{\lambda M_0}$. Using the above definitions, we may write

$$1 - \lambda\eta(t)\sum_{i=1}^{n} h_i\chi_i(\alpha) = 1 - \lambda\eta(t)\left(\sum_{i\in\mathcal{I}_{max}} M_0\chi_i(\alpha) + \sum_{i\notin\mathcal{I}_{max}} h_i\chi_i(\alpha)\right) \tag{3.2.9}$$

and

$$\sum_{i=1}^{n}(1-\lambda\eta(t)h_i)^{-\frac{1}{\lambda}}\mu(\Omega_i) = \sum_{i\in\mathcal{I}_{max}}(1-\lambda\eta(t)M_0)^{-\frac{1}{\lambda}}\mu(\overline{\Omega}_i) + \sum_{i\notin\mathcal{I}_{max}}(1-\lambda\eta(t)h_i)^{-\frac{1}{\lambda}}\mu(\Omega_i). \tag{3.2.10}$$

Then, for fixed $i \in \mathcal{I}_{max}$ choosing $\overline{\alpha} \in \overline{\Omega}_i$ and substituting into (3.2.9), we find

$$1-\lambda\eta(t)\sum_{i=1}^{n}h_i\chi_i(\overline{\alpha}) = 1-\lambda\eta(t)M_0. \tag{3.2.11}$$

Using (3.2.6), (3.2.10) and (3.2.11) we see that, for $\eta \in [0,\eta_*)$,

$$\gamma_\alpha(\overline{\alpha},t) = \left[\sum_{i\in\mathcal{I}_{max}}\mu(\overline{\Omega}_i) + (1-\lambda\eta(t)M_0)^{\frac{1}{\lambda}}\sum_{i\notin\mathcal{I}_{max}}(1-\lambda\eta(t)h_i)^{-\frac{1}{\lambda}}\mu(\Omega_i)\right]^{-1}. \tag{3.2.12}$$

Since $1-\lambda\eta(t)u_0'(\alpha) > 0$ for all $\eta \in [0,\eta_*)$ and $\alpha \in [0,1]$, (3.2.12) implies

$$\lim_{t\uparrow t_*}\gamma_\alpha(\overline{\alpha},t) = \left(\sum_{i\in\mathcal{I}_{max}}\mu(\overline{\Omega}_i)\right)^{-1} > 0 \tag{3.2.13}$$

for some $t_* > 0$. However, (2.1.14), (2.1.16) and (3.2.5) give

$$dt = \left(1-\lambda\eta(t)\sum_{i=1}^{n}h_i\chi_i(\alpha)\right)^{-2}\gamma_\alpha(\alpha,t)^{-2\lambda}d\eta \tag{3.2.14}$$

and so, for $\eta_* - \eta > 0$ small, (3.2.6), (3.2.10) and the above observation on the term $1 - \lambda\eta(t)u_0'(\alpha)$ yield, after integration,

$$t_* - t \sim C\int_{\eta}^{\eta_*}(1-\lambda M_0\sigma)^{-2}d\sigma.$$

Consequently, $t_* = +\infty$. Finally, since $\dot{\gamma}_\alpha = (u_x(\gamma(\alpha,t),t))\gamma_\alpha$,

$$\gamma_\alpha(\alpha,t) = \exp\left(\int_0^t u_x(\gamma(\alpha,s),s)ds\right). \tag{3.2.15}$$

Then (3.0.4)i), (3.2.13) and (3.2.15) yield

$$\lim_{t\to+\infty}\int_0^t M(s)\,ds = -\ln\left(\sum_{i\in\mathcal{I}_{max}}\mu(\overline{\Omega}_i)\right) > 0.$$

If $\alpha = \widetilde{\alpha} \in \Omega_i$ for some index $i \notin \mathcal{I}_{max}$, so that

$$1 - \lambda\eta(t)u_0'(\widetilde{\alpha}) = 1 - \lambda\eta(t)\tilde{h}, \qquad \tilde{h} < M_0,$$

then (3.2.6) implies

$$\gamma_\alpha(\widetilde{\alpha}, t) \sim C(1 - \lambda\eta(t)M_0)^{\frac{1}{\lambda}} \to 0$$

as $t \to +\infty$. Thus, by (3.2.15), we obtain

$$\lim_{t\to+\infty} \int_0^t u_x(\gamma(\widetilde{\alpha}, s), s)ds = -\infty.$$

We refer to appendix A for the case $\lambda = 0$.

L^p **regularity for** $p \in [1, +\infty]$ **and** $\lambda < 0$

Suppose $\lambda < 0$ so that $\eta_* = \frac{1}{\lambda m_0}$. We now write

$$1 - \lambda\eta(t)\sum_{i=1}^{n} h_i\chi_i(\alpha) = 1 - \lambda\eta(t)\left(\sum_{i\in\mathcal{I}_{min}} m_0\chi_i(\alpha) + \sum_{i\notin\mathcal{I}_{min}} h_i\chi_i(\alpha)\right) \tag{3.2.16}$$

and

$$\begin{aligned}\sum_{i=1}^{n}(1 - \lambda\eta(t)h_i)^{\frac{1}{|\lambda|}}\mu(\Omega_i) = &\sum_{i\in\mathcal{I}_{min}} (1 - \lambda\eta(t)m_0)^{\frac{1}{|\lambda|}}\mu(\underline{\Omega}_i) \\ &+ \sum_{i\notin\mathcal{I}_{min}} (1 - \lambda\eta(t)h_i)^{\frac{1}{|\lambda|}}\mu(\Omega_i).\end{aligned} \tag{3.2.17}$$

Choose $\underline{\alpha} \in \underline{\Omega}_i$ for some $i \in \mathcal{I}_{min}$ and substitute into (3.2.16) to obtain

$$1 - \lambda\eta(t)\sum_{i=1}^{n} h_i\chi_i(\underline{\alpha}) = 1 - \lambda\eta(t)m_0. \tag{3.2.18}$$

Using (3.2.17) and (3.2.18) with (3.2.6) gives

$$\gamma_\alpha(\underline{\alpha}, t) = \left[\sum_{i\in\mathcal{I}_{min}} \mu(\underline{\Omega}_i) + \frac{\sum_{i\notin\mathcal{I}_{min}}(1 - \lambda\eta(t)h_i)^{\frac{1}{|\lambda|}}\mu(\Omega_i)}{(1 - \lambda\eta(t)m_0)^{\frac{1}{|\lambda|}}}\right]^{-1} \tag{3.2.19}$$

for $\eta \in [0, \eta_*)$. Then, since $1 - \lambda\eta(t)u_0'(\alpha) > 0$ for $\eta \in [0, \eta_*)$, $\alpha \in [0, 1]$ and $\lambda < 0$, we have that

$$\lim_{t\uparrow t_*} \gamma_\alpha(\underline{\alpha}, t) = 0$$

for some $t_* > 0$ or, equivalently,

$$\lim_{t\to t_*} \int_0^t m(s)ds = -\infty$$

by (3.0.5) and (3.2.15). The blow-up time $t_* > 0$ is now finite. Indeed, (3.2.6), (3.2.14) and (3.2.17) yield the estimate

$$dt \sim \left(\sum_{i \notin \mathcal{I}_{min}} (1 - \lambda\eta(t)h_i)^{\frac{1}{|\lambda|}} \mu(\Omega_i) \right)^{2\lambda} d\eta$$

for $\eta_* - \eta > 0$ small and $\lambda < 0$. Since $h_i > m_0$ for any $i \notin \mathcal{I}_{min}$, integration of the above implies a finite $t_* > 0$.

Now, if $\alpha = \alpha' \in \Omega_i$ for some $i \notin \mathcal{I}_{min}$, then $u_0'(\alpha') = h'$ for $h' > m_0$. Following the argument in the $\lambda > 0$ case yields

$$\gamma_\alpha(\alpha', t) = \left(\sum_{i \notin \mathcal{I}_{min}} (1 - \lambda\eta(t)h_i)^{\frac{1}{|\lambda|}} \mu(\Omega_i) \right)^{-1} (1 - \lambda\eta(t)h')^{\frac{1}{|\lambda|}},$$

consequently

$$\lim_{t \uparrow t_*} \gamma_\alpha(\alpha', t) = C \in \mathbb{R}^+$$

and so, by (3.2.15), $\int_0^t u_x(\gamma(\alpha, s), s)\, ds$ remains finite as $t \uparrow t_*$ for every $\alpha' \neq \underline{\alpha}$ and $\lambda < 0$.

Lastly, we look at L^p regularity of u_x for $p \in [1, +\infty)$ and $\lambda < 0$. From (2.1.14) and (2.1.19),

$$|u_x(\gamma(\alpha, t), t)|^p \, \gamma_\alpha(\alpha, t) = \frac{\mathcal{K}_0(\alpha, t) |\mathcal{J}(\alpha, t)^{-1} - \bar{\mathcal{K}}_0(t)^{-1} \bar{\mathcal{K}}_1(t)|^p}{|\lambda\eta(t)|^p \, \bar{\mathcal{K}}_0(t)^{2\lambda p + 1}}$$

for $t \in [0, t_*)$ and $p \in \mathbb{R}$. Then, integrating in α and using (3.2.3) gives

$$\|u_x(\cdot, t)\|_p^p = \frac{1}{|\lambda\eta(t)|^p} \left(\sum_{i=1}^n (1 - \lambda\eta(t)h_i)^{-\frac{1}{\lambda}} \mu(\Omega_i) \right)^{-(2\lambda p + 1)}$$
$$\sum_{j=1}^n \left\{ (1 - \lambda\eta(t)h_j)^{-\frac{1}{\lambda}} \left| (1 - \lambda\eta(t)h_j)^{-1} - \frac{\sum_{i=1}^n (1 - \lambda\eta(t)h_i)^{-1-\frac{1}{\lambda}} \mu(\Omega_i)}{\sum_{i=1}^n (1 - \lambda\eta(t)h_i)^{-\frac{1}{\lambda}} \mu(\Omega_i)} \right|^p \mu(\Omega_j) \right\}$$

for $p \in [1, +\infty)$. Splitting each sum above into the indexes $i, j \in \mathcal{I}_{min}$ and $i, j \notin \mathcal{I}_{min}$, we obtain, for $\eta_* - \eta > 0$ small,

$$\|u_x(\cdot, t)\|_p^p \sim C \mathcal{J}(\underline{\alpha}, t)^{-\frac{1}{\lambda}} \left| \mathcal{J}(\underline{\alpha}, t)^{-1} - C \left(\mathcal{J}(\underline{\alpha}, t)^{-1-\frac{1}{\lambda}} + C \right) \right|^p$$
$$+ C \sum_{j \notin \mathcal{I}_{min}} \left\{ (1 - \lambda\eta h_j)^{-\frac{1}{\lambda}} \left| (1 - \lambda\eta h_j)^{-1} - C \left(\mathcal{J}(\underline{\alpha}, t)^{-1-\frac{1}{\lambda}} + C \right) \right|^p \mu(\Omega_j) \right\}$$

where $\lambda < 0$, $\mathcal{J}(\underline{\alpha}, t) = 1 - \lambda\eta(t)m_0$ and $C \in \mathbb{R}^+$ may now also depend on $p \in [1, +\infty)$. Suppose $\lambda \in [-1, 0)$, then $-1 - \frac{1}{\lambda} \geq 0$ and the above implies

$$\|u_x(\cdot, t)\|_p^p \sim C \mathcal{J}(\underline{\alpha}, t)^{-\left(p + \frac{1}{\lambda}\right)} + g(t) \tag{3.2.20}$$

for $g(t)$ a bounded function on $[0,t_*)$ with finite, non-negative limit as $t \uparrow t_*$. On the other hand, if $\lambda < -1$ then $-1-\frac{1}{\lambda} < 0$ and

$$\|u_x(\cdot,t)\|_p^p \sim C\mathcal{J}(\underline{\alpha},t)^{-\left(p+\frac{1}{\lambda}\right)} \tag{3.2.21}$$

holds instead. The last part of the Theorem follows from (3.2.20) and (3.2.21) as $t \uparrow t_*$. See section 4.1.2 for examples. □

3.3 Initial Data with Arbitrary Curvature Near M_0 or m_0

As motivation for this section, consider the following example with periodic, piecewise linear u_0'. Let

$$u_0(\alpha) = \begin{cases} 2\alpha^2 - \alpha, & \alpha \in [0,1/2], \\ -2\alpha^2 + 3\alpha - 1, & \alpha \in (1/2,1]. \end{cases} \tag{3.3.1}$$

Then

$$u_0'(\alpha) = \begin{cases} 4\alpha - 1, & \alpha \in [0,1/2], \\ -4\alpha + 3, & \alpha \in (1/2,1] \end{cases} \tag{3.3.2}$$

attains its greatest and least values, $M_0 = 1$ and $m_0 = -1$, at $\overline{\alpha} = 1/2$ and $\underline{\alpha} = \{0,1\}$ respectively. As a result, (3.0.6) implies that $\eta_* = \frac{1}{|\lambda|}$ for $\lambda \neq 0$.

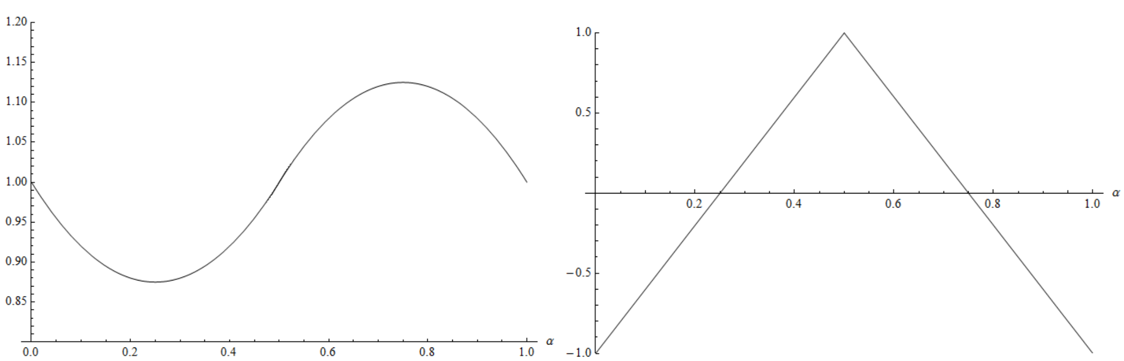

Figure 3.1: $u_0(\alpha)$ and $u_0'(\alpha)$ in (3.3.1) and (3.3.2).

Using (3.3.2), we find

$$\bar{\mathcal{K}}_0(t) = \begin{cases} \frac{1}{2(1-\lambda)\eta(t)}\left(\mathcal{J}(\overline{\alpha},t)^{1-\frac{1}{\lambda}} - \mathcal{J}(\underline{\alpha},t)^{1-\frac{1}{\lambda}}\right), & \lambda \in \mathbb{R}\backslash\{0,1\}, \\ \frac{1}{2\eta(t)}\ln\left(\frac{\eta_*+\eta(t)}{\eta_*-\eta(t)}\right), & \lambda = 1 \end{cases} \tag{3.3.3}$$

and

$$\bar{\mathcal{K}}_1(t) = \frac{\mathcal{J}(\overline{\alpha},t)^{-\frac{1}{\lambda}} - \mathcal{J}(\underline{\alpha},t)^{-\frac{1}{\lambda}}}{2\eta(t)}, \qquad \lambda \neq 0 \tag{3.3.4}$$

where

$$\mathcal{J}(\overline{\alpha},t) = 1 - \lambda\eta(t), \qquad \mathcal{J}(\underline{\alpha},t) = 1 + \lambda\eta(t).$$

If $\lambda < 0$, then

$$\begin{cases} \bar{\mathcal{K}}_0(t) \to 2^{\frac{1}{|\lambda|}} \frac{|\lambda|}{1-\lambda}, \\ \bar{\mathcal{K}}_1(t) \to |\lambda|\, 2^{\frac{1}{|\lambda|}-1} \end{cases} \tag{3.3.5}$$

as $\eta \uparrow \eta_* = -\frac{1}{\lambda}$ and so both integral terms are finite (and nonzero) for all $\eta \in [0, \eta_*]$. Consequently, when $\alpha = \underline{\alpha}$, $u_x(\gamma(\alpha,t),t)$ undergoes a one-sided discrete blow-up due to the space-dependent term in (2.1.19). We find that

$$m(t) \to -\infty$$

as $\eta \uparrow -\frac{1}{\lambda}$ for all $\lambda < 0$. The existence of a finite $t_* > 0$ follows from (2.1.16) and (3.3.3)i). On the other hand, if $\lambda > 0$ and $\eta_* - \eta > 0$ is small,

$$\bar{\mathcal{K}}_0(t) \sim \begin{cases} \frac{\lambda}{2(1-\lambda)} \mathcal{J}(\overline{\alpha},t)^{1-\frac{1}{\lambda}}, & \lambda \in (0,1), \\ \frac{\lambda}{2^{\frac{1}{\lambda}}(\lambda-1)}, & \lambda \in (1,+\infty), \\ -C\log(\eta_* - \eta(t)), & \lambda = 1 \end{cases} \tag{3.3.6}$$

and

$$\bar{\mathcal{K}}_1(t) \sim \frac{\lambda}{2\mathcal{J}(\overline{\alpha},t)^{\frac{1}{\lambda}}}. \tag{3.3.7}$$

If $\alpha = \overline{\alpha}$, the above estimates and (2.1.19) imply that, as $\eta \uparrow \eta_*$,

$$M(t) = u_x(\gamma(\overline{\alpha},t),t) \to 0, \qquad \lambda \in (0,1/2),$$

but

$$M(t) = u_x(\gamma(\overline{\alpha},t),t) \to +\infty, \qquad \lambda > 1/2.$$

Furthermore, for $\alpha \neq \overline{\alpha}$,

$$u_x(\gamma(\alpha,t),t) \to 0, \qquad \lambda \in (0,1/2),$$

while

$$u_x(\gamma(\alpha,t),t) \to -\infty, \qquad \lambda > 1/2.$$

For the threshold parameter $\lambda = 1/2$,

$$\begin{cases} u_x(\gamma(\alpha,t),t) \to -1 \quad \text{as} \quad \eta \uparrow \eta_* = 2 \quad \text{for} \quad \alpha \notin \{\overline{\alpha}, \underline{\alpha}\}, \\ M(t) = u_x(\gamma(\overline{\alpha},t),t) \equiv 1, \\ m(t) = u_x(\gamma(\underline{\alpha},t),t) \equiv -1. \end{cases} \tag{3.3.8}$$

Finally, from (2.1.16) and (3.3.6),

$$t_* - t \sim \begin{cases} C\int_\eta^{\eta_*} (1-\lambda\mu)^{2(\lambda-1)} d\mu, & \lambda \in (0,1), \\ C(\eta_* - \eta)(2 - 2\log(\eta_* - \eta) + \ln^2(\eta_* - \eta)), & \lambda = 1, \\ C(\eta_* - \eta), & \lambda > 1, \end{cases}$$

and so $t_* = +\infty$ for $\lambda \in (0, 1/2]$ but $0 < t_* < +\infty$ when $\lambda > 1/2$.

In summary, for the choice of data (3.3.2), $u_x(\gamma(\alpha,t),t)$ undergoes a two-sided, everywhere blow-up in finite-time for $\lambda > 1/2$, whereas, if $\lambda < 0$, a one-sided discrete blow-up occurs instead, $m(t) \to -\infty$ as $t \uparrow t_*$. In contrast, the solution persists for all time when $\lambda \in (0, 1/2]$, that is, $u_x \to 0$ as $t \to +\infty$ for $\lambda \in (0, 1/2)$, while a nontrivial steady-state is reached if $\lambda = 1/2$.

Remark 3.3.9. We recall that if $\lambda \in [1/2, 1)$ and $u_0'''(x) \in L_{\mathbb{R}}^{\frac{1}{2(1-\lambda)}}(0,1)$, then u persists globally in time ([38]). This result does not contradict the above blow-up example. Indeed, if $u_0''' \in L_{\mathbb{R}}^{\frac{1}{2(1-\lambda)}}$ for $\lambda \in [1/2, 1)$, then u_0'' is an absolutely continuous function on $[0,1]$, and hence continuous. However, in the case just considered, u_0'' is, of course, not continuous.

As opposed to the results from sections 3.1 and 3.2, where u_0' had either *quadratic* or *constant* local behaviour near the points $\overline{\alpha}_i$ and/or $\underline{\alpha}_j$, we find that the above choice of u_0' with *linear* local behaviour instead leads to different blow-up behaviour. More particularly, (3.3.2) implies finite-time blow-up for $\lambda \in (1/2, 1]$; parameter values for which no blow-up occurred in the cases previously considered. Furthermore, for other values of the parameter the nature of the blow-up, or its occurrence at all, differs as well from the results in Theorems 3.1.1 and 3.2.4.

3.3.1 The Data Classes

In light of the above observations, we conclude that relative to the sign of $\lambda \neq 0$, the curvature of u_0 near $\overline{\alpha}_i$ and/or $\underline{\alpha}_j$ plays a decisive role in the finite-time blow-up of solutions to (1.1.1). The purpose of the remaining sections is to further examine this interaction by studying a larger class of initial data in which u_0' admits other than quadratic, or piecewise constant, behaviour near the locations in question. Specifically, suppose u_0' is bounded, at least $C^0(0,1)$ *a.e.*, and assume that for $\lambda > 0$ there is $q \in \mathbb{R}^+$ and $C_1 \in \mathbb{R}^-$, such that

$$u_0'(\alpha) \sim M_0 + C_1 \left|\alpha - \overline{\alpha}_i\right|^q \tag{3.3.10}$$

for $0 \leq |\alpha - \overline{\alpha}_i| \leq r,\ 1 \leq i \leq m$, and small enough $0 < r \leq 1$, $r \equiv \min_{1\leq i\leq m}\{r_i\}$. Similarly, if $\lambda < 0$, suppose

$$u_0'(\alpha) \sim m_0 + C_2 \left|\alpha - \underline{\alpha}_j\right|^q \tag{3.3.11}$$

for $0 \leq \left|\alpha - \underline{\alpha}_j\right| \leq r$, $C_2 \in \mathbb{R}^+$ and $1 \leq j \leq n$. See Figure 3.2 below. Also, for $q \in \mathbb{R}^+$ and either $\lambda > 0$ or $\lambda < 0$, we will assume there are a finite number of locations $\overline{\alpha}_i$ or $\underline{\alpha}_j$, respectively. Particularly, this rules out the possibility of having initial data for which u_0' oscillates infinitely many times through its greatest value $M_0 > 0$ when $\lambda > 0$, or through its minimum value $m_0 < 0$ for $\lambda < 0$. Moreover, for $q \in (0,1)$, the above local estimates may lead to cusp singularities in u_0', namely, jump discontinuities in u_0'' of *infinite* magnitude. In contrast, a jump discontinuity of *finite* magnitude in u_0'' may occur if $q = 1$. As we will see in the coming sections, the either finite or infinite character in the size of this jump along with the corresponding set of boundary conditions plays a decisive role, particularly, in the formation of spontaneous singularities in stagnation point-form solutions to the two and three dimensional incompressible Euler equations that arise from smooth initial data. Finally, observe that (3.3.10) and/or (3.3.11) generalize the class of smooth data studied in section 3.1 characterized by functions u_0' with quadratic local behaviour near $\overline{\alpha}_i$ and/or $\underline{\alpha}_j$. Now, for $0 < r \leq 1$ as specified above, define

$$\mathcal{D}_i \equiv [\overline{\alpha}_i - r, \overline{\alpha}_i + r], \qquad \mathcal{D}_j \equiv [\underline{\alpha}_j - r, \underline{\alpha}_j + r].$$

Below, we list some of the data classes that admit the asymptotic behaviour (3.3.10) and/or (3.3.11) for particular values of $q > 0$.

- $u_0(x) \in C^\infty(0,1)$ for $q = 2k$ and $k \in \mathbb{Z}^+$ (see definition 3.3.117).

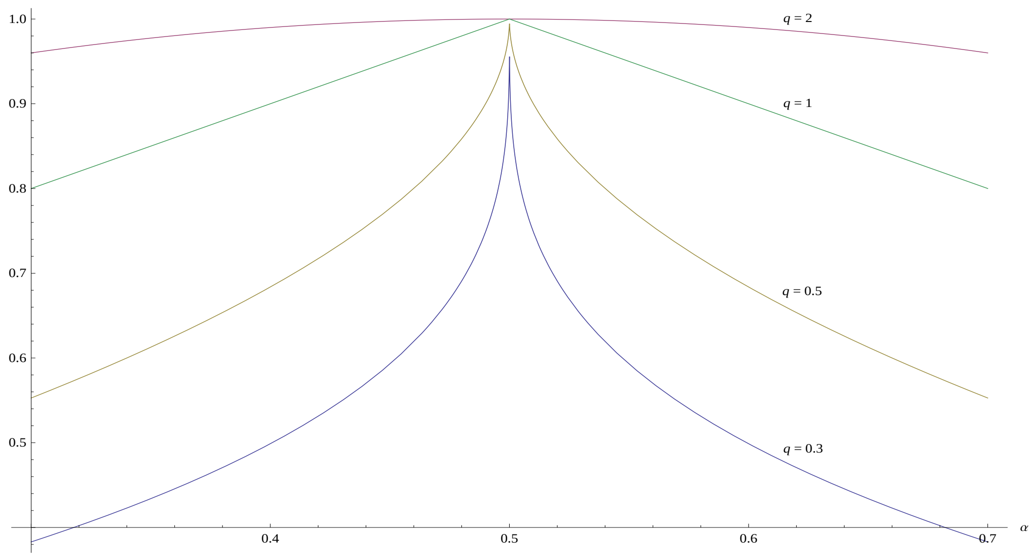

Figure 3.2: Local behaviour of $u_0'(\alpha)$ satisfying (3.3.10) for several values of $q > 0$, $\overline{\alpha} = 1/2$, $M_0 = 1$ and $C_1 = -1$.

- If $q = 1$, $u_0''(x) \in PC(\mathcal{D}_i)$ for $\lambda > 0$, or $u_0''(x) \in PC(\mathcal{D}_j)$ if $\lambda < 0$.

- In the limit as $q \to +\infty$, $u_0'(x) \in PC(\mathcal{D}_i)$ for $\lambda > 0$, or $u_0'(x) \in PC(\mathcal{D}_j)$ if $\lambda < 0$.

- From (3.3.10), we see that the quantity

$$[u_0']_{q;\overline{\alpha}_i} = \sup_{\alpha \in \mathcal{D}_i} \frac{|u_0'(\alpha) - u_0'(\overline{\alpha}_i)|}{|\alpha - \overline{\alpha}_i|^q} \tag{3.3.12}$$

  is finite. As a result, for $0 < q \leq 1$ and $\lambda > 0$, u_0' is Hölder continuous at $\overline{\alpha}_i$. Analogously for $\lambda < 0$, since

$$[u_0']_{q;\underline{\alpha}_j} = \sup_{\alpha \in \mathcal{D}_j} \frac{|u_0'(\alpha) - u_0'(\underline{\alpha}_j)|}{|\alpha - \underline{\alpha}_j|^q} \tag{3.3.13}$$

  is defined by (3.3.11).

- For $\lambda > 0$ and either $s < q < s + 1$, $s \in \mathbb{N}$, or $q > 0$ odd, $u_0'(\alpha) \in C^{s+1}(\mathcal{D}_i)$. Similarly for $\lambda < 0$.

The outline of this section is as follows. In section 3.3.2, we examine L^p, $p \in [1, +\infty]$, regularity of u_x with bounded $u_0'(x)$ that is, at least, $C^0(0,1)$ *a.e.* and satisfies (3.3.10) and/or (3.3.11) for $q = 1$. Then, in section 3.3.3, a similar analysis follows for arbitrary

$q > 0$. Amongst other results, we note that in section 3.3.3 we generalize the results from section 3.1 to arbitrary smooth initial data.

3.3.2 Global Estimates and Blow-up for $\lambda \neq 0$ and $q = 1$

In this section, we consider initial data satisfying (3.3.10) and/or (3.3.11) for $q = 1$. One main reason for discussing the $q = 1$ case separately from arbitrary $q > 0$, is that the argument we will use for the latter, see Lemma 3.3.41, excludes the study, particularly, of stagnation point-form solutions to the 2D incompressible Euler equations ($\lambda = 1$) whenever u_0' satisfies (3.3.10) for $q = 1$. We begin by studying the L^∞ regularity of u_x for $\lambda \in \mathbb{R}$, then, for the cases where finite-time blow-up in the L^∞ norm is established, we examine further properties of L^p regularity for arbitrary $p \in [1, +\infty)$. For the case $\lambda = 0$, the reader may refer to appendix A.

L^∞ Regularity for $\lambda \neq 0$ and $q = 1$

Theorem 3.3.14. *Consider the initial boundary value problem (1.1.1)-(1.1.2) or (1.1.3) for $u_0'(\alpha)$ bounded and, at least, $C^0(0,1)$ a.e..*

1. *Suppose $\lambda > 1/2$ and u_0' satisfies (3.3.10) with $q = 1$. Then, there exists a finite $t_* > 0$ such that both the maximum $M(t)$ and the minimum $m(t)$ diverge to $+\infty$ and respectively to $-\infty$ as $t \uparrow t_*$. Moreover, for every $\alpha \notin \{\overline{\alpha}_i, \underline{\alpha}_j\}$, $\lim_{t\uparrow t_*} u_x(\gamma(\alpha, t), t) = -\infty$ (two-sided, everywhere blow-up).*

2. *Suppose $\lambda \in (0, 1/2]$ and u_0' satisfies (3.3.10) with $q = 1$. Then solutions exist globally in time. More particularly, these vanish as $t \uparrow t_* = +\infty$ for $\lambda \in (0, 1/2)$ but converge to a non-trivial steady-state if $\lambda = 1/2$.*

3. *Suppose $\lambda < 0$ and u_0' satisfies (3.3.11) with $q = 1$. Then, there is a finite $t_* > 0$ such that only the minimum diverges, $m(t) \to -\infty$, as $t \uparrow t_*$ (one-sided, discrete blow-up). Further, if only Dirichlet boundary conditions (1.1.3) are considered and/or u_0 is odd about the midpoint, then for every $\underline{\alpha}_j \in [0,1]$ there exists a unique $\underline{x}_j \in [0,1]$ given by (3.1.2) such that $\lim_{t\uparrow t_*} u_x(\underline{x}_j, t) = -\infty$.*

Proof. Let C denote a positive constant which may depend on $\lambda \neq 0$.

Proof of Statements *1* and *2*

For simplicity, we prove 1 and 2 for the case where M_0 occurs at a single location $\overline{\alpha} \in (0,1)$.[17] According to (3.3.10), there is $0 < r \leq 1$ small enough such that $u_0'(\alpha) \sim M_0 + C_1 |\alpha - \overline{\alpha}|$ for $0 \leq |\alpha - \overline{\alpha}| \leq r$ and $C_1 < 0$. Then

$$\epsilon + M_0 - u_0'(\alpha) \sim \epsilon - C_1 |\alpha - \overline{\alpha}| \tag{3.3.15}$$

for $\epsilon > 0$, so that

$$\begin{aligned} \int_{\overline{\alpha}-r}^{\overline{\alpha}+r} \frac{d\alpha}{(\epsilon + M_0 - u_0'(\alpha))^{\frac{1}{\lambda}}} &\sim \int_{\overline{\alpha}-r}^{\overline{\alpha}+r} \frac{d\alpha}{(\epsilon - C_1 |\alpha - \overline{\alpha}|)^{\frac{1}{\lambda}}} \\ &= \int_{\overline{\alpha}-r}^{\overline{\alpha}} \frac{d\alpha}{(\epsilon + C_1(\alpha - \overline{\alpha}))^{\frac{1}{\lambda}}} + \int_{\overline{\alpha}}^{\overline{\alpha}+r} \frac{d\alpha}{(\epsilon - C_1(\alpha - \overline{\alpha}))^{\frac{1}{\lambda}}} \\ &= \frac{2\lambda}{|C_1|(1-\lambda)} \left(\epsilon^{1-\frac{1}{\lambda}} - (\epsilon + |C_1| r)^{1-\frac{1}{\lambda}} \right) \end{aligned} \tag{3.3.16}$$

for $\lambda \in (0, +\infty) \backslash \{1\}$.[18] Consequently, setting $\epsilon = \frac{1}{\lambda \eta} - M_0$ into (3.3.16) we find that

$$\bar{\mathcal{K}}_0(t) \sim \begin{cases} C, & \lambda > 1, \\ \frac{2\lambda M_0}{|C_1|(1-\lambda)} \mathcal{J}(\overline{\alpha}, t)^{1-\frac{1}{\lambda}}, & \lambda \in (0,1) \end{cases} \tag{3.3.17}$$

for $\eta_* - \eta > 0$ small, $\eta_* = \frac{1}{\lambda M_0}$ and $\mathcal{J}(\overline{\alpha}, t) = 1 - \lambda \eta(t) M_0$. In a similar fashion, we can estimate

$$\bar{\mathcal{K}}_1(t) \sim \frac{2\lambda M_0}{|C_1|} \mathcal{J}(\overline{\alpha}, t)^{-\frac{1}{\lambda}} \tag{3.3.18}$$

for any $\lambda > 0$. Suppose $\lambda > 1$. Then, (2.1.19), (3.3.17)i) and (3.3.18) give

$$u_x(\gamma(\alpha, t), t) \sim C \left(\frac{1}{\mathcal{J}(\alpha, t)} - \frac{C}{\mathcal{J}(\overline{\alpha}, t)^{\frac{1}{\lambda}}} \right) \tag{3.3.19}$$

for $\eta_* - \eta > 0$ small. Setting $\alpha = \overline{\alpha}$ into (3.3.19) and using (3.0.4) implies that

$$M(t) \sim \frac{C}{\mathcal{J}(\overline{\alpha}, t)} \to +\infty$$

as $\eta \uparrow \eta_*$. However, if $\alpha \neq \overline{\alpha}$, the second term in (3.3.19) dominates and

$$u_x(\gamma(\alpha, t), t) \sim -\frac{C}{\mathcal{J}(\overline{\alpha}, t)^{\frac{1}{\lambda}}} \to -\infty.$$

[17] By a similar argument, the Theorem can be established for the case of several $\overline{\alpha}_i \in [0,1]$.

[18] The case $\lambda = 1$ is considered separately.

The existence of a finite $t_* > 0$ for all $\lambda > 1$ follows from (2.1.16) and (3.3.17)i), which imply

$$t_* - t \sim C(\eta_* - \eta).$$

Now let $\lambda \in (0,1)$. Using (3.3.17)ii) and (3.3.18) on (2.1.19), yields

$$u_x(\gamma(\alpha,t),t) \sim C\left(\frac{1}{\mathcal{J}(\alpha,t)} - \frac{1-\lambda}{\mathcal{J}(\overline{\alpha},t)}\right)\mathcal{J}(\overline{\alpha},t)^{2(1-\lambda)} \tag{3.3.20}$$

for $\eta_* - \eta > 0$ small. Setting $\alpha = \overline{\alpha}$ into (3.3.20) implies

$$M(t) \sim C\mathcal{J}(\overline{\alpha},t)^{1-2\lambda} \to \begin{cases} 0^+, & \lambda \in (0,1/2), \\ +\infty, & \lambda \in (1/2,1) \end{cases} \tag{3.3.21}$$

as $\eta \uparrow \eta_*$. If instead $\alpha \neq \overline{\alpha}$,

$$u_x(\gamma(\alpha,t),t) \sim -C\mathcal{J}(\overline{\alpha},t)^{1-2\lambda} \to \begin{cases} 0^-, & \lambda \in (0,1/2), \\ -\infty, & \lambda \in (1/2,1) \end{cases} \tag{3.3.22}$$

as $\eta \uparrow \eta_*$. For the threshold parameter $\lambda = 1/2$, we keep track of the constants and find that, as $\eta \uparrow \eta_*$,

$$u_x(\gamma(\alpha,t),t) \to \begin{cases} \frac{|C_1|}{4}, & \alpha = \overline{\alpha} \\ -\frac{|C_1|}{4}, & \alpha \neq \overline{\alpha}. \end{cases} \tag{3.3.23}$$

Finally, (2.1.16) and (3.3.17)ii) imply

$$dt \sim C\mathcal{J}(\overline{\alpha},t)^{2(\lambda-1)}d\eta$$

so that

$$t_* = \lim_{\eta\uparrow\eta_*} t(\eta) \sim \begin{cases} \frac{C}{2\lambda-1}\left(C - \lim_{\eta\uparrow\eta_*}(\eta_*-\eta)^{2\lambda-1}\right), & \lambda \in (0,1)\backslash\{1/2\}, \\ -C\lim_{\eta\uparrow\eta_*}\log(\eta_*-\eta), & \lambda = \frac{1}{2}. \end{cases} \tag{3.3.24}$$

As a result, $t_* = +\infty$ for $\lambda \in (0,1/2]$ while $0 < t_* < +\infty$ when $\lambda \in (1/2,1)$. Lastly, if $\lambda = 1$

$$\bar{\mathcal{K}}_0(t) \sim -\frac{2M_0}{|C_1|}\log(\eta_* - \eta) \tag{3.3.25}$$

for $0 < \eta_* - \eta = \frac{1}{M_0} - \eta << 1$ small. Then, a two-sided everywhere blow-up in finite-time follows just as above from (2.1.19), (2.1.16), (3.3.18) and (3.3.25).

Proof of Statement *3*

For $\lambda < 0$, set $\eta_* = \frac{1}{\lambda m_0}$ and suppose $u_0'(\alpha)$ is at least $C^0(0,1)$ *a.e.* and satisfies (3.3.11) for $q = 1$. Then $\bar{\mathcal{K}}_0(t)$ remains finite, and positive, for all $\eta \in [0, \eta_*]$. In fact, $\bar{\mathcal{K}}_0(t)$ satisfies (3.1.53) if $\lambda \in [-1, 0)$ while (3.1.40) holds for $\lambda < -1$. Similarly, when $\lambda \in [-1, 0)$, $\bar{\mathcal{K}}_1(t)$ satisfies (3.1.55). See sections 3.1.2 and (3.1.3) for details. However, if $\lambda < -1$, we still need to estimate the behaviour of $\bar{\mathcal{K}}_1(t)$ for $\eta_* - \eta > 0$ small. For simplicity, we do so for $u_0'(\alpha)$ achieving its smallest value $m_0 < 0$ at a single point $\underline{\alpha} \in (0,1)$. Then, (3.3.11) with $q = 1$ yields

$$\begin{aligned}\int_{\underline{\alpha}-r}^{\underline{\alpha}+r} \frac{d\alpha}{(\epsilon + u_0'(\alpha) - m_0)^{1+\frac{1}{\lambda}}} &\sim \int_{\underline{\alpha}-r}^{\underline{\alpha}+r} \frac{d\alpha}{(\epsilon + C_2\,|\alpha - \underline{\alpha}|)^{1+\frac{1}{\lambda}}} \\ &= \int_{\underline{\alpha}-r}^{\underline{\alpha}} \frac{d\alpha}{(\epsilon - C_2(\alpha - \underline{\alpha}))^{1+\frac{1}{\lambda}}} + \int_{\underline{\alpha}}^{\underline{\alpha}+r} \frac{d\alpha}{(\epsilon + C_2(\alpha - \underline{\alpha}))^{1+\frac{1}{\lambda}}} \\ &= \frac{2\,|\lambda|}{C_2}\left((\epsilon + C_2 r)^{\frac{1}{|\lambda|}} - \epsilon^{\frac{1}{|\lambda|}}\right).\end{aligned} \tag{3.3.26}$$

Substituting $\epsilon = m_0 - \frac{1}{\lambda\eta}$ into the above, we find that $\bar{\mathcal{K}}_1(t)$ has a finite, positive limit as $\eta \uparrow \eta_*$ for any $\lambda < -1$. Therefore, for $\lambda < 0$, every time-dependent integral in (2.1.19) remains bounded and positive for all $\eta \in [0, \eta_*]$. As a result, blow-up of (2.1.19), as $\eta \uparrow \eta_*$, will follow from the space-dependent term, $\mathcal{J}(\alpha, t)^{-1}$, evaluated at $\alpha = \underline{\alpha}$. In this way, we set $\alpha = \underline{\alpha}$ into (2.1.19) and use (3.2.7)ii) to obtain

$$m(t) \sim \frac{C m_0}{\mathcal{J}(\overline{\alpha}, t)} \to -\infty$$

as $\eta \uparrow \eta_*$. On the other hand, for $\alpha \neq \underline{\alpha}$, the definition of m_0 implies that the space-dependent term now remains bounded for all $\eta \in [0, \eta_*]$, and so (2.1.19) stays finite as $\eta \uparrow \eta_*$. Finally, the existence of a finite blow-up time $t_* > 0$ for the minimum as well as formula (3.1.2) follow from (2.1.16) and (2.2.1), respectively, along with the above estimates on $\bar{\mathcal{K}}_0(t)$. See section 4.2 for examples. □

Remark 3.3.27. Recall from Theorem 3.1.1, which examines a family of smooth initial data, that $\lambda_* = 1$ acts as the threshold parameter between solutions that vanish at $t = +\infty$ for $\lambda \in (0, \lambda_*)$ and those which blow-up in finite-time when $\lambda \in (\lambda_*, +\infty)$, while for $\lambda_* = 1$, u_x converges to a nontrivial steady-state as $t \to +\infty$. According to Theorem 3.3.14 above, if u_0' behaves linearly near $\overline{\alpha}_i$, we now have the corresponding behavior at $\lambda_* = 1/2$ instead.

Particularly, this means that if $\overline{\alpha}_i \in (0,1)$, the jump discontinuity of finite magnitude in u_0'' at $\overline{\alpha}_i$ leads to finite-time blow-up when $\lambda = 1$, while solutions persist globally in time if $\lambda = 1/2$. Interestingly enough, recall that for $\lambda = 1/2$ or $\lambda = 1$, equation (1.1.1) i), iii) models stagnation point-form solutions to the 3D or 2D incompressible Euler equations respectively. In section 3.3.3, we show that jump discontinuities in u_0'' of infinite magnitude instead (cusps in the graph of u_0'), lead to finite-time blow-up for $\lambda = 1/2$. Also, see Remark 3.3.28 below and Corollary 3.3.29 for the case of smooth data with linear behaviour near the boundary.

For $\lambda \neq 0$, Remark 3.3.28 below discusses the role that both periodic and Dirichlet boundary conditions play in the finite-time blow-up of solutions to (1.1.1) which arise from smooth initial data having linear local behaviour near $\overline{\alpha}_i$ and/or $\underline{\alpha}_j$. The main results concerning the regularity of stagnation point-form solutions to the 2D incompressible Euler equations are summarized in Corollary 3.3.29.

Remark 3.3.28. Suppose there are a finite number of $\overline{\alpha}_i$ lying in the interior $(0,1)$ and consider either periodic or Dirichlet boundary conditions. Then, no function $u_0'(\alpha)$ can be both smooth in $[0,1]$ and satisfy (3.3.10) for $q = 1$. Indeed, since $q = 1$ and there are $\overline{\alpha}_i \in (0,1)$, $u_0''(\alpha)$ has jump discontinuities of *finite* magnitude at those locations. Therefore, if $u_0'(\alpha)$ is smooth and behaves linearly near $\overline{\alpha}_i$, then these points must lie strictly on the boundary. An example for the Dirichlet setting is given by $u_0(\alpha) = \alpha(1-\alpha)$ with $\overline{\alpha}_1 = 0$.[19] On the other hand, suppose a periodic function $u_0(\alpha)$ satisfies (3.3.10) with $q = 1$ and $M_0 = u_0'(0) = u_0'(1) > u_0'(\alpha)$ for all $\alpha \in (0,1)$. Then $0 > u_0''(0) = u_0''(1)$, by periodicity. But using (3.3.10) for $q = 1$ gives

$$0 > u_0''(1) = \lim_{h \to 0^-} \frac{u_0'(1+h) - M_0}{h} \sim \lim_{h \to 0^-} \frac{(M_0 + |C_1|\, h) - M_0}{h} = |C_1|\,,$$

a contradiction. We conclude that if a periodic function $u_0'(\alpha)$ behaves linearly near $\overline{\alpha}_i$, then these points must lie somewhere in the interior, and thus, u_0 cannot be smooth. Using these results along with Theorem 3.3.14, we deduce that finite-time blow-up in u_x for smooth initial data and $\lambda > 1/2$ can only occur under Dirichlet, not periodic boundary conditions. This includes, particularly, breakdown in stagnation point-form solutions to the 2D Euler equations ($\lambda = 1$). Moreover, by using $\underline{\alpha}_j$ and (3.3.11) instead, the same conclusion follows for $\lambda < 0$. Finally, we note that the blow-up, at least for $\lambda \in (1/2, 1]$, may be suppressed

[19]Smooth data similar to this was used in [9] to construct a blow-up solution for $\lambda = 1$ (2D Euler).

in the Dirichlet setting if we further assume that u_0' admits a smooth, periodic extension to the entire real line, which would prevent linear behaviour near the boundary.

Corollary 3.3.29. *Consider the IVP (1.1.1) for $\lambda = 1$. Suppose the initial data is smooth and u_0' satisfies (3.3.10) for $q = 1$. Then, there exists a finite $t_* > 0$ such that stagnation point-form solutions (1.2.3) to the 2D incompressible Euler equations will diverge* ONLY *under Dirichlet boundary conditions. More particularly, as $t \uparrow t_*$,*

$$\begin{cases} u_x(\overline{\alpha}_i, t) \to +\infty, & \overline{\alpha}_i \in \{0, 1\}, \\ u_x(x, t) \to -\infty, & x \neq \overline{\alpha}_i, \\ \|u_x(\cdot, t)\|_p \to +\infty, & p > 1. \end{cases}$$

In contrast, if periodic boundary conditions are considered, solutions persist for all time.

Proof. See Remark 3.3.28 and Theorem 3.3.30(*1*) below. □

Further L^p Regularity for $\lambda \neq 0$, $p \in [1, +\infty)$ and $q = 1$

From Theorem 3.3.14, $u_x \in L^\infty$ for all time if $\lambda \in [0, 1/2]$ and the data satisfies (3.3.10) for $q = 1$. Therefore, for these values of the parameter and $p \geq 1$, $\|u_x\|_p$ exist globally. In contrast, there is a finite $t_* > 0$ such that $\|u_x\|_\infty$ diverges as $t \uparrow t_*$ when $\lambda \in \mathbb{R}\backslash[0, 1/2]$. In this section, we use the upper and lower bounds (3.1.63) and (3.1.64) to study further $L^p(0, 1)$ regularity properties of u_x as $t \uparrow t_*$ for $p \in [1, +\infty)$ and $\lambda \in \mathbb{R}\backslash[0, 1/2]$.

Theorem 3.3.30. *Consider the initial boundary value problem (1.1.1)-(1.1.2) or (1.1.3) for $u_0'(\alpha)$ bounded and, at least, $C^0(0, 1)$ a.e. Also, let $t_* > 0$ denote the finite L^∞ blow-up time for u_x in Theorem 3.3.14. It follows:*

1. *Suppose u_0' satisfies (3.3.10) with $q = 1$. Then, $\lim_{t \uparrow t_*} \|u_x\|_p = +\infty$ for all $\lambda > 1/2$ and $p > 1$.*

2. *Suppose u_0' satisfies (3.3.11) with $q = 1$. Then, $u_x \in L^1$ for all $\lambda < 0$ and $t \in [0, t_*]$, while $u_x \in L^p$ for $\frac{1}{1-p} < \lambda < 0$, $p > 1$ and $t \in [0, t_*]$.*

3. *The energy $E(t) = \|u_x\|_2^2$ diverges if $\lambda \in (-\infty, -1] \cup (1/2, +\infty)$ as $t \uparrow t_*$ but remains finite for $t \in [0, t_*]$ and $\lambda \in (-1, 0)$. Also, $\lim_{t \uparrow t_*} \dot{E}(t) = +\infty$ when $\lambda \in (-\infty, -1/2) \cup (1/2, +\infty)$, whereas $\dot{E}(t) \equiv 0$ if $\lambda = -1/2$ and $\dot{E}(t)$ stays bounded for $t \in [0, t_*]$ and $\lambda \in (-1/2, 0)$.*

Proof. Let C denote a positive constant that may depend on λ and $p \in [1, +\infty)$.

Proof of Statement *1*

First, suppose $\lambda > 0$ and set $\eta_* = \frac{1}{\lambda M_0}$. For simplicity, we prove 1 under the assumption $M_0 > 0$ occurs at a single point $\overline{\alpha} \in (0,1)$. As a result, for some $\epsilon > 0$, (3.3.10) implies that

$$\epsilon + M_0 - u_0'(\alpha) \sim \epsilon - C_1 |\alpha - \overline{\alpha}|$$

for $0 \leq |\alpha - \overline{\alpha}| \leq r$, $0 < r \leq 1$ small enough and $C_1 < 0$. Accordingly, we have

$$\begin{aligned}\int_{\overline{\alpha}-r}^{\overline{\alpha}+r} \frac{d\alpha}{(\epsilon + M_0 - u_0'(\alpha))^{1+\frac{1}{\lambda p}}} &\sim \int_{\overline{\alpha}-r}^{\overline{\alpha}+r} \frac{d\alpha}{(\epsilon - C_1 |\alpha - \overline{\alpha}|)^{1+\frac{1}{\lambda p}}} \\ &= \int_{\overline{\alpha}-r}^{\overline{\alpha}} \frac{d\alpha}{(\epsilon + C_1(\alpha - \overline{\alpha}))^{1+\frac{1}{\lambda p}}} + \int_{\overline{\alpha}}^{\overline{\alpha}+r} \frac{d\alpha}{(\epsilon - C_1(\alpha - \overline{\alpha}))^{1+\frac{1}{\lambda p}}} \\ &= \frac{2\lambda p}{|C_1|}\left(\epsilon^{-\frac{1}{\lambda p}} - (\epsilon - C_1 r)^{-\frac{1}{\lambda p}}\right)\end{aligned}$$

for $p \geq 1$, and so

$$\int_{\overline{\alpha}-r}^{\overline{\alpha}+r} \frac{d\alpha}{(\epsilon + M_0 - u_0'(\alpha))^{1+\frac{1}{\lambda p}}} \sim C\epsilon^{-\frac{1}{\lambda p}} \tag{3.3.31}$$

for small $\epsilon > 0$. Then, setting $\epsilon = \frac{1}{\lambda \eta} - M_0$ into (3.3.31) we conclude that

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda p}}} \sim \frac{C}{\mathcal{J}(\overline{\alpha},t)^{\frac{1}{\lambda p}}} \tag{3.3.32}$$

for $\eta_* - \eta > 0$ small, $\lambda > 0$, $p \geq 1$ and $\mathcal{J}(\overline{\alpha},t) = 1 - \lambda\eta(t)M_0$. Next, we use a similar argument to obtain, for $p \geq 1$, the following estimates

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{\frac{1}{\lambda p}}} \sim \begin{cases} C\mathcal{J}(\overline{\alpha},t)^{1-\frac{1}{\lambda p}}, & \lambda \in (0, 1/p), \\ -C\log(\eta_* - \eta), & \lambda = 1/p, \\ C, & \lambda > 1/p \end{cases} \tag{3.3.33}$$

and

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{p+\frac{1}{\lambda}}} \sim C\mathcal{J}(\overline{\alpha},t)^{1-p-\frac{1}{\lambda}}, \qquad \lambda > 0. \tag{3.3.34}$$

In (3.3.32), (3.3.33)i) and (3.3.34) above, the positive constants C are given by

$$\frac{2\lambda p M_0}{|C_1|}, \qquad \frac{2\lambda p M_0}{|C_1|(1-\lambda p)}, \qquad \frac{2\lambda M_0}{|C_1|(\lambda(p-1)+1)} \tag{3.3.35}$$

respectively, for λ and p as specified in the corresponding estimate.

Suppose $\lambda, p > 1$ so that $\lambda > 1/p$. Then, using (3.3.17)i), (3.3.18), (3.3.32) and (3.3.33)iii) on (3.1.64) implies that

$$\begin{aligned}\|u_x(\cdot,t)\|_p &\geq \frac{1}{|\lambda\eta(t)|\,\bar{\mathcal{K}}_0(t)^{2\lambda+\frac{1}{p}}}\left|\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda p}}} - \frac{\bar{\mathcal{K}}_1(t)}{\bar{\mathcal{K}}_0(t)}\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{\frac{1}{\lambda p}}}\right| \\ &\sim C\left|C\mathcal{J}(\overline{\alpha},t)^{-\frac{1}{\lambda p}} - \mathcal{J}(\overline{\alpha},t)^{-\frac{1}{\lambda}}\right| \\ &\sim C\mathcal{J}(\overline{\alpha},t)^{-\frac{1}{\lambda}}\end{aligned}$$

for $\eta_* - \eta > 0$ small. Therefore, $\|u_x\|_p \to +\infty$ as $\eta \uparrow \eta_*$ for all $\lambda, p > 1$.

Now, suppose $\lambda \in (1/2, 1/p)$ for $p \in (1,2)$, so that, relative to the value of p, $\lambda \in (1/2, 1)$. Then, using (3.3.17)ii), (3.3.18), (3.3.32), (3.3.33)i) and (3.3.35) on (3.1.64) we now have

$$\begin{aligned}\|u_x(\cdot,t)\|_p &\geq \frac{1}{|\lambda\eta(t)|\,\bar{\mathcal{K}}_0(t)^{2\lambda+\frac{1}{p}}}\left|\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda p}}} - \frac{\bar{\mathcal{K}}_1(t)}{\bar{\mathcal{K}}_0(t)}\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{\frac{1}{\lambda p}}}\right| \\ &\sim C\left|1 - \frac{1-\lambda}{1-\lambda p}\right|\mathcal{J}(\overline{\alpha},t)^{\rho(\lambda,p)} \\ &= C\mathcal{J}(\overline{\alpha},t)^{\rho(\lambda,p)}\end{aligned}$$

for $\eta_* - \eta > 0$ small and $\rho(\lambda,p) = 2(1-\lambda) - \frac{1}{p}$. However, for λ and p as specified above, we have that $\rho(\lambda,p) < 0$ for $1 - \frac{1}{2p} < \lambda < \frac{1}{p}$ and $p \in (1, 3/2)$. Therefore, for any $\lambda \in (1/2, 1)$, there is $1 - p > 0$ arbitrarily small such that $\|u_x\|_p \to +\infty$ as $\eta \uparrow \eta_*$. Finally, if $\lambda = 1$ we have $\lambda > 1/p$ for $p > 1$, therefore (3.3.18), (3.3.25), (3.3.32) and (3.3.33)iii) imply that for $0 < \eta_* - \eta << 1$ small

$$\begin{aligned}\|u_x(\cdot,t)\|_p &\geq \frac{1}{|\lambda\eta(t)|\,\bar{\mathcal{K}}_0(t)^{2\lambda+\frac{1}{p}}}\left|\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda p}}} - \frac{\bar{\mathcal{K}}_1(t)}{\bar{\mathcal{K}}_0(t)}\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{\frac{1}{\lambda p}}}\right| \\ &\sim C\mathcal{J}(\overline{\alpha},t)^{-1}(-\log(\eta_* - \eta))^{-3-\frac{1}{p}},\end{aligned}$$

and so $\|u_x\|_p \to +\infty$ as $\eta \uparrow \eta_*$. The existence of a finite blow-up time $t_* > 0$ follows from Theorem 3.3.14. This concludes the proof of statement *1*.

Proof of Statement *2*

Suppose $\lambda < 0$, set $\eta_* = \frac{1}{\lambda m_0}$ and assume that $u_0'(\alpha)$ is bounded, at least $C^0(0,1)$ *a.e.*, and satisfies (3.3.11) with $q = 1$. First of all, recall from the proof of Theorem 3.3.14 that both integral terms $\bar{\mathcal{K}}_i(t)$, $i = 0, 1$ remain finite, and positive, for all $\eta \in [0, \eta_*]$ and $\lambda < 0$. Furthermore, in Theorem 3.3.14, we established the existence of a finite $t_* > 0$ such that

$\lim_{t\uparrow t_*} \|u_x\|_\infty = +\infty$ for all $\lambda < 0$.[20] These remarks, along with the upper bound (3.1.63), imply that

$$\lim_{t\uparrow t_*} \|u_x\|_p < +\infty \quad \Leftrightarrow \quad \lim_{t\uparrow t_*} \int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{p+\frac{1}{\lambda}}} < +\infty, \qquad p \geq 1. \tag{3.3.36}$$

However, if $p = 1$,

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{p+\frac{1}{\lambda}}} = \bar{\mathcal{K}}_1(t),$$

which remains finite as $t \uparrow t_*$. As a result

$$\lim_{t\uparrow t_*} \|u_x(\cdot,t)\|_1 < +\infty$$

for all $\lambda < 0$. If $p > 1$, we need to estimate the integral. Assume for simplicity that u_0' attains its least value $m_0 < 0$ only at one location $\underline{\alpha} \in (0,1)$. Then for $q = 1$ and some $\epsilon > 0$, (3.3.11) implies

$$\begin{aligned}
\int_{\underline{\alpha}-r}^{\underline{\alpha}+r} \frac{d\alpha}{(\epsilon + u_0'(\alpha) - m_0)^{p+\frac{1}{\lambda}}} &\sim \int_{\underline{\alpha}-r}^{\underline{\alpha}+r} \frac{d\alpha}{(\epsilon + C_2\,|\alpha - \underline{\alpha}|)^{p+\frac{1}{\lambda}}} \\
&= \int_{\underline{\alpha}-r}^{\underline{\alpha}} \frac{d\alpha}{(\epsilon - C_2(\alpha - \underline{\alpha}))^{p+\frac{1}{\lambda}}} + \int_{\underline{\alpha}}^{\underline{\alpha}+r} \frac{d\alpha}{(\epsilon + C_2(\alpha - \underline{\alpha}))^{p+\frac{1}{\lambda}}} \\
&= \frac{2\,|\lambda|}{C_2(1+\lambda(p-1))} \left((\epsilon + C_2 r)^{1-p-\frac{1}{\lambda}} - \epsilon^{1-p-\frac{1}{\lambda}}\right).
\end{aligned}$$

Substituting $\epsilon = m_0 - \frac{1}{\lambda\eta}$ into the above, we obtain

$$\int_{\underline{\alpha}-r}^{\underline{\alpha}+r} \frac{d\alpha}{\mathcal{J}(\alpha,t)^{p+\frac{1}{\lambda}}} \sim \frac{2\,|\lambda|}{C_2(1+\lambda(p-1))} \left(C - |m_0|\,\mathcal{J}(\underline{\alpha},t)^{1-p-\frac{1}{\lambda}}\right) \tag{3.3.37}$$

for $\eta_* - \eta > 0$ small. Suppose $\frac{1}{1-p} < \lambda < 0$ for $p > 1$. Then $1 - p - \frac{1}{\lambda} > 0$ and the integral remains finite as $t \uparrow t_*$. Consequently, (3.3.36) implies that

$$\lim_{t\uparrow t_*} \|u_x(\cdot,t)\|_p < +\infty$$

for all $\frac{1}{1-p} < \lambda < 0$ and $p > 1$. This establishes *2.* We remark that the lower bound (3.1.64) yields no information regarding L^p blow-up of u_x, as $t \uparrow t_*$, for parameter values $-\infty < \lambda < \frac{1}{1-p}$ $p > 1$. However, we may still use (3.1.60) and (3.1.61) in section 3.1.4 to obtain additional blow-up information on energy-related quantities.

[20] More particularly, we showed that only the minimum blows up, $m(t) \to -\infty$, as $t \uparrow t_*$.

Proof of Statement *3*

From Theorem 3.3.14, $u_x \in L^\infty$ for all time when $\lambda \in [0, 1/2]$. Therefore, $E(t)$ exist globally for these values of the parameter. Likewise, *3.1.78* implies that $\dot{E}(t)$ persists globally for $\lambda \in [0, 1/2]$. Now, blow-up of $E(t)$ and $\dot{E}(t)$ to $+\infty$, as $t \uparrow t_*$, for $\lambda > 1/2$ is a consequence of *1* above. Furthermore, setting $p = 2$ into part *2* implies that $E(t)$ remains bounded for all $\lambda \in (-1, 0)$ and $t \in [0, t_*]$. Similarly for $p = 3$, we use part *2* and (3.1.78) to conclude that $\dot{E}(t)$ remains finite when $\lambda \in [-1/2, 0)$ and $t \in [0, t_*]$. According to these results, we have yet to determine the behaviour of $E(t)$ as $t \uparrow t_*$ for $\lambda \leq -1$, as well as that of $\dot{E}(t)$ when $\lambda < -1/2$. To do so, we will use formulas (3.1.60) and (3.1.61). Following the usual argument[21], the details of which we omit this time, we derive the following estimates

$$\bar{\mathcal{K}}_2(t) \sim \begin{cases} C\mathcal{J}(\underline{\alpha}, t)^{-1-\frac{1}{\lambda}}, & \lambda < -1, \\ -C\log(\eta_* - \eta), & \lambda = -1, \\ C, & \lambda \in (-1, 0) \end{cases} \tag{3.3.38}$$

and

$$\bar{\mathcal{K}}_3(t) \sim \begin{cases} C\mathcal{J}(\underline{\alpha}, t)^{-2-\frac{1}{\lambda}}, & \lambda < -1/2, \\ -C\log(\eta_* - \eta), & \lambda = -1/2, \\ C, & \lambda \in (-1/2, 0) \end{cases} \tag{3.3.39}$$

for $\eta_* - \eta > 0$ small. The constants $C \in \mathbb{R}^+$ in (3.3.38)i) and (3.3.39)i) are given by

$$\frac{2\lambda\,|m_0|}{C_2(1+\lambda)}, \qquad \frac{2\lambda\,|m_0|}{C_2(1+2\lambda)},$$

respectively, for λ as specified by the corresponding estimate. Since both $\bar{\mathcal{K}}_i(t)$, $i = 0, 1$ stay finite and positive for all $\eta \in [0, \eta_*]$, formula (3.1.60) tells us that blow-up in $\bar{\mathcal{K}}_2(t)$ leads to a diverging $E(t)$. Then, (3.3.38)i) implies that for $\lambda < -1$,

$$E(t) \sim C\mathcal{J}(\underline{\alpha}, t)^{-1-\frac{1}{\lambda}} \to +\infty$$

as $\eta \uparrow \eta_*$. Similarly for $\lambda = -1$ by using (3.3.38)ii) instead. Clearly, this also implies blow-up of $\dot{E}(t)$ to $+\infty$ as $t \uparrow t_*$ for all $\lambda \leq -1$. Finally, from (3.1.61)iii), (3.3.38)iii) and (3.3.39)i),

$$\dot{E}(t) \sim \frac{Cm_0^3(1+2\lambda)}{\mathcal{J}(\underline{\alpha}, t)^{2+\frac{1}{\lambda}}} \to +\infty$$

as $\eta \uparrow \eta_*$ for all $\lambda \in (-1, -1/2)$. The existence of a finite $t_* > 0$ follows from *3* in Theorem 3.3.14. □

[21] See for instance the argument that led to estimates (3.3.26) and (3.3.37).

Remark 3.3.40. Notice from Theorem 3.3.14 that the values of λ for which u_x undergoes its "strongest" type of L^∞ blow-up, the two-sided everywhere blow-up, agrees with those λ in Theorem 3.3.30 for which the "strongest" form of L^p blow-up takes place, an L^p blow-up for $1-p>0$ arbitrarily small. On the other hand, in Theorem 3.3.14 we also showed that, for $\lambda<0$, u_x undergoes its "weakest" type of L^∞ blow-up, a one-sided, discrete blow-up. In this case, however, Theorem 3.3.30 tells us that u_x remains integrable for $t\in[0,t_*]$, while, for $p>1$ and $\frac{1}{1-p}\leq\lambda<0$, it stays in L^p for all $t\in[0,t_*]$. As we will see in the remaining sections, this type of interaction between the "strength" of the L^∞ blow-up and the $L^p, p\in[1,+\infty)$ regularity of u_x also holds in the general case of $q>0$.

3.3.3 Global Estimates and Blow-up for $\lambda\neq 0$ and $q>0$

In this last section, we treat the more general case of initial data satisfying (3.3.10) and/or (3.3.11) for arbitrary $q\in\mathbb{R}^+$. Amongst other results, we will examine the L^p regularity of u_x for $\lambda\in\mathbb{R}\backslash\{0\}$, $q>0$ and $p\in[1,+\infty]$. More particularly, depending on the sign of $\lambda\neq 0$, regularity of u_x in the L^∞ norm is first examined. Then, for the cases leading to L^∞ blow-up as t approaches some finite $t_*>0$, the behaviour of $\lim_{t\uparrow t_*}\|u_x\|_p$ for $p\in[1,+\infty)$ is studied. Moreover, the jacobian (2.1.14) is also considered. Finally, a larger class of initial data than the one examined in section 3.1 is discussed. Before stating and proving our results, we first establish Lemma 3.3.41 below which we use to obtain estimates on the behaviour of several time-dependent integrals for $\eta_*-\eta>0$ small.

Lemma 3.3.41. *Suppose $u_0'(\alpha)$ is bounded, at least $C^0(0,1)$ a.e., and for some $q\in\mathbb{R}^+$ satisfies (3.3.10) when $\lambda\in\mathbb{R}^+$, or (3.3.11) if $\lambda\in\mathbb{R}^-$. It holds:*

1. *If $\lambda\in\mathbb{R}^+$ and $b>\frac{1}{q}$,*

$$\int_0^1\frac{d\alpha}{\mathcal{J}(\alpha,t)^b}\sim\frac{C}{\mathcal{J}(\overline{\alpha}_i,t)^{b-\frac{1}{q}}}\tag{3.3.42}$$

for $\eta_*-\eta>0$ small, $\eta_*=\frac{1}{\lambda M_0}$ and positive constants C given by

$$C=\frac{2m\Gamma\left(1+\frac{1}{q}\right)\Gamma\left(b-\frac{1}{q}\right)}{\Gamma\left(b\right)}\left(\frac{M_0}{|C_1|}\right)^{\frac{1}{q}}.\tag{3.3.43}$$

Here, $m\in\mathbb{N}$ denotes the finite number of locations $\overline{\alpha}_i$ in $[0,1]$.

2. *If $\lambda\in\mathbb{R}^-$ and $b>\frac{1}{q}$,*

$$\int_0^1\frac{d\alpha}{\mathcal{J}(\alpha,t)^b}\sim\frac{C}{\mathcal{J}(\underline{\alpha}_j,t)^{b-\frac{1}{q}}}\tag{3.3.44}$$

for $\eta_* - \eta > 0$ *small,* $\eta_* = \frac{1}{\lambda m_0}$ *and positive constants* C *determined by*

$$C = \frac{2n\Gamma\left(1+\frac{1}{q}\right)\Gamma\left(b-\frac{1}{q}\right)}{\Gamma(b)}\left(\frac{|m_0|}{C_2}\right)^{\frac{1}{q}}. \tag{3.3.45}$$

Above, $n \in \mathbb{N}$ *represents the finite number of points* $\underline{\alpha}_j$ *in* $[0,1]$*.*

3. Suppose $q > 1/2$ *and* $b \in (0, 1/q)$*, or* $q \in (0, 1/2)$ *and* $b \in (0,2)$*, satisfy* $\frac{1}{q}$*,* b*,* $b-\frac{1}{q} \notin \mathbb{Z}$*. Then for* $\lambda \neq 0$ *and* η_* *as defined in (3.0.6),*

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^b} \sim C \tag{3.3.46}$$

for $\eta_* - \eta > 0$ *small and positive constants* C *that depend on the choice of* λ*,* b *and* q*. Similarly, the integral remains bounded, and positive, for all* $\eta \in [0, \eta_*]$ *and* $\lambda \neq 0$ *when* $b \leq 0$ *and* $q > 0$*.*

Proof.

Proof of Statement *1*

For simplicity, we prove statement *1* for functions u_0' that attain their greatest value $M_0 > 0$ at a single location $\overline{\alpha} \in (0,1)$. By a slight modification of the argument below, the Lemma can be shown to hold for several $\overline{\alpha}_i \in [0,1]$. Using (3.3.10), there is $0 < r \leq 1$ small enough such that

$$\epsilon + M_0 - u_0'(\alpha) \sim \epsilon - C_1 |\alpha - \overline{\alpha}|^q$$

for $q \in \mathbb{R}^+$, $\epsilon > 0$ and $0 \leq |\alpha - \overline{\alpha}| \leq r$. Therefore

$$\begin{aligned}\int_{\overline{\alpha}-r}^{\overline{\alpha}+r} \frac{d\alpha}{(\epsilon + M_0 - u_0'(\alpha))^b} &\sim \int_{\overline{\alpha}-r}^{\overline{\alpha}+r} \frac{d\alpha}{(\epsilon - C_1|\alpha-\overline{\alpha}|^q)^b} \\ &= \epsilon^{-b}\left[\int_{\overline{\alpha}-r}^{\overline{\alpha}} \left(1 + \frac{|C_1|}{\epsilon}(\overline{\alpha}-\alpha)^q\right)^{-b} d\alpha + \int_{\overline{\alpha}}^{\overline{\alpha}+r} \left(1 + \frac{|C_1|}{\epsilon}(\alpha-\overline{\alpha})^q\right)^{-b} d\alpha\right]\end{aligned}$$

for $b \in \mathbb{R}$. Making the change of variables

$$\sqrt{\frac{|C_1|}{\epsilon}}(\overline{\alpha}-\alpha)^{\frac{q}{2}} = \tan\theta, \qquad \sqrt{\frac{|C_1|}{\epsilon}}(\alpha-\overline{\alpha})^{\frac{q}{2}} = \tan\theta$$

in the first and second integrals inside the bracket, respectively, we find after simplification that

$$\int_{\overline{\alpha}-r}^{\overline{\alpha}+r} \frac{d\alpha}{(\epsilon + M_0 - u_0'(\alpha))^b} \sim \frac{4}{q|C_1|^{\frac{1}{q}}\epsilon^{b-\frac{1}{q}}} \int_0^{\frac{\pi}{2}} \frac{(\cos\theta)^{2b-\frac{2}{q}-1}}{(\sin\theta)^{1-\frac{2}{q}}} d\theta \tag{3.3.47}$$

for small $\epsilon > 0$. Suppose $b > \frac{1}{q}$, then setting $\epsilon = \frac{1}{\lambda\eta} - M_0$ into (3.3.47) implies

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^b} \sim \frac{C}{\mathcal{J}(\overline{\alpha},t)^{b-\frac{1}{q}}} \tag{3.3.48}$$

for $\eta_* - \eta > 0$ small, $\eta_* = \frac{1}{\lambda M_0}$, $\mathcal{J}(\overline{\alpha},t) = 1 - \lambda\eta(t)M_0$ and

$$C = \frac{4}{q}\left(\frac{M_0}{|C_1|}\right)^{\frac{1}{q}} \int_0^{\frac{\pi}{2}} \frac{(\cos\theta)^{2b-\frac{2}{q}-1}}{(\sin\theta)^{1-\frac{2}{q}}} d\theta. \tag{3.3.49}$$

Now, recall that for $p, s, y > 0$ (see for instance [24]),

$$\int_0^1 t^{p-1}(1-t)^{s-1}dt = \frac{\Gamma(p)\Gamma(s)}{\Gamma(p+s)}, \qquad \Gamma(1+y) = y\Gamma(y), \tag{3.3.50}$$

where (3.3.50)i) is commonly known as the Beta function. Therefore, letting $t = \sin^2\theta$, $p = \frac{1}{q}$ and $s = b - \frac{1}{q}$ into (3.3.50)i), and using (3.3.50)ii), one gets

$$2\int_0^{\frac{\pi}{2}} \frac{(\cos\theta)^{2b-\frac{2}{q}-1}}{(\sin\theta)^{1-\frac{2}{q}}} d\theta = \frac{q\,\Gamma\left(1+\frac{1}{q}\right)\Gamma\left(b-\frac{1}{q}\right)}{\Gamma(b)}, \qquad b > \frac{1}{q}. \tag{3.3.51}$$

The result follows from (3.3.48), (3.3.49) and (3.3.51).

Proof of Statement *2*

Follows from an argument analogous to the one above by using (3.3.11) instead of (3.3.10).

Proof of Statement *3*

The last claim in statement *3* follows trivially if $b \leq 0$ and $q \in \mathbb{R}^+$ due to the boundedness and almost everywhere continuity of u_0' in $[0,1]$. To establish the remaining claims, we make use of Lemmas 3.0.9 and 3.0.11. However, in order to use the latter, we require that $b \in (0,2)$ and $b \neq 1/q$. Since $b > 0$ and the case $b > 1/q$ was established in parts (1) and (2), suppose that $b \in (0, 1/q)$ and $b \in (0,2)$. This is equivalent to having either $q > 1/2$ and $b \in (0, 1/q)$, or $q \in (0, 1/2)$ and $b \in (0,2)$.

First, for q and b as above, we consider the case of $\lambda > 0$. Also, for simplicity, suppose that u_0' attains its greatest value at a single point $\overline{\alpha} \in (0,1)$. Then, by (3.3.10) and Lemma 3.0.11, there is $0 < r \leq 1$ small enough such that

$$\begin{aligned}\int_{\overline{\alpha}-r}^{\overline{\alpha}+r} \frac{d\alpha}{(\epsilon + M_0 - u_0'(\alpha))^b} &\sim \int_{\overline{\alpha}-r}^{\overline{\alpha}+r} \frac{d\alpha}{(\epsilon - C_1\,|\alpha-\overline{\alpha}|^q)^b} \\ &= 2r\epsilon^{-b}\,{}_2F_1\left[\frac{1}{q}, b, 1+\frac{1}{q}, \frac{C_1 r^q}{\epsilon}\right]\end{aligned} \tag{3.3.52}$$

for $\epsilon \geq |C_1| \geq |C_1| r^q > 0$ and $0 \leq |\alpha - \overline{\alpha}| \leq r$. Now, the restriction on ϵ implies that $-1 \leq \frac{C_1 r^q}{\epsilon} < 0$. However, our ultimate goal is to let ϵ vanish, so that, eventually, the argument $\frac{C_1 r^q}{\epsilon}$ of the series in (3.3.52)ii) will leave the unit circle, particularly $\frac{C_1 r^q}{\epsilon} < -1$. At that point, definition (3.0.7) for the series no longer holds and we turn to its analytic continuation in Lemma 3.0.9. Accordingly, taking $\epsilon > 0$ small enough such that $|C_1| r^q > \epsilon > 0$, we apply Lemma 3.0.9 to (3.3.52) and obtain

$$\frac{2r}{\epsilon^b} {}_2F_1\left[\frac{1}{q}, b, 1+\frac{1}{q}, \frac{C_1 r^q}{\epsilon}\right] = \frac{2r^{1-qb}}{(1-bq)|C_1|^b} + \frac{2\Gamma\left(1+\frac{1}{q}\right)\Gamma\left(b-\frac{1}{q}\right)}{\Gamma(b)|C_1|^{\frac{1}{q}}\epsilon^{b-\frac{1}{q}}} + \psi(\epsilon) \tag{3.3.53}$$

for $\psi(\epsilon) = o(1)$ as $\epsilon \to 0$, and either $q > 1/2$ and $b \in (0, 1/q)$, or $q \in (0, 1/2)$ and $b \in (0, 2)$. In addition, due to the assumptions in Lemma 3.0.9 we also require that $\frac{1}{q}$, b, $b - \frac{1}{q} \notin \mathbb{Z}$. Finally, since $b - \frac{1}{q} < 0$, upon substituting $\epsilon = \frac{1}{\lambda\eta} - M_0$ into (3.3.52) and (3.3.53), we conclude that

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha, t)^b} \sim C \tag{3.3.54}$$

for $\eta_* - \eta > 0$ small, $\eta_* = \frac{1}{\lambda M_0}$ and positive constants C that depend on the choice of $\lambda > 0$, b and q as above. An analogous argument may be used if $\lambda < 0$ by using (3.3.11) instead of (3.3.10). □

Estimates for $\bar{\mathcal{K}}_i(t)$, $i = 0, 1$ with $\lambda \neq 0$ and $q \in \mathbb{R}^+$

For parameters $\lambda > 0$

Setting $b = \frac{1}{\lambda}$ into *1* and *3* of Lemma 3.3.41, we find that for $\lambda > 0$ and $\eta_* - \eta > 0$ small,

$$\bar{\mathcal{K}}_0(t) \sim \begin{cases} C, & \lambda > q > \frac{1}{2}, \quad \text{or} \quad q \in (0, 1/2), \quad \lambda > \frac{1}{2}, \\ C_6 \mathcal{J}(\overline{\alpha}_i, t)^{\frac{1}{q} - \frac{1}{\lambda}}, & q > 0, \quad \lambda \in (0, q) \end{cases} \tag{3.3.55}$$

where the positive constants $C_6 > 0$ are given by

$$C_6 = \frac{2m\Gamma\left(1+\frac{1}{q}\right)\Gamma\left(\frac{1}{\lambda}-\frac{1}{q}\right)}{\Gamma\left(\frac{1}{\lambda}\right)} \left(\frac{M_0}{|C_1|}\right)^{\frac{1}{q}}, \tag{3.3.56}$$

and for (3.3.55)i) we assume that λ and q satisfy, when applicable,

$$\lambda \neq \frac{q}{1-nq}, \quad q \neq \frac{1}{n} \quad \forall \quad n \in \mathbb{N}. \tag{3.3.57}$$

Similarly, by letting $b = 1 + \frac{1}{\lambda}$, one finds that

$$\bar{\mathcal{K}}_1(t) \sim \begin{cases} C, & q \in (1/2, 1), \quad \lambda > \frac{q}{1-q} \quad \text{or} \quad q \in (0, 1/2),\ \lambda > 1, \\ C_7 \mathcal{J}(\overline{\alpha}_i, t)^{\frac{1}{q} - \frac{1}{\lambda} - 1}, & q \in (0,1), \quad 0 < \lambda < \frac{q}{1-q} \quad \text{or} \quad q \geq 1, \quad \lambda > 0 \end{cases} \tag{3.3.58}$$

with positive constants C_7 determined by

$$C_7 = \frac{2m\Gamma\left(1 + \frac{1}{q}\right)\Gamma\left(1 + \frac{1}{\lambda} - \frac{1}{q}\right)}{\Gamma\left(1 + \frac{1}{\lambda}\right)} \left(\frac{M_0}{|C_1|}\right)^{\frac{1}{q}}. \tag{3.3.59}$$

Additionally, for (3.3.58)i) we assume that λ and q satisfy (3.3.57).

For parameters $\lambda < 0$

For $\lambda < 0$ and $b = \frac{1}{\lambda}$, we use *3* of Lemma 3.3.41 to conclude that

$$\bar{\mathcal{K}}_0(t) \sim C \tag{3.3.60}$$

for $\eta_* - \eta > 0$ small, $q > 0$ and $\lambda < 0$. Moreover, *2* and *3* of Lemma 3.3.41, now with $b = 1 + \frac{1}{\lambda}$ and $\lambda < 0$, imply that

$$\bar{\mathcal{K}}_1(t) \sim C \tag{3.3.61}$$

for either

$$\begin{cases} q > 0, & \lambda \in [-1, 0), \\ q \in (0,1), & \lambda < -1 \quad \text{satisfying} \ \ (3.3.57), \\ q > 1, & \frac{q}{1-q} < \lambda < -1, \end{cases} \tag{3.3.62}$$

whereas

$$\bar{\mathcal{K}}_1(t) \sim C_8 \mathcal{J}(\underline{\alpha}_j, t)^{\frac{1}{q} - \frac{1}{\lambda} - 1} \tag{3.3.63}$$

for $q > 1,\ \lambda < \frac{q}{1-q}$ and positive constants C_8 determined by

$$C_8 = \frac{2n\Gamma\left(1 + \frac{1}{q}\right)\Gamma\left(1 + \frac{1}{\lambda} - \frac{1}{q}\right)}{\Gamma\left(1 + \frac{1}{\lambda}\right)} \left(\frac{|m_0|}{C_2}\right)^{\frac{1}{q}}. \tag{3.3.64}$$

L^∞ **Regularity for** $\lambda, q \in \mathbb{R}^+$

In this section, we use the estimates obtained for $\bar{\mathcal{K}}_i(t)$, $i = 0, 1$ in the previous section to examine the L^∞ regularity of u_x for $\lambda > 0$ and bounded, at least continuous *a.e.* u_0' satisfying (3.3.10) for some $q \in \mathbb{R}^+$. Furthermore, the behaviour of the jacobian (2.1.14) is also studied.

Theorem 3.3.65. *Consider the initial boundary value problem (1.1.1)-(1.1.2) or (1.1.3) for* $u_0'(\alpha)$ *bounded, at least* $C^0(0,1)$ *a.e., and satisfying estimate (3.3.10).*

1. *For* $q \in \mathbb{R}^+$ *and* $\lambda \in [0, q/2]$*, solutions exist globally in time. More particularly, these vanish as* $t \uparrow t_* = +\infty$ *for* $\lambda \in (0, q/2)$ *but converge to a nontrivial steady state if* $\lambda = q/2$*.*

2. *For* $q \in \mathbb{R}^+$ *and* $\lambda \in (q/2, q)$*, there exists a finite* $t_* > 0$ *such that both the maximum* $M(t)$ *and the minimum* $m(t)$ *diverge to* $+\infty$ *and respectively to* $-\infty$ *as* $t \uparrow t_*$*. Additionally, for* $\alpha \notin \{\overline{\alpha}_i, \underline{\alpha}_j\}$*,* $\lim_{t\uparrow t_*} u_x(\gamma(\alpha,t),t) = -\infty$ *(two-sided, everywhere blow-up).*

3. *For* $q \in (0, 1/2)$ *and* $\lambda > 1$ *such that* $q \neq \frac{1}{n}$ *and* $\lambda \neq \frac{q}{1-nq}$ *for all* $n \in \mathbb{N}$*, there is a finite* $t_* > 0$ *such that only the maximum blows up,* $M(t) \to +\infty$*, as* $t \uparrow t_*$ *(one-sided, discrete blow-up). Further, if* $\frac{1}{2} < \lambda < \frac{q}{1-q}$ *for* $q \in (1/3, 1/2)$*, a two-sided, everywhere blow-up (as described in (2) above) occurs as* t *approaches some finite* $t_* > 0$*.*

4. *Suppose* $q \in (1/2, 1)$*. Then for* $q < \lambda < \frac{q}{1-q}$*, there exists a finite* $t_* > 0$ *such that, as* $t \uparrow t_*$*, two-sided, everywhere blow-up develops. If instead* $\lambda > \frac{q}{1-q}$*, only the maximum diverges,* $M(t) \to +\infty$*, as* $t \uparrow t_* < +\infty$*.*

5. *For* $\lambda > q > 1$*, there is a finite* $t_* > 0$ *such that* u_x *undergoes a two-sided, everywhere blow-up as* $t \uparrow t_*$*.*

Proof. Suppose $\lambda, q > 0$, let C denote a positive constant which may depend on λ and q, and set $\eta_* = \frac{1}{\lambda M_0}$.

Proof of Statements *1* **and** *2*

Suppose $\lambda \in (0, q)$ for some $q > 0$. Then, for $\eta_* - \eta > 0$ small, $\bar{\mathcal{K}}_0(t)$ satisfies (3.3.55)ii) while $\bar{\mathcal{K}}_1(t)$ obeys (3.3.58)ii). Consequently, (2.1.19) implies that

$$u_x(\gamma(\alpha,t),t) \sim \frac{M_0}{C_6^{2\lambda}} \left(\frac{\mathcal{J}(\overline{\alpha}_i,t)}{\mathcal{J}(\alpha,t)} - \frac{C_7}{C_6} \right) \mathcal{J}(\overline{\alpha}_i,t)^{1-\frac{2\lambda}{q}} \tag{3.3.66}$$

for positive constants C_6 and C_7 given by (3.3.56) and (3.3.59). But for $y_1 = \frac{1}{\lambda} - \frac{1}{q}$ and $y_2 = \frac{1}{\lambda}$, (3.3.50)ii), (3.3.56) and (3.3.59) yield

$$\frac{C_7}{C_6} = \frac{\Gamma(y_1+1)\,\Gamma(y_2)}{\Gamma(y_1)\,\Gamma(y_2+1)} = \frac{y_1}{y_2} = 1 - \frac{\lambda}{q} \in (0,1), \qquad \lambda \in (0,q). \tag{3.3.67}$$

As a result, setting $\alpha = \overline{\alpha}_i$ into (3.3.66) and using (3.0.4) implies that

$$M(t) \sim \frac{M_0}{C_6^{2\lambda}} \left(\frac{\lambda}{q}\right) \mathcal{J}(\overline{\alpha}_i, t)^{1-\frac{2\lambda}{q}} \tag{3.3.68}$$

for $\eta_* - \eta > 0$ small, whereas, if $\alpha \neq \overline{\alpha}_i$,

$$u_x(\gamma(\alpha,t),t) \sim -\frac{M_0}{C_6^{2\lambda}} \left(1 - \frac{\lambda}{q}\right) \mathcal{J}(\overline{\alpha}_i, t)^{1-\frac{2\lambda}{q}}. \tag{3.3.69}$$

Clearly, when $\lambda = q/2$,

$$M(t) \to \frac{M_0}{2C_6^q} > 0$$

as $\eta \uparrow \eta_*$, while, for $\alpha \neq \overline{\alpha}_i$,

$$u_x(\gamma(\alpha,t),t) \to -\frac{M_0}{2C_6^q} < 0.$$

If $\lambda \in (0, q/2)$, (3.3.68) now implies that

$$M(t) \to 0^+$$

as $\eta \uparrow \eta_*$, whereas, using (3.3.69) for $\alpha \neq \overline{\alpha}_i$,

$$u_x(\gamma(\alpha,t),t) \to 0^-.$$

In contrast, if $\lambda \in (q/2, q)$, $1 - \frac{2\lambda}{q} < 0$. Then (3.3.68) and (3.3.69) yield

$$M(t) \to +\infty \tag{3.3.70}$$

as $\eta \uparrow \eta_*$, but

$$u_x(\gamma(\alpha,t),t) \to -\infty \tag{3.3.71}$$

for $\alpha \neq \overline{\alpha}_i$. Lastly, rewriting (2.1.16) as

$$dt = \bar{\mathcal{K}}_0(t)^{2\lambda} d\eta \tag{3.3.72}$$

and using (3.3.55)ii), we obtain

$$t_* - t \sim C \int_\eta^{\eta_*} (1 - \lambda\mu M_0)^{\frac{2\lambda}{q}-2} d\mu \tag{3.3.73}$$

or equivalently

$$t_* - t \sim \begin{cases} \frac{C}{2\lambda - q}\left(C(\eta_* - \eta)^{\frac{2\lambda}{q}-1} - \lim_{\mu\uparrow\eta_*}(\eta_* - \mu)^{\frac{2\lambda}{q}-1}\right), & \lambda \in (0,q)\backslash\{q/2\}, \\ C\left(\log(\eta_* - \eta) - \lim_{\mu\uparrow\eta_*}\log(\eta_* - \mu)\right), & \lambda = q/2. \end{cases} \tag{3.3.74}$$

Consequently, $t_* = +\infty$ for $\lambda \in (0, q/2]$ while $0 < t_* < +\infty$ if $\lambda \in (q/2, q)$.

Proof of Statement *3*

First, suppose $q \in (0, 1/2)$ and $\lambda > 1$ satisfy (3.3.57). Then $\bar{\mathcal{K}}_0(t)$ and $\bar{\mathcal{K}}_1(t)$ satisfy (3.3.55)i) and (3.3.58)i), respectively. Therefore, (2.1.19) implies that

$$u_x(\gamma(\alpha,t),t) \sim C\left(\frac{1}{\mathcal{J}(\alpha,t)} - C\right) \tag{3.3.75}$$

for $\eta_* - \eta > 0$ small. Set $\alpha = \overline{\alpha}_i$ into (3.3.75) and use (3.2.7)i) to find that

$$M(t) \sim \frac{C}{\mathcal{J}(\overline{\alpha}_i,t)} \to +\infty$$

as $\eta \uparrow \eta_*$. However, if $\alpha \neq \overline{\alpha}_i$, $u_x(\gamma(\alpha,t),t)$ remains finite for all $\eta \in [0,\eta_*]$ due to the definition of M_0. The existence of a finite blow-up time $t_* > 0$ for the maximum is guaranteed by (3.3.55)i) and (3.3.72), which lead to

$$t_* - t \sim C(\eta_* - \eta). \tag{3.3.76}$$

Next, suppose $\frac{1}{2} < \lambda < \frac{q}{1-q}$ for $q \in (1/3, 1/2)$, so that $\frac{q}{1-q} \in (1/2, 1)$. Then, using (3.3.55)i) and (3.3.58)ii) on (2.1.19), we find that

$$u_x(\gamma(\alpha,t),t) \sim C\left(\frac{C}{\mathcal{J}(\alpha,t)} - \mathcal{J}(\overline{\alpha}_i,t)^{\frac{1}{q}-\frac{1}{\lambda}-1}\right) \tag{3.3.77}$$

for $\eta_* - \eta > 0$ small. Consequently, setting $\alpha = \overline{\alpha}_i$ into the above and using $\lambda > q$, we find that

$$M(t) \sim \frac{C}{\mathcal{J}(\overline{\alpha}_i,t)} \to +\infty \tag{3.3.78}$$

as $\eta \uparrow \eta_*$. On the other hand, for $\alpha \neq \overline{\alpha}_i$, the space-dependent in (3.3.77) now remains bounded and positive for all $\eta \in [0, \eta_*]$. As a result, the second term dominates and

$$u_x(\gamma(\alpha,t),t) \sim -C\mathcal{J}(\overline{\alpha}_i,t)^{\frac{1}{q}-\frac{1}{\lambda}-1} \to -\infty \tag{3.3.79}$$

as $\eta \uparrow \eta_*$. The existence of a finite blow-up time $t_* > 0$, follows, as in the previous case, from (3.3.72) and (3.3.55)i).

Proof of Statement *4*

Part *4* follows from an argument analogous to the one above. Briefly, if $q < \lambda < \frac{q}{1-q}$ for $q \in (1/2, 1)$, we use estimates (3.3.55)i) and (3.3.58)ii) on (2.1.19) to get (3.3.77), with different positive constants C. A two-sided, everywhere blow-up in finite-time then follows just as above. If instead $\lambda > \frac{q}{1-q}$ for $q \in (1/2, 1)$, then (3.3.55)i) still holds but $\bar{\mathcal{K}}_1(t)$ now remains finite for all $\eta \in [0, \eta_*]$; it satisfies (3.3.58)i). Therefore, up to different positive constants C, (2.1.19) leads to (3.3.75), and so only the maximum diverges, $M(t) \to +\infty$, as t approaches some finite $t_* > 0$ whose existence is guaranteed by (3.3.76).

Proof of Statement *5*

For $\lambda > q > 1$, (3.3.55)i), (3.3.58)ii) and (2.1.19) imply (3.3.77). Then, we follow the argument used to establish the second part of *3* to show that two-sided, everywhere blow-up occurs at a finite time. See section 4.2 for examples. □

Remark 3.3.80. Theorems 3.3.14 and 3.3.65 allow us to predict the regularity of stagnation point-form (SPF) solutions to the two ($\lambda = 1$) and three ($\lambda = 1/2$) dimensional incompressible Euler equations assuming we know something about the curvature of the initial data u_0 near $\overline{\alpha}_i$. Setting $\lambda = 1$ into Theorem 3.3.65(1) implies that SPF solutions in the 2D setting persist for all time if u_0' is, at least, $C^0(0,1)$ *a.e.* and satisfies (3.3.10) for arbitrary $q \geq 2$. On the contrary, Theorems 3.3.14 and 3.3.65(2)-(4), tell us that if $q \in (1/2, 2)$, two-sided, everywhere blow-up in finite-time occurs instead. Analogously, solutions to the corresponding 3D problem exist globally in time for $q \geq 1$, whereas, two-sided, everywhere blow-up develops when $q \in (1/2, 1)$. See Table 3.2 below.

Table 3.2: Regularity of SPF solutions to Euler equations

q	2D Euler	3D Euler
$(1/2, 1)$	Finite time blow up	Finite time blow up
$[1, 2)$	Finite time blow up	Global in time
$[2, +\infty)$	Global in time	Global in time

Finally, we remark that finite-time blow-up in u_x is expected for both the two and three dimensional equations if $q \in (0, 1/2]$. See for instance section 4.2 for a blow-up example in the 3D case with $q = 1/3$.

Behaviour of the Jacobian for $\lambda, q \in \mathbb{R}^+$

Corollary 3.3.81 below briefly examines the behaviour, as $t \uparrow t_*$, of the jacobian (2.1.14) for $t_* > 0$ is as in Theorem 3.3.65.

Corollary 3.3.81. *Consider the initial boundary value problem (1.1.1)-(1.1.2) or (1.1.3) for $u_0'(\alpha)$, at least $C^0(0,1)$ a.e., satisfying (3.3.10) for some $q \in \mathbb{R}^+$. Furthermore, let $t_* > 0$ be as in Theorem 3.3.65. It follows,*

1. *For $q \in \mathbb{R}^+$ and $\lambda \in (0, q)$,*

$$\lim_{t \uparrow t_*} \gamma_\alpha(\alpha, t) = \begin{cases} +\infty, & \alpha = \overline{\alpha}_i, \\ 0, & \alpha \neq \overline{\alpha}_i \end{cases} \tag{3.3.82}$$

where $t_* = +\infty$ for $\lambda \in (0, q/2]$, while $0 < t_* < +\infty$ if $\lambda \in (q/2, q)$.

2. *Suppose $\lambda > q > 1/2$, or $q \in (0, 1/2)$ and $\lambda > 1/2$, satisfy (3.3.57). Then, there exists a finite $t_* > 0$ such that*

$$\lim_{t \uparrow t_*} \gamma_\alpha(\alpha, t) = \begin{cases} +\infty, & \alpha = \overline{\alpha}_i, \\ C, & \alpha \neq \overline{\alpha}_i \end{cases} \tag{3.3.83}$$

where the positive constants C depend on λ, q and $[0,1] \ni \alpha \neq \overline{\alpha}_i$.

Proof. Set $\eta_* = \frac{1}{\lambda M_0}$ for $\lambda > 0$.

Proof of Statement *1*

Suppose $\lambda \in (0, q)$ for $q > 0$. Then (2.1.14) and (3.3.55)ii) imply

$$\gamma_\alpha(\alpha, t) \sim \frac{1}{C_6} \frac{\mathcal{J}(\overline{\alpha}_i, t)^{\frac{1}{\lambda} - \frac{1}{q}}}{\mathcal{J}(\alpha, t)^{\frac{1}{\lambda}}}$$

for $\eta_* - \eta > 0$ small. Setting $\alpha = \overline{\alpha}_i$ then gives

$$\gamma_\alpha(\overline{\alpha}_i, t) \sim \frac{1}{C_6 \mathcal{J}(\overline{\alpha}_i, t)^{\frac{1}{q}}} \to +\infty$$

as $\eta \uparrow \eta_*$, whereas, for $\alpha \neq \overline{\alpha}_i$,

$$\gamma_\alpha(\alpha, t) \sim C \mathcal{J}(\overline{\alpha}_i, t)^{\frac{1}{\lambda} - \frac{1}{q}} \to 0.$$

The either finite or infinite character of $t_* > 0$ follows from Theorem 3.3.65.

Proof of Statement *2*

Now suppose $\lambda > q > 1/2$, or $\lambda > 1/2$ for any $q \in (0, 1/2)$, satisfy (3.3.57). Then (2.1.14) and (3.3.55)i) imply that

$$\gamma_\alpha(\alpha, t) \sim \frac{C}{\mathcal{J}(\alpha, t)^{\frac{1}{\lambda}}}$$

for $\eta_* - \eta > 0$ small. If $\alpha = \overline{\alpha}_i$, then $\gamma_\alpha(\overline{\alpha}_i, t) \to +\infty$ as $\eta \uparrow \eta_*$, whereas, for $\alpha \neq \overline{\alpha}_i$, the definition of M_0 implies that γ_α converges to some finite, positive constant C as $\eta \uparrow \eta_*$. Finally, the existence of a finite $t_* > 0$ follows from Theorem 3.3.65. □

Further L^p Regularity for $\lambda > q/2$, $p \in [1, +\infty)$ and $q \in \mathbb{R}^+$

Recall from Theorem 3.3.65 that for $q \in \mathbb{R}^+$, $\|u_x\|_\infty$ exists for all time if $\lambda \in [0, q/2]$. Therefore, for these values of the parameter and $p \geq 1$, $u_x \in L^p$ for all $t \in [0, +\infty]$. On the other hand, blow-up of u_x in the L^∞ norm occurs as t approaches some finite $t_* > 0$ for $\lambda > q/2$. In this section, we study further properties of L^p regularity in u_x, as $t \uparrow t_*$, for $\lambda > q/2$, $p \in [1, +\infty)$ and initial data $u_0'(\alpha)$ satisfying (3.3.10) for some $q \in \mathbb{R}^+$. To do so, we will make use of the upper and lower bounds, (3.1.63) and (3.1.64), derived in section 3.3.2 for $\|u_x\|_p$. As a result, we require estimates on the behaviour of the time-dependent integrals

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha, t)^{\frac{1}{\lambda p}}}, \qquad \int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha, t)^{1+\frac{1}{\lambda p}}}, \qquad \int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha, t)^{p+\frac{1}{\lambda}}} \tag{3.3.84}$$

for $\eta_* - \eta > 0$ small, which may be obtained directly from parts (1) and (3) of Lemma 3.3.41. Since applying the Lemma is a rather straight-forward exercise, we omit the details and state our findings below.

For $p \geq 1$,

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha, t)^{\frac{1}{\lambda p}}} \sim \begin{cases} C, & q \in (0, 1/2), \ \lambda > \frac{1}{2p} \quad \text{or} \quad q > \frac{1}{2}, \ \lambda > \frac{q}{p}, \\ C_9 \mathcal{J}(\overline{\alpha}_i, t)^{\frac{1}{q} - \frac{1}{\lambda p}}, & q > 0, \quad \lambda \in (0, q/p) \end{cases} \tag{3.3.85}$$

for $\eta_* - \eta > 0$ small and positive constants

$$C_9 = \frac{2m\Gamma\left(1 + \frac{1}{q}\right)\Gamma\left(\frac{1}{\lambda p} - \frac{1}{q}\right)}{\Gamma\left(\frac{1}{\lambda p}\right)} \left(\frac{M_0}{|C_1|}\right)^{\frac{1}{q}}.$$

Also,

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha, t)^{1+\frac{1}{\lambda p}}} \sim C \tag{3.3.86}$$

for $p\geq 1$ and either

$$\begin{cases} q\in(0,1/2), \quad \lambda>\frac{1}{p}, \\ q\in(1/2,1), \quad \lambda>\frac{q}{p(1-q)}, \end{cases} \tag{3.3.87}$$

whereas

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda p}}} \sim C_{10}\mathcal{J}(\overline{\alpha}_i,t)^{\frac{1}{q}-\frac{1}{\lambda p}-1} \tag{3.3.88}$$

for $p\geq 1$ and

$$\begin{cases} q\in(0,1), \quad 0<\lambda<\frac{q}{p(1-q)}, \\ q\geq 1, \quad \lambda>0. \end{cases} \tag{3.3.89}$$

The positive constant C_{10} in (3.3.88) can be obtained by simply substituting every $\frac{1}{\lambda p}$ term in C_9 above, by $1+\frac{1}{\lambda p}$. We also point out that due to part (3) of Lemma 3.3.41, estimates (3.3.85)i) and (3.3.86) are valid for

$$\lambda\neq\frac{q}{p(1-nq)}, \qquad q\neq\frac{1}{n} \;\; \forall \;\; n\in\mathbb{N}\cup\{0\}. \tag{3.3.90}$$

Finally,

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{p+\frac{1}{\lambda}}} \sim C \tag{3.3.91}$$

for either

$$\begin{cases} q\in(0,1/2), \quad p\in[1,2), \qquad \lambda>\frac{1}{2-p}, \\ q\in(1/2,1), \quad p\in[1,1/q), \quad \lambda>\frac{q}{1-pq}. \end{cases} \tag{3.3.92}$$

whereas

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{p+\frac{1}{\lambda}}} \sim C\mathcal{J}(\overline{\alpha}_i,t)^{\frac{1}{q}-\frac{1}{\lambda}-p} \tag{3.3.93}$$

for

$$\begin{cases} q\in(0,1], \quad p\in[1,1/q), \quad 0<\lambda<\frac{q}{1-pq}, \\ q\in(0,1], \quad p\geq\frac{1}{q}, \quad \lambda>0, \\ q>1, \qquad p\geq 1, \quad \lambda>0. \end{cases} \tag{3.3.94}$$

Estimate (3.3.91) is in turn valid for

$$\lambda\neq\frac{q}{1+q(n-p)}, \qquad q\neq\frac{1}{n} \;\; \forall \;\; n\in\mathbb{N}. \tag{3.3.95}$$

Now, in what follows, $t_*>0$ will denote the L^∞ blow-up time in Theorem 3.3.65. Also, we will assume that (3.3.57), (3.3.90) and (3.3.95) hold whenever their corresponding estimates are used. For simplicity, the restrictions placed on λ and q by these conditions are only stated in the main Theorem 3.3.100 at the end of this section, which summarizes our results. We begin by considering the lower bound (3.1.64). In particular, we will show that two-sided, everywhere blow-up in Theorem 3.3.65 corresponds to a diverging L^p norm of u_x for $p > 1$. Then, we consider the upper bound (3.1.63). In that case, we will find that if $q \in \mathbb{R}^+$ and $\lambda > q$ are such that only the maximum diverges at a finite $t_* > 0$, then u_x remains integrable for all $t \in [0, t_*]$, whereas, its regularity in other L^p spaces for $t \in [0, t_*]$ and $p \in (1, +\infty)$ will be determined from the parameter λ as a function of either p, q, or both.

Suppose $q/2 < \lambda < q/p$ for $q \in \mathbb{R}^+$ and $p \in (1, 2)$. This implies that estimate (3.3.85)ii) holds. Also, since $(q/2, q/p) \subset (0, q)$, (3.3.55)ii) applies as well. Now, if $q \in (0, 1)$, then

$$0 < \frac{q}{2} < \lambda < \frac{q}{p} < q < \frac{q}{1-q},$$

and so (3.3.58)ii) follows, otherwise, (3.3.58)ii) is also known to hold for all $q \geq 1$ and $\lambda > 0$. Similarly for $q \in (0, 1)$, we have that

$$0 < \frac{q}{2} < \lambda < \frac{q}{p} < \frac{q}{p(1-q)}$$

so that (3.3.88) holds. Alternatively, this last estimate is also valid if $q \geq 1$ for any $\lambda > 0$. Accordingly, using these estimates on (3.1.64) yields, after simplification,

$$\begin{aligned} \|u_x(\cdot, t)\|_p &\geq \frac{1}{|\lambda\eta(t)|\, \bar{\mathcal{K}}_0(t)^{2\lambda+\frac{1}{p}}} \left| \int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha, t)^{1+\frac{1}{\lambda p}}} - \frac{\bar{\mathcal{K}}_1(t)}{\bar{\mathcal{K}}_0(t)} \int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha, t)^{\frac{1}{\lambda p}}} \right| \\ &\sim C(p-1)\mathcal{J}(\overline{\alpha}, t)^{\sigma(p,q,\lambda)} \end{aligned}$$

for $\eta_* - \eta > 0$ small and

$$\sigma(p, q, \lambda) = 1 + \frac{1}{q}\left(1 - \frac{1}{p} - 2\lambda\right).$$

Then, $\|u_x\|_p$ will diverge, as $\eta \uparrow \eta_*$, for $\sigma(p, q, \lambda) < 0$, or equivalently, $p(1+q-2\lambda)-1 < 0$. Since $q/2 < \lambda < q/p$ for $q > 0$ and $p \in (1, 2)$, we find this to be the case if

$$q \in \mathbb{R}^+, \qquad 1 < p < 1 + \frac{q}{1+q}, \qquad \frac{1}{2}\left(q + 1 - \frac{1}{p}\right) < \lambda < \frac{q}{p}.$$

Therefore, by taking $p - 1 > 0$ arbitrarily small, we find that

$$\lim_{t \uparrow t_*} \|u_x\|_p = +\infty$$

for $\lambda \in (q/2, q)$ and $q \in \mathbb{R}^+$. The existence of a finite blow-up time $t_* > 0$ follows from (2) in Theorem 3.3.65, while the embedding

$$L^s \hookrightarrow L^p, \qquad s \geq p, \tag{3.3.96}$$

yields L^p blow-up for any $p > 1$. Next, for $q \in (1/3, 1/2)$ we consider values of λ lying between stagnation point-form solutions to the 2D ($\lambda = 1$) and 3D ($\lambda = 1/2$) incompressible Euler equations. Suppose $\frac{1}{2} < \lambda < \frac{q}{p(1-q)}$ for $1 < p < \frac{2q}{1-q}$ and $q \in (1/3, 1/2)$. The condition on p simply guarantees that $\frac{q}{p(1-q)} > \frac{1}{2}$ for q as specified. Furthermore, we have that

$$0 < \frac{1}{2p} < \frac{1}{2} < \lambda < \frac{q}{p(1-q)} < \frac{q}{1-q} \in (1/2, 1),$$

so that relative to our choice of λ and q, $\lambda \in (1/2, 1)$. Using the above, we find that (3.3.55)i), (3.3.58)ii), (3.3.85)i) and (3.3.88) hold, and so (3.1.64) leads to

$$\begin{aligned} \|u_x(\cdot,t)\|_p &\geq \frac{1}{|\lambda\eta(t)|\,\bar{\mathcal{K}}_0(t)^{2\lambda+\frac{1}{p}}} \left| \int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda p}}} - \frac{\bar{\mathcal{K}}_1(t)}{\bar{\mathcal{K}}_0(t)} \int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{\frac{1}{\lambda p}}} \right| \\ &\sim C \left| C\mathcal{J}(\overline{\alpha},t)^{\frac{1}{q}-\frac{1}{\lambda p}-1} - \mathcal{J}(\overline{\alpha},t)^{\frac{1}{q}-\frac{1}{\lambda}-1} \right| \\ &\sim C\mathcal{J}(\overline{\alpha},t)^{\frac{1}{q}-\frac{1}{\lambda}-1} \end{aligned} \tag{3.3.97}$$

for $\eta_* - \eta > 0$ small. Therefore, as $\eta \uparrow \eta_*$, $\|u_x\|_p$ will diverge for all $\frac{1}{2} < \lambda < \frac{q}{p(1-q)}$, $q \in (1/3, 1/2)$ and $1 < p < \frac{2q}{1-q}$. Here, we can take $p - 1 > 0$ arbitrarily small and use (3.3.96) to conclude the finite-time blow-up, as $t \uparrow t_*$, of $\|u_x\|_p$ for all $\frac{1}{2} < \lambda < \frac{q}{1-q}$, $q \in (1/3, 1/2)$ and $p > 1$. The existence of a finite blow-up time $t_* > 0$ is guaranteed by the second part of (3) in Theorem 3.3.65. Now suppose $q \in (1/2, 1)$ and $q < \lambda < \frac{q}{p(1-q)}$ for $1 < p < \frac{1}{1-q}$. This means that $\lambda > q > 1/2$ and

$$0 < \frac{q}{p} < q < \lambda < \frac{q}{p(1-q)} < \frac{q}{1-q}. \tag{3.3.98}$$

Consequently, using (3.3.55)i), (3.3.58)ii), (3.3.85)i) and (3.3.88) on (3.1.64), implies (3.3.97), possibly with distinct positive constants C. Then, as $\eta \uparrow \eta_*$,

$$\|u_x(\cdot,t)\|_p \to +\infty$$

for all $q < \lambda < \frac{q}{p(1-q)}$, $q \in (1/2, 1)$ and $1 < p < \frac{1}{1-q}$. Similarly, if q and p are as above, but

$\frac{q}{p(1-q)} < \lambda < \frac{q}{1-q}$, (3.3.55)i), (3.3.58)ii), (3.3.85)i) and (3.3.86) imply

$$\begin{aligned}\|u_x(\cdot,t)\|_p &\geq \frac{1}{|\lambda\eta(t)|\,\bar{\mathcal{K}}_0(t)^{2\lambda+\frac{1}{p}}}\left|\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda p}}} - \frac{\bar{\mathcal{K}}_1(t)}{\bar{\mathcal{K}}_0(t)}\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{\frac{1}{\lambda p}}}\right| \\ &\sim C\left|C - \mathcal{J}(\overline{\alpha},t)^{\frac{1}{q}-\frac{1}{\lambda}-1}\right| \\ &\sim C\mathcal{J}(\overline{\alpha},t)^{\frac{1}{q}-\frac{1}{\lambda}-1} \to +\infty\end{aligned}$$

as $\eta \uparrow \eta_*$. From these last two results and (3.3.96), we see that

$$\|u_x\|_p \to +\infty \quad \text{as} \quad \eta \uparrow \eta_*$$

for all $q < \lambda < \frac{q}{1-q}$, $q \in (1/2, 1)$ and $p > 1$. The existence of a finite $t_* > 0$ follows from part 4 of Theorem 3.3.65. Lastly, suppose $\lambda > q > 1$ and $p > 1$. Then, estimates (3.3.55)i), (3.3.58)ii), (3.3.85)i) and (3.3.88) hold for $\eta_* - \eta > 0$ small. As a result, (3.1.64) implies (3.3.97), which in turn leads to L^p blow-up of u_x for any $\lambda > q > 1$ and $p > 1$, as $\eta \uparrow \eta_*$. The existence of a finite $t_* > 0$ is due to (5) in Theorem 3.3.65.

Notice from the results established so far, that some values of $\lambda > q/2$ for $q \in \mathbb{R}^+$ are missing. These are precisely the cases for which the lower bound (3.1.64) yields inconclusive information about the L^p regularity of u_x for $p \in (1, +\infty)$. To examine some aspects of the L^p regularity of u_x for $t \in [0, t_*]$ and $p \in [1, +\infty)$, in these particular cases, we consider the upper bound (3.1.63) instead. First, suppose $q \in (0, 1/2)$ and $\lambda > \frac{1}{2-p}$ for $p \in [1, 2)$. Then, we have that $\lambda > \frac{1}{2-p} > 1 > \frac{q}{1-q} > q$. As a result, (3.3.55)i), (3.3.58)i) and (3.3.91) imply that all the integral terms in (3.1.63) remain bounded, and nonzero, for $\eta \in [0, \eta_*]$. We conclude that

$$\lim_{t\uparrow t_*} \|u_x(\cdot,t)\|_p < +\infty \tag{3.3.99}$$

for all $\lambda > \frac{1}{2-p}$, $q \in (0, 1/2)$ and $p \in [1, 2)$. Here, $t_* > 0$ denotes the finite L^∞ blow-up time for u_x established in the first part of (3) in Theorem 3.3.65. Particularly, this result implies that even though

$$\lim_{t\uparrow t_*} \|u_x\|_\infty = +\infty$$

for all $\lambda > 1$ when $q \in (0, 1/2)$, u_x remains integrable for $t \in [0, t_*]$.

Finally, suppose $q \in (1/2, 1)$ and $\lambda > \frac{q}{1-pq}$ for $p \in [1, 1/q)$. Then

$$\lambda > \frac{q}{1-pq} \geq \frac{q}{1-q} > 1 > q > \frac{1}{2},$$

and so (3.3.55)i), (3.3.58)i) and (3.3.91) hold. Consequently, (3.1.63) implies that

$$\lim_{t \uparrow t_*} \|u_x\|_p < +\infty$$

for all $\lambda > \frac{q}{1-pq}$, $q \in (1/2, 1)$ and $p \in [1, 1/q)$. This time, $t_* > 0$ stands as the finite L^∞ blow-up time for u_x established in the second part of (4) in Theorem 3.3.65. Furthermore, this result tells us that although

$$\lim_{t \uparrow t_*} \|u_x\|_\infty = +\infty$$

for $\lambda > \frac{q}{1-q}$ and $q \in (1/2, 1)$, u_x stays integrable for all $t \in [0, t_*]$. These last two results on the integrability of u_x, for $t \in [0, t_*]$, become evident by setting $p = 1$ into (3.1.63) to obtain

$$\|u_x\|_1 \leq \frac{2\bar{\mathcal{K}}_1(t)}{|\lambda \eta(t)|\, \bar{\mathcal{K}}_0(t)^{1+2\lambda}}.$$

The result then follows from the above inequality and estimates (3.3.55)i) and (3.3.58)i). Theorem 3.3.100 below summarizes the above results.

Theorem 3.3.100. *Consider the initial boundary value problem (1.1.1)-(1.1.2) or (1.1.3) for $u_0'(\alpha)$ bounded, at least $C^0(0,1)$ a.e., and satisfying (3.3.10). Also, let $t_* > 0$ be as in Theorem 3.3.65.*

1. *For $q > 0$ and $\lambda \in [0, q/2]$, $\lim_{t \to +\infty} \|u_x\|_p < +\infty$ for all $p \geq 1$. More particularly, $\lim_{t \to +\infty} \|u_x\|_p = 0$ for $\lambda \in (0, q/2)$, while, as $t \to +\infty$, u_x converges to a nontrivial, L^∞ function when $\lambda = q/2$.*

2. *Let $p > 1$. Then, there exists a finite $t_* > 0$ such that for all $q > 0$ and $\lambda \in (q/2, q)$, $\lim_{t \uparrow t_*} \|u_x\|_p = +\infty$. Similarly if $\lambda > q > 1$, or $\frac{1}{2} < \lambda < \frac{q}{1-q}$, $q \in (1/3, 1/2)$.*

3. *For all $q \in (0, 1/2)$, $\lambda > \frac{1}{2-p}$ and $p \in [1, 2)$, there exists a finite $t_* > 0$ such that $\lim_{t \uparrow t_*} \|u_x\|_p < +\infty$ (see (3) in Theorem 3.3.65).*

4. *Suppose $q \in (1/2, 1)$. Then, there exists a finite $t_* > 0$ such that $\lim_{t \uparrow t_*} \|u_x\|_p = +\infty$ for $q < \lambda < \frac{q}{1-q}$ and $p > 1$, whereas, if $\lambda > \frac{q}{1-pq}$ and $p \in [1, 1/q)$, $\lim_{t \uparrow t_*} \|u_x\|_p < +\infty$ (see (4) in Theorem 3.3.65).*

L^∞ Regularity for $\lambda < 0$ and $q \in \mathbb{R}^+$

We now examine L^∞ regularity of u_x for $\lambda < 0$. We prove Theorem 3.3.101 below.

Theorem 3.3.101. *Consider the initial boundary value problem (1.1.1)-(1.1.2) or (1.1.3) for $u_0'(\alpha)$ bounded, at least $C^0(0,1)$ a.e., and satisfying (3.3.11). It holds,*

1. *Suppose $\lambda \in [-1, 0)$ and $q \in \mathbb{R}^+$. Then, there exists a finite $t_* > 0$ such that only the minimum diverges, $m(t) \to -\infty$, as $t \uparrow t_*$ (one-sided, discrete blow-up).*

2. *Suppose that $\lambda < -1$ and $q \in (0, 1)$ satisfy $\lambda \neq \frac{q}{1-nq}$ and $q \neq \frac{1}{n}$ $\forall$ $n \in \mathbb{N}$. Then, a one-sided discrete blow-up, as described in (1) above, occurs in finite-time. Similarly for $\frac{q}{1-q} < \lambda < -1$ and $q > 1$.*

3. *Suppose $\lambda < \frac{q}{1-q}$ and $q > 1$. Then, there is a finite $t_* > 0$ such that both the maximum $M(t)$ and the minimum $m(t)$ diverge to $+\infty$ and respectively to $-\infty$ as $t \uparrow t_*$. Moreover, $\lim_{t\uparrow t_*} u_x(\gamma(\alpha, t), t) = +\infty$ for $\alpha \notin \{\overline{\alpha}_i, \underline{\alpha}_j\}$ (two-sided, everywhere blow-up).*

4. *For $\lambda < 0$, assume only Dirichlet boundary conditions (1.1.3) and/or suppose u_0 is odd about the midpoint. Then, for every $\underline{\alpha}_j \in [0, 1]$ there exists a unique $\underline{x}_j \in [0, 1]$ given by formula (3.1.2) such that $\lim_{t\uparrow t_*} u_x(\underline{x}_j, t) = -\infty$. Finally, the jacobian (2.1.14) satisfies*

$$\lim_{t\uparrow t_*} \gamma_\alpha(\alpha, t) = \begin{cases} 0, & \alpha = \underline{\alpha}_j, \\ C, & \alpha \neq \underline{\alpha}_j \end{cases} \tag{3.3.102}$$

for all $\lambda < 0$, $q \in \mathbb{R}^+$ and where the positive constants C depend on the choice of λ, q and $\alpha \neq \underline{\alpha}_j$.

Proof. Let C be a positive constant depending on $\lambda < 0$ and $q > 0$, and set

$$\eta_* = \frac{1}{\lambda m_0}.$$

Proof of Statement *1*

Suppose $\lambda \in [-1, 0)$ and, given $q \in \mathbb{R}^+$, assume $u_0'(\alpha)$ is bounded, at least $C^0(0,1)$ *a.e.*, and satisfies (3.3.11). Then, from (3.3.60) and (3.3.61) both integral terms $\bar{\mathcal{K}}_i(t)$, $i = 0, 1$ in (2.1.19) remain finite and nonzero as $\eta \uparrow \eta_*$. More particularly, one can show that (3.1.53) and (3.1.55) hold for all $\eta \in [0, \eta_*]$. Then, by setting $\alpha = \underline{\alpha}_j$ into (2.1.19) and using (3.0.5) we find that, due to the space-dependent term in (2.1.19), the minimum diverges, $m(t) \to -\infty$, as $\eta \uparrow \eta_*$. However, if $\alpha \neq \underline{\alpha}_j$, the definition of m_0 implies that the space-dependent term now remains bounded, and positive, for all $\eta \in [0, \eta_*]$. As a result, $u_x(\gamma, t)$ stays finite for $\alpha \neq \underline{\alpha}_j$ and $\eta \in [0, \eta_*]$. We conclude that as $\eta \uparrow \eta_*$, a one-sided, discrete blow-up occurs. The existence of a finite blow-up time $t_* > 0$ and formula (3.1.2), the latter under Dirichlet boundary conditions, follow from (2.1.16) and (2.2.1), respectively, along with (3.3.60). In fact, we may use (2.1.16) and (3.1.53) to obtain estimate (3.1.56).

Proof of Statements *2* and *3*

Now suppose $\lambda < -1$. As in the previous case, the term $\bar{\mathcal{K}}_0(t)$ remains finite and positive for all $\eta \in [0, \eta_*]$. Particularly, $\bar{\mathcal{K}}_0(t)$ satisfies (3.1.40) for all $\eta \in [0, \eta_*]$. On the other hand, $\bar{\mathcal{K}}_1(t)$ now either converges or diverges as $\eta \uparrow \eta_*$ according to (3.3.61) or (3.3.63). If $\lambda < -1$ and $q \in \mathbb{R}^+$ are such that (3.3.61) holds, then (2) follows just as part (1). However, if $q > 1$ and $\lambda < \frac{q}{1-q}$, we use (3.3.60) and (3.3.63) on (2.1.19) to obtain

$$u_x(\gamma(\alpha, t), t) \sim C m_0 \left(\frac{1}{\mathcal{J}(\alpha, t)} - C \mathcal{J}(\underline{\alpha}_j, t)^{\frac{1}{q} - \frac{1}{\lambda} - 1} \right)$$

for $\eta_* - \eta > 0$ small. Setting $\alpha = \underline{\alpha}_j$ into the above and using (3.0.5) then implies

$$m(t) \sim \frac{C m_0}{\mathcal{J}(\underline{\alpha}_j, t)} \to -\infty$$

as $\eta \uparrow \eta_*$. On the other hand, if $\alpha \neq \underline{\alpha}_j$, so that the space-dependent term $\mathcal{J}(\alpha, t)^{-1}$ now remains bounded, we use $q > 1$ and $\lambda < \frac{q}{1-q}$ to conclude that the second term dominates and

$$u_x(\gamma(\alpha, t), t) \sim C \left| m_0 \right| \mathcal{J}(\underline{\alpha}_j, t)^{\frac{1}{q} - \frac{1}{\lambda} - 1} \to +\infty$$

as $\eta \uparrow \eta_*$. The existence of a finite blow-up time $t_* > 0$ and formula (3.1.2), the latter for the Dirichlet setting (1.1.3) and/or odd data u_0, follow just as in the $\lambda \in [-1, 0)$ case. In fact, (2.1.16) and (3.1.40) yield the lower bound $\eta_* \leq t_*$.[22] Finally, (3.3.102) is derived directly from (2.1.14) and (3.3.60). See section 4.2 for examples. □

Further L^p Regularity for $\lambda < 0$, $p \in [1, +\infty)$ and $q \in \mathbb{R}^+$

Let $t_* > 0$ denote the finite L^∞ blow-up time for u_x in Theorem 3.3.101 and recall that $\eta_* = \frac{1}{\lambda m_0}$. In this last section, we briefly study the L^p regularity of u_x as $t \uparrow t_*$ for $\lambda < 0$

[22] Which we may compare to (3.1.56). From (2.1.16), we see that the two coincide, $t_* = \eta_*$, in the case of Burgers' equation $\lambda = -1$.

and $p \in [1, +\infty)$. Also, the behaviour of the jacobian is considered and a class of smooth functions larger than the one studied in section 3.1 is discussed at the end.

As in section 3.3.3, our study of L^p regularity requires the use of the upper and lower bounds (3.1.63) and (3.1.64). First of all, by the last part of (3) in Lemma 3.3.41,

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{\frac{1}{\lambda p}}} d\alpha \sim C \tag{3.3.103}$$

for $\eta_* - \eta > 0$ small, $\lambda < 0$, $q \in \mathbb{R}^+$ and $p \geq 1$. Similarly by the same result,

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{p+\frac{1}{\lambda}}} \sim C \tag{3.3.104}$$

for $\eta_* - \eta > 0$ small, $-\frac{1}{p} \leq \lambda < 0$, $q \in \mathbb{R}^+$ and $p \geq 1$. Moreover, due to the first part of (3) in the Lemma, estimate (3.3.104) is also seen to hold, with different positive constants C, for $\lambda < -\frac{1}{p}$, $p \geq 1$ and $q \in \mathbb{R}^+$ satisfying either of the following

$$\begin{cases} q \in (0,1/2), & p \in [1,2], & \lambda < -\frac{1}{p}, \\ q \in (0,1/2), & p > 2, & \frac{1}{2-p} < \lambda < -\frac{1}{p}, \\ q \in (1/2,1), & p \in [1,1/q], & \lambda < -\frac{1}{p}, \\ q \in (1/2,1), & p > \frac{1}{q}, & \frac{q}{1-pq} < \lambda < -\frac{1}{p}, \\ q > 1, & p \geq 1, & \frac{q}{1-pq} < \lambda < -\frac{1}{p}, \end{cases} \tag{3.3.105}$$

as well as

$$\lambda \notin \left\{ \frac{q}{1-q(p+n)}, \frac{1}{1-p} \right\}, \qquad q \neq \frac{1}{n} \quad \forall \quad n \in \mathbb{N}. \tag{3.3.106}$$

If (3.3.104) diverges instead, then it dominates the other terms in the upper bound (3.1.63), regardless of whether these converge or diverge, and so no information on the behaviour of $\|u_x\|_p$ is obtained. Finally, using (2) in Lemma 3.3.41, one finds that

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda p}}} \sim C\mathcal{J}(\underline{\alpha}_j,t)^{\frac{1}{q}-\frac{1}{\lambda p}-1} \tag{3.3.107}$$

for $\eta_* - \eta > 0$ small, $q > 1$, $p \geq 1$ and $\lambda < \frac{q}{p(1-q)}$. We remark that for the cases where the above integral converges, the lower bound (3.1.64) yields no information. For the remaining of this section, we will assume that (3.3.57) holds whenever (3.3.61) is being used for $\lambda < -1$ and $q \in (0,1)$. Also, (3.3.106) will be valid in those cases where estimate (3.3.104) is considered for λ, p and q as in (3.3.105).

Suppose $\frac{q}{1-q} < \lambda < \frac{q}{p(1-q)}$ for $q > 1$ and $p > 1$. Then, using (3.3.60), (3.3.61), (3.3.103) and (3.3.107) on the lower bound (3.1.64) implies that

$$\lim_{t\uparrow t_*} \|u_x\|_p = +\infty.$$

If instead, $\lambda < \frac{q}{1-q}$ for $q > 1$ and $p > 1$, then (3.3.60), (3.3.63), (3.3.103) and (3.3.107) give

$$\begin{aligned}
\|u_x(\cdot,t)\|_p &\geq \frac{1}{|\lambda\eta(t)|\,\bar{\mathcal{K}}_0(t)^{2\lambda+\frac{1}{p}}}\left|\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda p}}} - \frac{\bar{\mathcal{K}}_1(t)}{\bar{\mathcal{K}}_0(t)}\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{\frac{1}{\lambda p}}}\right| \\
&\sim C\left|C\mathcal{J}(\underline{\alpha},t)^{\frac{1}{q}-\frac{1}{\lambda p}-1} - \mathcal{J}(\underline{\alpha},t)^{\frac{1}{q}-\frac{1}{\lambda}-1}\right| \\
&\sim C\mathcal{J}(\underline{\alpha},t)^{\frac{1}{q}-\frac{1}{\lambda p}-1} \to +\infty
\end{aligned}$$

as $\eta \uparrow \eta_*$. For the upper bound (3.1.63), we simply mention that estimates (3.3.60), (3.3.61) and (3.3.104) lead to several instances where $\|u_x\|_p$ remains finite for all $t \in [0, t_*]$. For simplicity, we omit the details and summarize the results from this section in Theorem (3.3.108) below.

Theorem 3.3.108. *Consider the initial boundary value problem (1.1.1)-(1.1.2) or (1.1.3) for $u_0'(\alpha)$ bounded, at least $C^0(0,1)$ a.e., and satisfying (3.3.11). In addition, let $t_* > 0$ denote the finite L^∞ blow-up time for u_x as described in Theorem 3.3.101. It follows,*

1. *Let $q \in (0, 1/2)$. Then $\lim_{t\uparrow t_*} \|u_x\|_p < +\infty$ for either $\lambda < 0$ and $p \in [1, 2]$, or $\frac{1}{2-p} < \lambda < 0$ and $p > 2$.*

2. *Let $q \in (1/2, 1)$. Then $\lim_{t\uparrow t_*} \|u_x\|_p < +\infty$ for either $\lambda < 0$ and $p \in [1, 1/q]$, or $\frac{q}{1-pq} < \lambda < 0$ and $p > 1/q$.*

3. *Let $q > 1$. Then $\lim_{t\uparrow t_*} \|u_x\|_p < +\infty$ for $\frac{q}{1-pq} < \lambda < 0$ and $p \geq 1$, whereas $\lim_{t\uparrow t_*} \|u_x\|_p = +\infty$ for $\lambda < \frac{q}{p(1-q)}$ and $p > 1$.*

Whenever applicable, conditions (3.3.57) and (3.3.106) apply to parts (1) and (2) above.

Remark 3.3.109. Suppose $\lambda > 0$. Then, Lemma 3.3.41 was established under the assumptions that the continuous, bounded function $u_0'(\alpha)$ attained its greatest value $M_0 > 0$ at several locations $\overline{\alpha}_i \in [0, 1]$, $i = 1, 2, ..., m$, and its local behaviour near each of these points is the same, i.e u_0' satisfies (3.3.10) for the same $q \in \mathbb{R}^+$ regardless of location $\overline{\alpha}_i$. Clearly, we may encounter functions u_0' with local behaviour that varies from one particular location $\overline{\alpha}_j$ to the next $\overline{\alpha}_k$, $j \neq k$. Formally, we can have u_0' which near $\overline{\alpha}_i$ satisfies

$$u_0'(\alpha) \sim M_0 + C_i \left|\alpha - \overline{\alpha}_i\right|^{q_i} \tag{3.3.110}$$

for all $0 \leq |\alpha - \overline{\alpha}_i| \leq r$, $0 < r \leq 1$, $q_i > 0$ and some $C_i < 0$. Here, r is chosen as small as needed to avoid overlapping amongst the intervals. Now, without loss of generality, suppose $q_1 \geq q_2 \geq ... \geq q_m > 0$ so that

$$\frac{1}{q_m} \geq \frac{1}{q_{m-1}} \geq ... \geq \frac{1}{q_1} > 0.$$

Then, applying the argument used to prove *1* in Lemma 3.3.41, we find that for $b > \frac{1}{q_m}$ and $\eta_* - \eta > 0$ small,

$$\begin{aligned} \sum_{i=1}^{m} \int_{\overline{\alpha}_i - r}^{\overline{\alpha}_i + r} \frac{d\alpha}{\mathcal{J}(\alpha, t)^b} &\sim \sum_{i=1}^{m} c_i \mathcal{J}(\overline{\alpha}_i, t)^{\frac{1}{q_i} - b} \\ &= \mathcal{J}(\overline{\alpha}_1, t)^{\frac{1}{q_1} - b} \left(c_1 + \sum_{i=2}^{m} c_i \mathcal{J}(\overline{\alpha}_i, t)^{\frac{1}{q_i} - \frac{1}{q_1}} \right) \end{aligned} \tag{3.3.111}$$

for the constants c_i given by

$$c_i = \frac{2N\Gamma\left(1 + \frac{1}{q_i}\right)\Gamma\left(b - \frac{1}{q_i}\right)}{\Gamma\left(b\right)} \left(\frac{M_0}{|C_i|}\right)^{\frac{1}{q_i}}, \tag{3.3.112}$$

and where the positive integer $N \geq 1$ denotes the multiplicity of the corresponding q_i in the set $\{q_1, q_2, ..., q_m\}$. Furthermore, since for every $1 \leq i \leq m$, $b > \frac{1}{q_i}$, the constants c_i are all positive and well-defined. Also, because $q_1 \geq q_i$, we have $\frac{1}{q_i} - \frac{1}{q_1} \geq 0$. As a result, using the continuity of u_0' implies that the integral will diverge, as $\eta \uparrow \eta_*$, at a rate

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha, t)^b} \sim c_1 \mathcal{J}(\overline{\alpha}_1, t)^{\frac{1}{q_1} - b} \tag{3.3.113}$$

for all $b > \frac{1}{q_m}$. This implies that the blow-up rate for the integral is determined by the greatest element in the set $\{q_i\}$, whereas the values of $b > 0$ for which the blow-up occurs depend on the smallest member. This interaction between q_g and q_s The above result is summarized in Corollary 3.3.114, which may be used for studying regularity of solutions in the more general case of u_0' with distinct local behaviours. Similar arguments are possible by following the above procedure, along with the one used in Lemma 3.3.41, to obtain corresponding integral estimates for $\lambda < 0$ and/or $b \leq \frac{1}{q_m}$.

Corollary 3.3.114. *For $\lambda > 0$ and $\eta_* = \frac{1}{\lambda M_0}$, suppose that $u_0'(\alpha)$ is bounded, at least $C^0(0,1)$ a.e., and satisfies (3.3.110). In addition, let $q_1 > 0$ denote the greatest element(s) in the set $\{q_i\}$, $i = 1, 2, ..., m$ having multiplicity N, and $q_m > 0$ its smallest member. Then for all $b > \frac{1}{q_m}$ and $\eta_* - \eta > 0$ small,*

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^b} \sim c_1 \mathcal{J}(\overline{\alpha}_1, t)^{\frac{1}{q_1}-b} \tag{3.3.115}$$

with positive constant

$$c_1 = \frac{2N\Gamma\left(1+\frac{1}{q_1}\right)\Gamma\left(b-\frac{1}{q_1}\right)}{\Gamma(b)} \left(\frac{M_0}{|C_1|}\right)^{\frac{1}{q_1}}. \tag{3.3.116}$$

3.3.4 Smooth Initial Data and the Order of $u_0''(x)$

Definition 3.3.117. Suppose a smooth function $f(x)$ satisfies $f(x_0) = 0$ but f is not identically zero. We say f has a zero of order $k \in \mathbb{N}$ at $x = x_0$ if

$$f(x_0) = f'(x_0) = ... = f^{(k-1)}(x_0) = 0, \qquad f^{(k)}(x_0) \neq 0.$$

In section 3.1, we examined a class of smooth initial data characterized by $u_0''(\alpha)$ having order $k = 1$ at a finite number of locations $\overline{\alpha}_i$ for $\lambda > 0$, or at $\underline{\alpha}_j$ if $\lambda < 0$, namely $u_0'''(\overline{\alpha}_i) < 0$ or $u_0'''(\underline{\alpha}_j) > 0$. As a result, in each case we were able to use an appropriate Taylor expansion, up to quadratic order, to account for the local behaviour of u_0' near these locations. By using definition 3.3.117 above and assuming that u_0'' has the same order k ($k \geq 2$) at every $\overline{\alpha}_i$ when $\lambda > 0$, or $\underline{\alpha}_j$ if $\lambda < 0$, we may apply the results established thus far to a larger class of smooth, periodic initial data than the one studied in section 3.1. We do this by simply substituting q in Theorems 3.3.65, 3.3.100, 3.3.101 and 3.3.108 by $2k$ in those cases where $q \geq 2$. The results are summarized in Corollary 3.3.118 below.

Corollary 3.3.118. *Consider the initial boundary value problem (1.1.1)-(1.1.2) for smooth, mean-zero initial data. Furthermore,*

1. *Suppose* $u_0''(\alpha)$ *has order* $k \geq 1$ *at every* $\overline{\alpha}_i$, $i = 1, 2, ..., m$. *Then*

   - *For* $\lambda \in [0, k]$, *solutions exist globally in time. More particularly, these vanish as* $t \uparrow t_* = +\infty$ *for* $\lambda \in (0, k)$ *but converge to a nontrivial steady state if* $\lambda = k$.
   - *For* $\lambda > k$, *there exists a finite* $t_* > 0$ *such that both the maximum* $M(t)$ *and the minimum* $m(t)$ *diverge to* $+\infty$ *and respectively to* $-\infty$ *as* $t \uparrow t_*$. *Furthermore,* $\lim_{t\uparrow t_*} u_x(\gamma(\alpha, t), t) = -\infty$ *if* $\alpha \notin \{\overline{\alpha}_i, \underline{\alpha}_j\}$ *and* $\lim_{t\uparrow t_*} \|u_x\|_p = +\infty$ *for all* $p > 1$.

2. *Suppose* $u_0''(\alpha)$ *has order* $k \geq 1$ *at each* $\underline{\alpha}_j$, $j = 1, 2, ..., n$. *Then*

   - *For* $\frac{2k}{1-2k} < \lambda < 0$, *there exists a finite* $t_* > 0$ *such that only the minimum diverges,* $m(t) \to -\infty$, *as* $t \uparrow t_*$, *whereas, for* $\frac{2k}{1-2kp} < \lambda < 0$ *and* $p \geq 1$, $\lim_{t\uparrow t_*} \|u_x\|_p < +\infty$.
   - *For* $\lambda < \frac{2k}{1-2k}$, *there is a finite* $t_* > 0$ *such that both* $M(t)$ *and* $m(t)$ *diverge to* $+\infty$ *and respectively to* $-\infty$ *as* $t \uparrow t_*$. *Additionally, if* $\alpha \notin \{\overline{\alpha}_i, \underline{\alpha}_j\}$, $\lim_{t\uparrow t_*} u_x(\gamma(\alpha, t), t) = +\infty$ *while* $\lim_{t\uparrow t_*} \|u_x\|_p = +\infty$ *for* $\lambda < \frac{2k}{p(1-2k)}$ *and* $p > 1$.

Remark 3.3.119. If there are $\overline{\alpha}_i \in \{0, 1\}$ when $\lambda > 0$, or $\underline{\alpha}_j \in \{0, 1\}$ for $\lambda < 0$, the results in Corollary 3.3.118 may be extended to the Dirichlet setting (1.1.3) by further assuming that $u_0'(\alpha)$ admits a periodic, smooth extension to the entire real line. Also, notice that letting $q \to +\infty$ in either (3.3.10) or (3.3.11) implies that $u_0' \sim M_0$ near $\overline{\alpha}_i$, or $u_0' \sim m_0$ for $\alpha \sim \underline{\alpha}_j$, respectively. If, in turn, we let $k \to +\infty$ in (1) of the above Corollary, we find that for this class of locally constant u_0', if a solution exist locally in time, it will persist for all time and $\lambda \geq 0$, a result, we remark, agrees with the regularity results derived in section 3.2 for piecewise constant u_0'.

Chapter 4

Examples

4.1 Examples for Sections 3.1 and 3.2

Examples 1-4 in §4.1.1 are instances of Theorem 3.1.1 for $\lambda \in \{3, -5/2, 1, -1/2\}$. In these cases, we will use formula (2.1.20) and the Mathematica software to aid in the closed-form evaluation of some of the integrals and the generation of plots. For simplicity, most details of the computations are omitted. Furthermore, examples 5 and 6 in §4.1.2 are representatives of Theorem 3.2.4 for $\lambda = 1$ and -2. Finally, due to the difficulty in solving the IVP (2.1.16), the plots in this section (except figure 4.2A)) will depict $u_x(\gamma(\alpha, t), t)$ for fixed $\alpha \in [0, 1]$ against the variable $\eta(t)$, not t. Figure 4.2A) however, will represent $u(x, t)$ for fixed $t \in [0, t_*)$ versus $x \in [0, 1]$.

4.1.1 For Theorem 3.1.1

For examples 1-3, let

$$u_0(\alpha) = -\frac{1}{4\pi}\cos(4\pi\alpha).$$

Then

$$u_0'(\alpha) = \sin(4\pi\alpha)$$

attains its maximum $M_0 = 1$ at $\overline{\alpha}_i = \{1/8, 5/8\}$, while $m_0 = -1$ occurs at $\underline{\alpha}_j = \{3/8, 7/8\}$.

Example 1. Two-sided Blow-up for $\lambda = 3$

Let $\lambda = 3$, then $\eta_* = \frac{1}{\lambda M_0} = 1/3$ and we have that

$$\bar{\mathcal{K}}_0(t) = {}_2F_1\left[\frac{1}{6}, \frac{2}{3}; 1; 9\,\eta(t)^2\right] \to \frac{\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{3}\right)\,\Gamma\left(\frac{5}{6}\right)} \sim 1.84 \tag{4.1.1}$$

and

$$\int_0^1 \frac{u_0'(\alpha)\,d\alpha}{\mathcal{J}(\alpha,t)^{\frac{4}{3}}} = 2\eta(t)\,{}_2F_1\left[\frac{7}{6},\frac{5}{3};2;9\,\eta(t)^2\right] \to +\infty \tag{4.1.2}$$

as $\eta \uparrow 1/3$. Using (4.1.1) and (4.1.2) on (2.1.20), and taking the limit as $\eta \uparrow 1/3$, we find that

$$M(t) = u_x(\gamma(\overline{\alpha}_i,t),t) \to +\infty$$

whereas, for $\alpha \neq \overline{\alpha}_i$,

$$u_x(\gamma(\alpha,t),t) \to -\infty.$$

The blow-up time $t_* \sim 0.54$ is obtained from (2.1.16) and (4.1.1). See figure 4.1(A).

Example 2. Two-sided Blow-up for $\lambda = -5/2$

For $\lambda = -5/2$ we have $\eta_* = \frac{1}{\lambda m_0} = 2/5$. Also,

$$\bar{\mathcal{K}}_0(t) = {}_2F_1\left[-\frac{1}{5},\frac{3}{10};1;\frac{25}{4}\eta(t)^2\right] \to \frac{\Gamma\left(\frac{9}{10}\right)}{\Gamma\left(\frac{7}{10}\right)\,\Gamma\left(\frac{6}{5}\right)} \sim 0.9 \tag{4.1.3}$$

and

$$\int_0^1 \frac{u_0'(\alpha)\,d\alpha}{\mathcal{J}(\alpha,t)^{\frac{3}{5}}} = -\frac{3}{4}\eta(t)\,{}_2F_1\left[\frac{4}{5},\frac{13}{10};2;\frac{25}{4}\eta(t)^2\right] \to -\infty \tag{4.1.4}$$

as $\eta \uparrow 2/5$. Plugging the above formulas into (2.1.20) and letting $\eta \uparrow 2/5$, we find that

$$m(t) = u_x(\gamma(\underline{\alpha}_j,t),t) \to -\infty$$

while, for $\alpha \neq \underline{\alpha}_j$,

$$u_x(\gamma(\alpha,t),t) \to +\infty.$$

The blow-up time $t_* \sim 0.46$ is obtained from (2.1.16) and (4.1.3). See figure 4.1(B).

The next example is an instance of global existence in stagnation point-form solutions (1.2.3) to the 2D incompressible Euler equations ($\lambda = 1$). We find that solutions converge to a nontrivial steady state as $t \to +\infty$.

Example 3. Global existence for $\lambda = 1$

Let $\lambda = 1$, then

$$\bar{\mathcal{K}}_0(t) = \frac{1}{\sqrt{1-\eta(t)^2}} \tag{4.1.5}$$

and

$$\int_0^1 \frac{u_0'(\alpha)\,d\alpha}{\mathcal{J}(\alpha,t)^2} = \frac{\eta(t)}{(1-\eta(t)^2)^{\frac{3}{2}}} \tag{4.1.6}$$

both diverge to $+\infty$ as $\eta \uparrow \eta_* = 1$. Also, (4.1.5) and (2.1.16) imply

$$\eta(t) = \tanh t,$$

which we use on (2.1.20), along with (4.1.5) and (4.1.6), to obtain

$$u_x(\gamma(\alpha,t),t) = \frac{\tanh t - \sin(4\pi\alpha)}{\tanh t \sin(4\pi\alpha) - 1}.$$

Clearly,

$$M(t) = u_x(\gamma(\overline{\alpha}_i,t),t) \equiv 1$$

and

$$m(t) = u_x(\gamma(\underline{\alpha}_j,t),t) \equiv -1$$

while, for $\alpha \notin \{\overline{\alpha}_i, \underline{\alpha}_j\}$,

$$u_x(\gamma(\alpha,t),t) \to -1$$

as $\eta \uparrow 1$. Finally, $\eta(t) = \tanh t$ yields

$$t_* = \lim_{\eta\uparrow 1} \operatorname{arctanh} \eta = +\infty.$$

It is also easy to see from (4.1.5) and the formulas in section 2.1 that the nonlocal term (1.1.1)iii) satisfies $I(t) \equiv -1$. See figure 4.1(C).

Example 4. One-sided Blow-up for $\lambda = -1/2$

For $\lambda = -1/2$ (HS equation), let

$$u_0 = \cos(2\pi\alpha) + 2\cos(4\pi\alpha).$$

Then, the least value $m_0 < 0$ of u_0', and the location $\underline{\alpha} \in [0,1]$ where m_0 occurs are given, approximately, by $m_0 \sim -30$ and $\underline{\alpha} \sim 0.13$, while $\eta_* = -\frac{2}{m_0} \sim 0.067$. For this choice of data, we find

$$\bar{\mathcal{K}}_0(t) = 1 + \frac{17\pi^2\eta(t)^2}{2} \tag{4.1.7}$$

and

$$\int_0^1 u_0'(\alpha)\mathcal{J}(\alpha,t)\,d\alpha = 17\pi^2\eta(t), \tag{4.1.8}$$

so that (2.1.16) and (4.1.7) give

$$\eta(t) = \sqrt{\frac{2}{17\pi^2}}\tan\left(\pi\sqrt{\frac{17}{2}}\,t\right).$$

Using these results on (2.1.20) yields, after simplification,

$$u_x(\gamma(\alpha,t),t) = \frac{\pi\left(2\sin(2\pi\alpha) + 8\sin(4\pi\alpha) + \sqrt{34}\tan\left(\pi\sqrt{\frac{17}{2}t}\right)\right)}{\sqrt{\frac{2}{17}}\tan\left(\pi\sqrt{\frac{17}{2}}\,t\right)(\sin(2\pi\alpha) + 4\sin(4\pi\alpha)) - 1}$$

for $0 \le \eta < \eta_*$. Setting $\alpha = \underline{\alpha}$ into the above formula, we see that

$$m(t) = u_x(\gamma(\underline{\alpha},t),t) \to -\infty$$

as $\eta \uparrow \eta_*$, whereas $u_x(\gamma(\alpha,t),t)$ remains finite for $\alpha \neq \underline{\alpha}$,. Finally, from the expression for $\eta(t)$ we obtain

$$t_* = t\,(-2/m_0) \sim 0.06.$$

See figure 4.1(D).

4.1.2 For Theorem 3.2.4

For examples 5 and 6 below, let

$$u_0(\alpha) = \begin{cases} -\alpha, & 0 \le \alpha < 1/4, \\ \alpha - 1/2, & 1/4 \le \alpha < 3/4. \\ 1 - \alpha, & 3/4 \le \alpha \le 1 \end{cases} \tag{4.1.9}$$

so that

$$u_0'(\alpha) = \begin{cases} -1, & 0 \le \alpha < 1/4, \\ 1, & 1/4 \le \alpha < 3/4. \\ -1, & 3/4 \le \alpha \le 1. \end{cases} \tag{4.1.10}$$

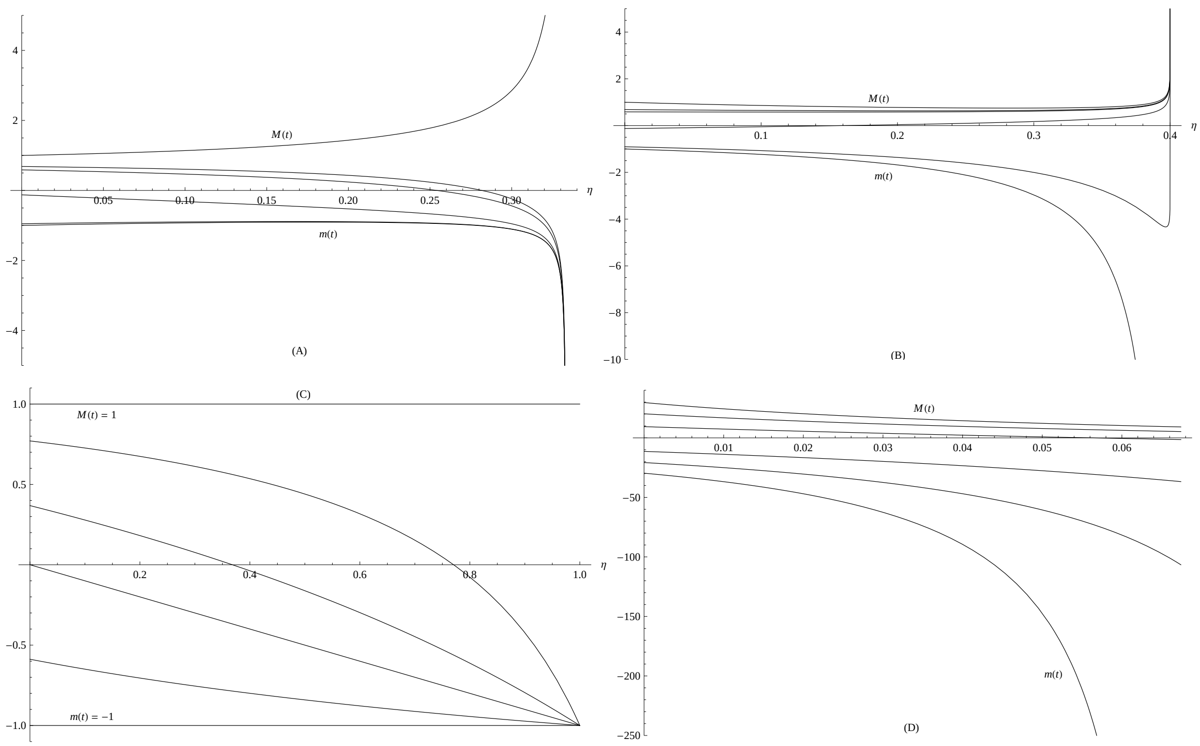

Figure 4.1: Figures A and B depict two-sided, everywhere blow-up of (2.1.20) for $\lambda = 3$ and $-5/2$ (Examples 1 and 2) as $\eta \uparrow 1/3$ and $2/5$, respectively. Figure C (Example 3) represents global existence in time for $\lambda = 1$; the solution converges to a nontrivial steady-state as $\eta \uparrow 1$ ($t \to +\infty$). Finally, figure D (Example 4) illustrates one-sided, discrete blow-up for $\lambda = -1/2$ as $\eta \uparrow 0.067$.

Then, $M_0 = 1$ occurs when $\alpha \in [1/4, 3/4)$, while $m_0 = -1$ for $\alpha \in [0, 1/4) \cup [3/4, 1]$ and

$$\eta_* = \frac{1}{|\lambda|}$$

for $\lambda \neq 0$. Also, notice that (4.1.9) is odd about the midpoint $\alpha = 1/2$ and vanishes at the end-points (as it should due to periodicity). As a result, uniqueness of solution to (2.0.1) implies that $\gamma(0, t) \equiv 0$ and $\gamma(1, t) \equiv 1$ for as long as u is defined. See also our discussion in section 2.2.2.

Example 5. Global existence for $\lambda = 1$

Using (4.1.10), we find that

$$\bar{\mathcal{K}}_0(t) = \frac{1}{1 - \eta(t)^2} \tag{4.1.11}$$

for $0 \leq \eta < \eta_* = 1$. Then (2.1.14) implies

$$\gamma_\alpha(\alpha, t) = \frac{1 - \eta(t)^2}{1 - \eta(t) u_0'(\alpha)},$$

or, after integrating and using (4.1.10) and $\gamma(0,t) \equiv 0$,

$$\gamma(\alpha,t) = \begin{cases} (1-\eta(t))\alpha, & 0 \le \alpha < 1/4, \\ \alpha + \eta(t)(\alpha - 1/2), & 1/4 \le \alpha < 3/4, \\ \alpha + \eta(t)(1-\alpha), & 3/4 \le \alpha \le 1. \end{cases} \tag{4.1.12}$$

Now, since $\dot{\gamma} = u \circ \gamma$, we have that

$$u(\gamma(\alpha,t),t) = \begin{cases} -\alpha\dot{\eta}(t), & 0 \le \alpha < 1/4, \\ (\alpha - 1/2)\dot{\eta}(t), & 1/4 \le \alpha < 3/4 \\ (1-\alpha)\dot{\eta}(t), & 3/4 \le \alpha \le 1 \end{cases} \tag{4.1.13}$$

where, by (2.1.16) and (4.1.11) above,

$$\dot{\eta}(t) = (1-\eta(t)^2)^2.$$

Notice that (4.1.12) let us solve for $\alpha = \alpha(x,t)$, the inverse Lagrangian map. We find

$$\alpha(x,t) = \begin{cases} \frac{x}{1-\eta(t)}, & 0 \le x < \frac{1-\eta(t)}{4}, \\ \frac{2x+\eta(t)}{2(1+\eta(t))}, & \frac{1-\eta(t)}{4} \le x < \frac{3+\eta(t)}{4}, \\ \frac{x-\eta(t)}{1-\eta(t)}, & \frac{3+\eta(t)}{4} \le x \le 1, \end{cases} \tag{4.1.14}$$

which we use on (4.1.13) to obtain the corresponding Eulerian representation

$$u(x,t) = \begin{cases} -(1-\eta(t))(1+\eta(t))^2 x, & 0 \le x < \frac{1-\eta(t)}{4}, \\ \frac{1}{2}(1+\eta(t))(1-\eta(t))^2(2x-1), & \frac{1-\eta(t)}{4} \le x < \frac{3+\eta(t)}{4}, \\ (1-\eta(t))(1+\eta(t))^2(1-x), & \frac{3+\eta(t)}{4} \le x \le 1, \end{cases} \tag{4.1.15}$$

which in turn yields

$$u_x(x,t) = \begin{cases} -(1-\eta(t))(1+\eta(t))^2, & 0 \le x < \frac{1-\eta(t)}{4}, \\ (1+\eta(t))(1-\eta(t))^2, & \frac{1-\eta(t)}{4} \le x < \frac{3+\eta(t)}{4}, \\ -(1-\eta(t))(1+\eta(t))^2, & \frac{3+\eta(t)}{4} \le x \le 1. \end{cases} \tag{4.1.16}$$

Finally, solving the IVP for η gives

$$t(\eta) = \frac{1}{2}\left(\operatorname{arctanh}(\eta) + \frac{\eta}{1-\eta^2}\right),$$

so that the blow-up time (2.1.18) is given by

$$t_* = \lim_{\eta \uparrow 1} t(\eta) = +\infty.$$

See figure 4.2(A) below.

Example 6. Finite-time blow-up for $\lambda = -2$

Using (4.1.9) and $\lambda = -2$ we find that

$$\bar{\mathcal{K}}_0(t) = \frac{\sqrt{1-2\eta(t)} + \sqrt{1+2\eta(t)}}{2} \tag{4.1.17}$$

and

$$\int_0^1 \frac{u_0'(\alpha)\,d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda}}} = \frac{d\bar{\mathcal{K}}_0(t)}{d\eta} = \frac{1}{2}\left(\frac{1}{\sqrt{1+2\eta(t)}} - \frac{1}{\sqrt{1-2\eta(t)}}\right)$$

for $\eta \in [0, \eta_*)$ and $\eta_* = 1/2$. Then, (2.1.20) yields

$$u_x(\gamma(\alpha,t),t) = \begin{cases} M(t) = \frac{\left(\sqrt{1-2\eta(t)}+\sqrt{1+2\eta(t)}\right)^3}{8(1+2\eta(t))\sqrt{1-2\eta(t)}}, & \alpha \in [1/4, 3/4), \\ m(t) = -\frac{\left(\sqrt{1-2\eta(t)}+\sqrt{1+2\eta(t)}\right)^3}{8(1-2\eta(t))\sqrt{1+2\eta(t)}}, & \alpha \in [0, 1/4) \cup [3/4, 1], \end{cases} \tag{4.1.18}$$

so that, as $\eta \uparrow 1/2$, $M(t) \to +\infty$ whereas $m(t) \to -\infty$. The finite blow-up time $t_* > 0$ is obtained from (2.1.16) and (4.1.17) above. We find

$$t(\eta) = \frac{1}{6\eta^3}\left(\eta^2\left(6 - 4\sqrt{1-4\eta^2}\right) + \sqrt{1-4\eta^2} - 1\right),$$

so that $t_* = t(1/2) = 2/3$. See figure 4.2(B) below.

4.2 Examples for Section 3.3

Examples for Theorems 3.3.14, 3.3.65 and 3.3.101 are now presented. For simplicity, only Dirichlet boundary conditions are considered. Given $\lambda \neq 0$, the time-dependent integrals in (2.1.20) are evaluated and pointwise plots are generated using the Mathematica software. Whenever possible, plots in the Eulerian variable x, instead of the Lagrangian coordinate α, are provided. For practical reasons, details of the computations in most examples are omitted. Also, due to the difficulty in solving the IVP (2.1.16) for the function $\eta(t)$ in terms of elementary functions, most plots for $u_x(\gamma(\alpha,t),t)$ are against the variable η rather than t.

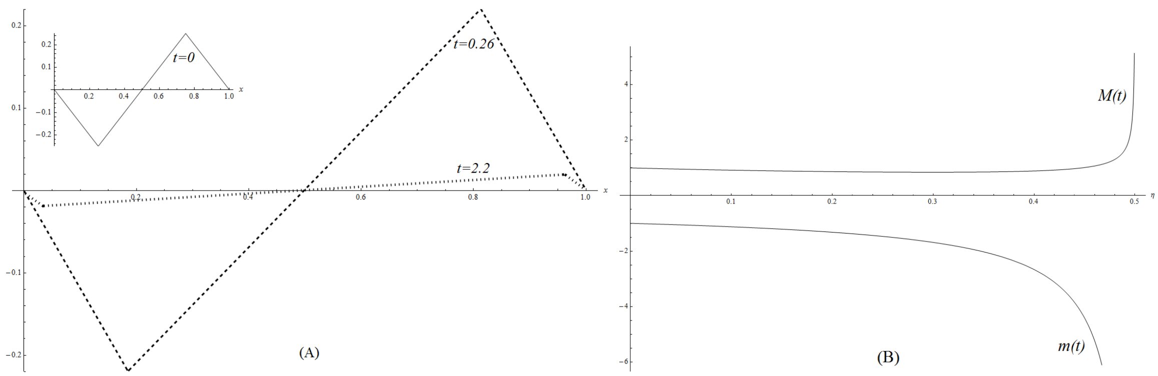

Figure 4.2: In figure A, (4.1.15) vanishes as $t \to +\infty$, while figure B depicts two-sided, everywhere blow-up of (4.1.18) as $\eta \uparrow \eta_* = 1/2$.

Example 1 below applies to stagnation point-form solutions to the incompressible 3D Euler equations, $\lambda = 1/2$. We consider two types of initial data, one satisfying (3.3.10) for $q \in (0, 1)$ and the other with $q > 1$. Recall from Table 3.2 that if $q \geq 1$, global existence in time follows, while, for $q \in (1/2, 1)$, we have finite-time blow-up instead. Below, we see that a spontaneous singularity may also form if $q = 1/3$.

Example 1. Regularity of stagnation point-form solutions to 3D Euler for $q = 1/3$ and $q = 6/5$

For $\lambda = 1/2$ and $\alpha \in [0, 1]$, first consider

$$u_0(\alpha) = \alpha(1 - \alpha^{\frac{1}{3}}). \tag{4.2.1}$$

Then

$$u_0'(\alpha) = 1 - \frac{4}{3}\alpha^{\frac{1}{3}}$$

achieves its maximum $M_0 = 1$ at $\overline{\alpha} = 0$. Also, $q = 1/3$ and $\eta_* = 2$. Furthermore, $u_0'(\alpha) \notin C^1(0, 1)$ at $\overline{\alpha}$, namely

$$\lim_{\alpha \to 0^+} u_0''(\alpha) = -\infty,$$

a jump discontinuity of infinite magnitude in u_0''. Evaluating the integrals in (2.1.20), we obtain

$$\bar{\mathcal{K}}_0(t) = -\frac{54(\eta(t) - 6)\eta(t) - 81(2 - \eta(t))(6 + \eta(t)) \operatorname{arctanh}\left(\frac{2\eta(t)}{\eta(t)-6}\right)}{4(6 + \eta(t))\eta(t)^3} \tag{4.2.2}$$

and

$$\int_0^1 \frac{u_0'(\alpha)\,d\alpha}{\mathcal{J}(\alpha,t)^3} = -\frac{27\left(9(2-\eta(t))(6+\eta(t))^2\,\log\left(\frac{24}{\eta(t)+6}-3\right)\right)}{8(6+\eta(t))^2\eta(t)^4} - \frac{27\left(8\eta(t)(54-(\eta(t)-9)\eta(t)) + 6\eta(t)(6+\eta(t))^2\,\text{arctanh}\left(\frac{2\eta(t)}{\eta(t)-6}\right)\right)}{8(6+\eta(t))^2\eta(t)^4} \tag{4.2.3}$$

for $0 \leq \eta < 2$. Furthermore, in the limit as $\eta \uparrow \eta_* = 2$,

$$\bar{\mathcal{K}}_0(t_*) = \frac{27}{16}, \qquad \int_0^1 \frac{u_0'(\alpha)\,d\alpha}{\mathcal{J}(\alpha,t)^3} \to +\infty.$$

Also, (2.1.16) and (4.2.2) yield

$$t(\eta) = -\frac{9\left(2\eta(6-5\eta)+9(\eta-2)^2\text{arctanh}\left(\frac{2\eta}{\eta-6}\right)\right)}{16\eta^2}$$

so that

$$t_* = \lim_{\eta\uparrow 2} t(\eta) = \frac{9}{4}.$$

Using (4.2.2) and (4.2.3) on (2.1.20), we find that $u_x(\gamma(\alpha,t),t)$ undergoes a two-sided, everywhere blow-up as $t \uparrow 9/4$. Now, if instead of $q = 1/3$ in (4.2.1) we let $q = 6/5$, then

$$u_0'(\alpha) = 1 - \frac{11}{5}\alpha^{\frac{6}{5}}$$

and u_0'' is now defined as $\alpha \downarrow 0$. In addition, for $q = 6/5$ we find that both integrals now diverge to $+\infty$ as $\eta \uparrow 2$, in contrast to the case $q = 1/3$ where $\bar{\mathcal{K}}_0(t)$ converged while $\bar{\mathcal{K}}_1(t)$ diverged. The diverging of the two integrals to $+\infty$ now causes a balancing effect amongst the terms in (2.1.20), which was absent for $q = 1/3$. Ultimately, we find that $u_x(\gamma(\alpha,t),t) \to 0$ as $\eta \uparrow 2$ for all $\alpha \in [0,1]$. Furthermore, using (2.1.16) we find that $t_* = +\infty$. See figure 4.3 below.

In Theorem 3.1.1, we showed that for a class of smooth initial data ($q = 2$), finite-time blow-up occurs for all $\lambda > 1$. Example 2 below is an instance of part 1 in Theorem 3.3.65. For $\lambda \in \{2, 5/4\}$, we consider initial data satisfying (3.3.10) for $q \in \{5, 5/2\}$, respectively, and find that solutions persist globally in time. Also, the example illustrates the two possible global behaviours: convergence of solutions, as $t \to +\infty$, to nontrivial or trivial steady states.

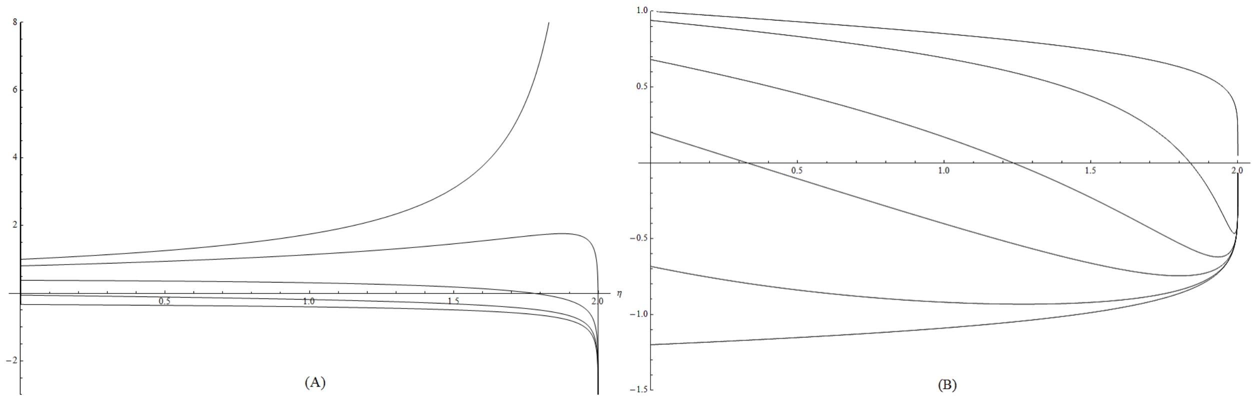

Figure 4.3: Example 1 for $\lambda = 1/2$ and $q \in \{1/3, 6/5\}$. Figure A depicts two-sided, everywhere blow-up of $u_x(\gamma(\alpha, t), t)$ for $q = 1/3$ as $\eta \uparrow 2$ $(t \uparrow 9/4)$, whereas, for $q = 6/5$, figure B represents its vanishing as $\eta \uparrow 2$ $(t \to +\infty)$.

Example 2. Global existence for $\lambda = 2$, $q = 5$ and $\lambda = 5/4$, $q = 5/2$

First, let $\lambda = 2$ and

$$u_0(\alpha) = \alpha(1 - \alpha^5). \tag{4.2.4}$$

Then

$$u_0'(\alpha) = 1 - 6\alpha^5$$

achieves its greatest value $M_0 = 1$ at $\overline{\alpha} = 0$ and $\eta_* = 1/2$. Since $\lambda = 2 \in [0, 5/2) = [0, q/2)$, part (1) of Theorem 3.3.65 implies global existence in time. Particularly

$$\lim_{t \to +\infty} u_x(\gamma(\alpha, t), t) = 0.$$

See figure 4.4(A). Now, suppose $\lambda = 5/4$ and replace $q = 5$ in (4.2.4) by $q = 5/2$. Then,

$$u_0'(\alpha) = 1 - \frac{7}{2}\alpha^{5/2}$$

attains $M_0 = 1$ at $\overline{\alpha} = 0$ and $\eta_* = 4/5$. Because $\lambda = 5/4 = q/2$, part 1 of Theorem 3.3.65 implies that u_x converges to a nontrivial steady-state as $t \to +\infty$. See figure 4.4(B).

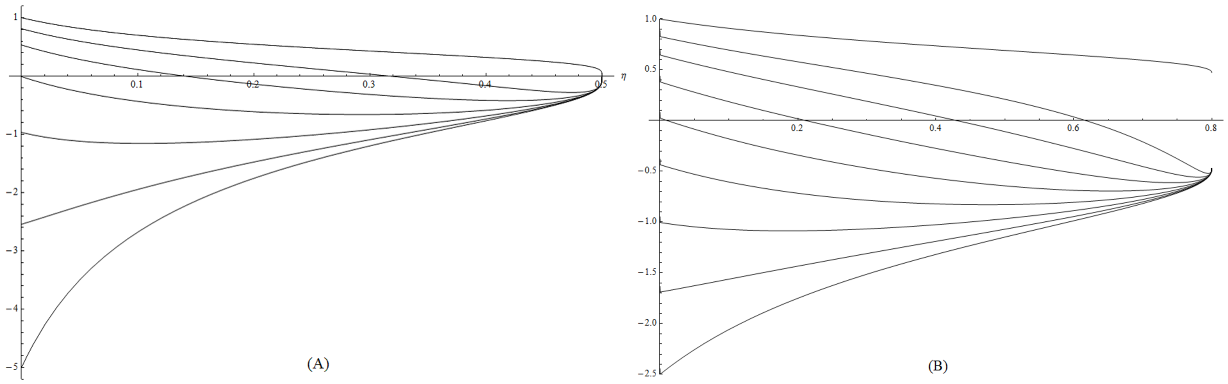

Figure 4.4: For example 2, figure A represents the vanishing of $u_x(\gamma(\alpha,t),t)$ as $\eta \uparrow 1/2$ $(t \to +\infty)$ for $\lambda = 2$ and $q = 5$, whereas, figure B illustrates its convergence to a nontrivial steady state as $\eta \uparrow 4/5$ $(t \to +\infty)$ if $q = 5/2$ and $\lambda = 5/4 = q/2$.

Example 3. Two-sided, everywhere blow-up for $\lambda = \frac{11}{2}$ and $q = 6$

Suppose $\lambda = 11/2$ and

$$u_0(\alpha) = \frac{\alpha}{11}(1-\alpha^6).$$

Then

$$u_0'(\alpha) = \frac{1}{11}(1-7\alpha^6)$$

attains its greatest value $M_0 = 1/11$ at $\overline{\alpha} = 0$. Also, $\eta_* = 2$ and $\lambda = 11/2 \in (q/2, q)$. According to 2 in Theorem 3.3.65, two-sided, everywhere blow-up takes place. The estimated blow-up time is $t_* \sim 22.5$. See figure 4.5(A).

Example 4. One-sided, discrete blow-up for $\lambda = -5/2$ and $q = 3/2$

Let $\lambda = -5/2$ and

$$u_0(\alpha) = \alpha(\alpha^{\frac{3}{2}} - 1).$$

Then u_0' attains its minimum $m_0 = -1$ at $\underline{\alpha} = 0$ and $\eta_* = 2/5$. Since $\frac{q}{1-q} < \lambda < -1$, part 2 of Theorem 3.3.101 implies one-sided, discrete blow-up. The estimated blow-up time is $t_* \sim 0.46$. See figure 4.5(B). We remark that in Theorem 3.1.1, the same value of λ for a class of smooth initial data with $q = 2$ led to two-sided, everywhere blow-up instead.

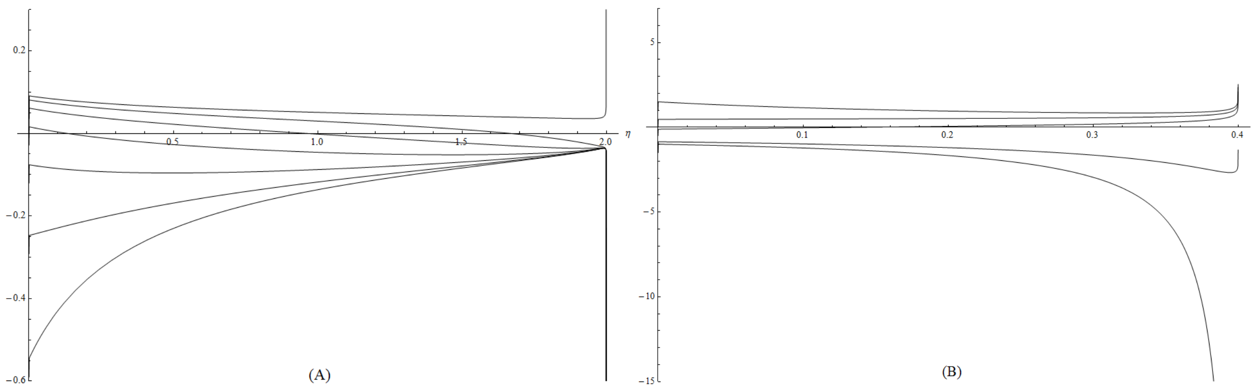

Figure 4.5: Figure A for example 3 depicts two-sided, everywhere blow-up of $u_x(\gamma(\alpha,t),t)$ as $\eta \uparrow 2$ $(t \uparrow 22.5)$ for $\lambda = 11/2$ and $q = 6$, while, figure B for example 4 illustrates one-sided, discrete blow-up, $m(t) = u_x(0,t) \to -\infty$, as $\eta \uparrow 2/5$ $(t \uparrow t_* \sim 0.46)$ for $\lambda = -5/2$ and $q = 3/2$.

In these last two examples, we consider smooth data with either mixed local behaviour near two distinct locations $\underline{\alpha}_j$ for $\lambda = -1/3$, or M_0 occurring at both endpoints for $\lambda = 1$.

Example 5. One-sided, discrete Blow-up for $\lambda = -1/3$ and $q = 1, 2$

For $\lambda = -1/3$, let

$$u_0(\alpha) = \alpha(1-\alpha)(\alpha - \frac{3}{4})\left(\alpha - \frac{1+4\sqrt{22}}{36}\right).$$

Then $m_0 \sim -0.113$ occurs at both $\underline{\alpha}_1 = 1$ and $\underline{\alpha}_2 = \frac{4+\sqrt{22}}{24} \sim 0.36$. Now, near $\underline{\alpha}_2$, u_0' behaves quadratically $(q = 2)$, whereas, for $1 - \alpha > 0$ small, it behaves linearly $(q = 1)$. The quadratic behaviour is due to u_0'' having order one at $\underline{\alpha}_2 \sim 0.36$, thus, Corollary 3.3.118 implies a discrete, one-sided blow-up. Similarly in the case of linear behaviour according to Theorem 3.3.14. After evaluating the integrals, we find that $m(t) \to -\infty$ as $t \uparrow t_* \sim 17.93$. Due to the Dirichlet boundary conditions, one blow-up location is the boundary $\underline{x}_1 = 1$, while the interior blow-up location, $\underline{x}_2$, is obtained by setting $\alpha = \underline{\alpha}_2$ into (2.2.1) and letting $\eta \uparrow \eta_* = \frac{3}{|m_0|}$. We find that $\underline{x}_2 \sim 0.885$. See figure 4.6($A$).

Example 6. Two-sided, everywhere blow-up for stagnation point-form solutions to the 2D incompressible Euler equations ($\lambda = 1$)

For $\lambda = 1$, let

$$u_0(\alpha) = \alpha(\alpha - 1)(\alpha - 1/2).$$

Then, $M_0 = 1/2$ occurs at both endpoints $\overline{\alpha}_i = \{0, 1\}$. Also $\eta_* = 2$ and since

$$u_0'(\alpha) = M_0 - 3\alpha + 3\alpha^2 = M_0 - 3\,|\alpha - 1| + 3(\alpha - 1)^2,$$

the local behaviour of u_0' near both endpoints is linear ($q = 1$). The integrals in (2.1.20) evaluate to

$$\bar{\mathcal{K}}_0(t) = \frac{2\,\mathrm{arctanh}(y(t))}{\sqrt{3\eta(t)(4 + \eta(t))}}$$

and

$$\int_0^1 \frac{u_0'(\alpha)\,d\alpha}{\mathcal{J}(\alpha, t)^2} = \frac{d\bar{\mathcal{K}}_0(t)}{d\eta}$$

for $0 \leq \eta < 2$ and where

$$y(t) = \frac{\sqrt{3\eta(t)(4 + \eta(t))}}{2(1 + \eta(t))}.$$

Using (2.1.11), we plug the above into (2.1.20) to find that

$$M(t) = u_x(0, t) = u_x(1, t) \to +\infty$$

as $\eta \uparrow 2$, while

$$u_x(x, t) \to -\infty$$

for all $x \in (0, 1)$. The blow-up time is estimated from (2.1.16) and $\bar{\mathcal{K}}_0(t)$ above as $t_* \sim 2.8$. See figure 4.6(B).

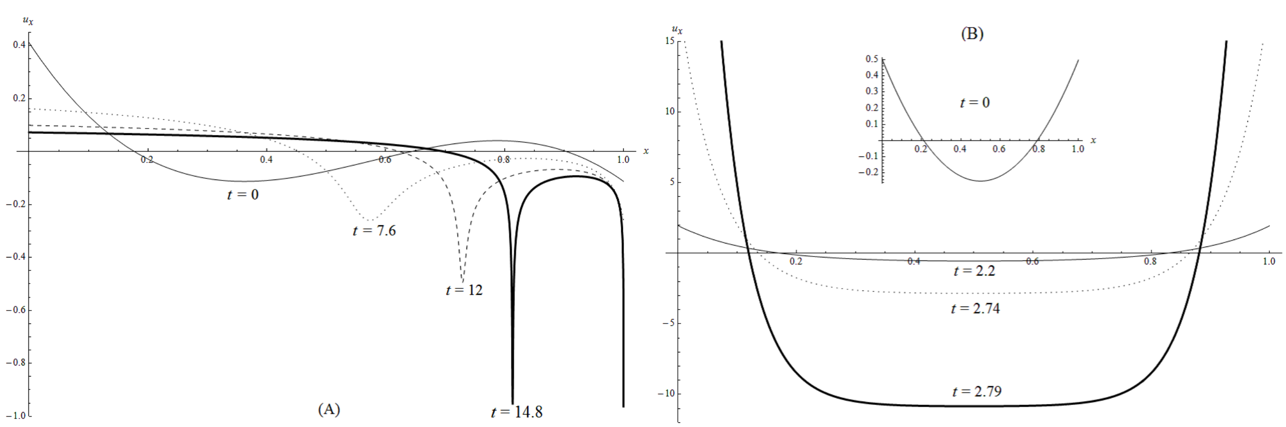

Figure 4.6: Figure A for example 5 with $\lambda = -1/3$ and $q = 1, 2$, depicts one-sided, discrete blow-up, $m(t) =\to -\infty$, as $t \uparrow 17.93$. The blow-up locations are $\underline{x}_1 = 1$ and $\underline{x}_2 \sim 0.885$. Then, figure B for example 6 with $\lambda = 1$ and $q = 1$, represents two-sided, everywhere blow-up of $u_x(x, t)$, as $t \uparrow 2.8$.

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Appendices

Appendix A - Global existence for $\lambda = 0$

To obtain the corresponding solution formulae for $\lambda = 0$, a limiting argument on (2.1.14) may be used.

Suppose $u_0'(\alpha)$ is, at least, $C^0(0,1)$ *a.e.*. Then for $n \in \mathbb{R}^+$, $\eta \in [0, \eta_*)$ and $\eta_* = \frac{n}{M_0}$ set

$$\psi_n = \left(1 - \frac{\eta u_0'}{n}\right)^n .$$

Observe that $\psi_n > 0$ for all $n > 0$ and $\alpha \in [0,1]$. As a result, since

$$\lim_{n\to+\infty} \psi_n = \exp\left\{ \lim_{n\to+\infty} \frac{\log\left(1 - \frac{\eta u_0'}{n}\right)}{\frac{1}{n}} \right\} = e^{-\eta u_0'}$$

then

$$\lim_{n\to+\infty} \int_0^1 \psi_n^{-1} d\alpha = \int_0^1 e^{\eta u_0'} d\alpha.$$

We conclude that

$$\lim_{n\to+\infty} \left\{ \frac{\psi_n^{-1}}{\int_0^1 \psi_n^{-1} d\alpha} \right\} = \frac{e^{\eta u_0'}}{\int_0^1 e^{\eta u_0'} d\alpha}. \tag{4.2.5}$$

But for $\lambda \neq 0$, the jacobian γ_α in (2.1.14) may be written as

$$\gamma_\alpha(\alpha, t) = \frac{\left(1 - \frac{\eta u_0'}{n}\right)^{-n}}{\int_0^1 \left(1 - \frac{\eta u_0'}{n}\right)^{-n} d\alpha} = \frac{\psi_n^{-1}}{\int_0^1 \psi_n^{-1} d\alpha} \tag{4.2.6}$$

for $n = \frac{1}{\lambda}$. Then, letting $n \to +\infty$ in (4.2.6) and using (4.2.5) implies that

$$\gamma_\alpha(\alpha, t) = \frac{e^{\eta u_0'}}{\int_0^1 e^{\eta u_0'} d\alpha} \tag{4.2.7}$$

in the limit as $\lambda \to 0^+$. But we know that $\dot{\gamma}_\alpha = (u_x(\gamma(\alpha,t),t))\gamma_\alpha$, so that

$$u_x(\gamma(\alpha,t),t) = u_0'(\alpha) - \frac{\int_0^1 u_0'(\alpha) e^{tu_0'(\alpha)} d\alpha}{\int_0^1 e^{tu_0'(\alpha)} d\alpha}. \tag{4.2.8}$$

The representation formula (4.2.8) is also valid if $\lambda \to 0^-$ by following an argument similar to the one above. Finally, (4.2.8) easily implies that

$$0 \leq u_0'(\alpha) - u_x(\gamma(\alpha,t),t) \leq \int_0^1 u_0'(\alpha)\, e^{tu_0'(\alpha)} d\alpha, \quad t \geq 0.$$

The global existence for the other types of initial data are analogous, and follow from the above.

Appendix B - Proof of Lemma 3.0.11

For the hypergeometric series (3.0.7), we have the following convergence results [21]:

- Absolute convergence for all $|z| < 1$.
- Suppose $|z| = 1$, then
  1. Absolute convergence if $Re(a + b - c) < 0$.
  2. Conditional convergence for $z \neq 1$ if $0 \leq Re(a + b - c) < 1$.
  3. Divergence if $1 \leq Re(a + b - c)$.

Furthermore, consider the identities [21]:

$$\frac{d}{dz} {}_2F_1\left[a, b; c; z\right] = \frac{ab}{c}\, {}_2F_1\left[a+1, b+1; c+1; z\right] \tag{4.2.9}$$

and

$${}_2F_1\left[a, b; b; z\right] = (1 - z)^{-a}, \tag{4.2.10}$$

as well as the contiguous relations

$$z\, {}_2F_1\left[a+1, b+1; c+1; z\right] = \frac{c}{a-b}\left({}_2F_1\left[a, b+1; c; z\right] - {}_2F_1\left[a+1, b; c; z\right]\right) \tag{4.2.11}$$

and

$${}_2F_1\left[a, b; c; z\right] = \frac{b}{b-a}\, {}_2F_1\left[a, b+1; c; z\right] - \frac{a}{b-a}\, {}_2F_1\left[a+1, b; c; z\right] \tag{4.2.12}$$

for $b \neq a$. Suppose $b < 2$, $0 \leq |\beta - \beta_0| \leq 1$ and $\epsilon \geq C_0$ for some $C_0 > 0$. We show that

$$\frac{1}{\epsilon^b}\frac{d}{d\beta}\left((\beta-\beta_0)\,{}_2F_1\left[\frac{1}{q}, b; 1+\frac{1}{q}; -\frac{C_0\,|\beta-\beta_0|^q}{\epsilon}\right]\right) = (\epsilon + C_0\,|\beta-\beta_0|^q)^{-b} \tag{4.2.13}$$

for all $q > 0$ and $b \neq 1/q$. For simplicity, let us denote ${}_2F_1$ by F. Also, all constants and variables are assumed to be real-valued. Set

$$a = 1/q, \qquad c = a+1, \qquad z = -\frac{C_0\,|\beta-\beta_0|^q}{\epsilon}.$$

Then $-1 \leq z \leq 0$, $a + b - c = b - 1 < 1$ and

$$\frac{dz}{d\beta} = -\frac{qC_0}{\epsilon}(\beta-\beta_0)\,|\beta-\beta_0|^{q-2}.$$

Therefore,

$$\begin{aligned}
&\frac{d}{d\beta}\left((\beta-\beta_0)F\left[a,b;c;z\right]\right) = (\beta-\beta_0)\frac{d}{d\beta}\left(F\left[a,b;c;z\right]\right) + F\left[a,b;c;z\right] \\
&= \frac{ab}{c}(\beta-\beta_0)F\left[a+1,b+1;c+1;z\right]\frac{dz}{d\beta} + F\left[a,b;c;z\right], && \text{by (4.2.9)} \\
&= \frac{b}{c}\left(zF\left[a+1,b+1;c+1;z\right]\right) + F\left[a,b;c;z\right], && \text{by } \frac{dz}{d\beta} \\
&= \frac{b}{a-b}\left(F\left[a,b+1;c;z\right] - F\left[a+1,b;c;z\right]\right) + F\left[a,b;c;z\right], && \text{by (4.2.11)} \\
&= \frac{b}{a-b}\left(F\left[a,b+1;c;z\right] - F\left[a+1,b;c;z\right]\right) + \frac{b}{b-a}F\left[a,b+1;c;z\right] \\
&- \frac{a}{b-a}F\left[a+1,b;c;z\right], && \text{by (4.2.12)} \\
&= F\left[a+1,b;c;z\right] = F\left[b,a+1;c;z\right], && \text{by (3.0.7)} \\
&= F\left[b,c;c;z\right], && \text{by } c = a+1 \\
&= \left(1 + \frac{C_0\,|\beta-\beta_0|^q}{\epsilon}\right)^{-b}, && \text{by (4.2.10)} \\
&= \epsilon^b\left(\epsilon + C_0\,|\beta-\beta_0|^q\right)^{-b}.
\end{aligned} \tag{4.2.14}$$

Multiplying both sides by ϵ^{-b} yields our result. $\square$

Notice that no issue arises in the use of identity (4.2.10) because, in our case, $-1 \leq z \leq 0$.

Appendix C - Proof of (3.0.4) and (3.0.5)

We prove (3.0.4) and (3.0.5) for $\lambda > 0$. The case of parameter values $\lambda < 0$ follows similarly.

Suppose $\lambda > 0$ and set $\eta_\epsilon = \frac{1}{\lambda M_0 + \epsilon}$ for arbitrary $\epsilon > 0$. Then $0 < \eta_\epsilon < \eta_*$ for $\eta_* = \frac{1}{\lambda M_0}$. Also, due to the definition of M_0,

$$1 - \lambda\eta_\epsilon u_0'(\alpha) = \frac{\epsilon + \lambda(M_0 - u_0'(\alpha))}{\lambda M_0 + \epsilon} > 0$$

for all $\alpha \in [0,1]$, while $1 - \lambda\eta_\epsilon u_0'(\alpha) = 0$ only if $\epsilon = 0$ and $\alpha = \overline{\alpha}_i$. We conclude that

$$1 - \lambda\eta(t)u_0'(\alpha) > 0 \tag{4.2.15}$$

for all $0 \leq \eta(t) < \eta_*$ and $\alpha \in [0,1]$. But $u_0'(\alpha) \leq M_0$, or equivalently

$$u_0'(\alpha)(1 - \lambda\eta(t)M_0) \leq M_0(1 - \lambda\eta(t)u_0'(\alpha)),$$

therefore (4.2.15) and $u_0'(\overline{\alpha}) = M_0$, yield

$$\frac{u_0'(\alpha)}{\mathcal{J}(\alpha,t)} \leq \frac{u_0'(\overline{\alpha})}{\mathcal{J}(\overline{\alpha},t)} \tag{4.2.16}$$

for $0 \leq \eta < \eta_*$ and

$$\mathcal{J}(\alpha,t) = 1 - \lambda\eta(t)u_0'(\alpha), \qquad \mathcal{J}(\overline{\alpha},t) = 1 - \lambda\eta(t)M_0.$$

The representation formula (2.1.20) and (4.2.16) then imply

$$u_x(\gamma(\overline{\alpha},t),t) \geq u_x(\gamma(\alpha,t),t) \tag{4.2.17}$$

for $0 \leq \eta(t) < \eta_*$ and $\alpha \in [0,1]$. Then (3.0.4) follows by using (2.1.27), definition (3.0.2) and (in)equality (4.2.17). Likewise, to establish (3.0.5) for $\lambda > 0$, notice that $u_0'(\alpha) \geq m_0 = u_0'(\underline{\alpha})$ gives

$$u_0'(\alpha)(1 - \lambda\eta(t)m_0) \geq m_0(1 - \lambda\eta(t)u_0'(\alpha)),$$

and so by (4.2.15),

$$\frac{u_0'(\alpha)}{\mathcal{J}(\alpha,t)} \geq \frac{u_0'(\underline{\alpha})}{\mathcal{J}(\underline{\alpha},t)} \tag{4.2.18}$$

for $0 \leq \eta < \eta_*$ and $\mathcal{J}(\underline{\alpha},t) = 1 - \lambda\eta(t)m_0$. The representation formula (2.1.20) and (4.2.18) then imply

$$u_x(\gamma(\alpha,t),t) \geq u_x(\gamma(\underline{\alpha},t),t) \tag{4.2.19}$$

for $0 \leq \eta(t) < \eta_*$ and $\alpha \in [0,1]$.

Similarly for $\lambda < 0$, (4.2.15) holds with $\eta_* = -\frac{1}{\lambda m_0} > 0$ instead. Both (3.0.4) and (3.0.5) then follow as above. $\square$

Appendix D - Inequalities

Definition 4.2.20. A function $f : \mathbb{R} \to \mathbb{R}$ is called convex provided that

$$f(rx + (1-r)y) \le rf(x) + (1-r)f(y)$$

for all $x, y \in \mathbb{R}$ and each $0 \le r \le 1$.

Proof of $(a+b)^p \le 2^{p-1}(a^p + b^p)$**:** For $p \ge 1$ and nonnegative reals a and b,

$$(a+b)^p \le 2^{p-1}(a^p + b^p)$$

Proof. Since $f(x) = x^p$ is convex for all $p \ge 1$, we use the above definition, with $r = 1/2$, to obtain

$$\left(\frac{a+b}{2}\right)^p = f\left(\frac{a}{2} + \left(1 - \frac{1}{2}\right) b\right) \le \frac{f(a)}{2} + \left(1 - \frac{1}{2}\right) f(b) = \frac{a^p + b^p}{2} \tag{4.2.21}$$

for nonnegative $(a, b) \in \mathbb{R}^2$. $\square$

Jensen's Inequality: Assume $f : \mathbb{R} \to \mathbb{R}$ is convex, and $\mathbf{U} \subset \mathbb{R}^n$ is open and bounded. In addition, let $u : \mathbf{U} \to \mathbb{R}$ be summable. Then

$$f\left(⨍_{\mathbf{U}} u\,dx\right) \le ⨍_{\mathbf{U}} f(u)\,dx \tag{4.2.22}$$

where $⨍_{\mathbf{U}} g\,dx = \frac{1}{|\mathbf{U}|}\int_{\mathbf{U}} g\,dx =$ average of g over $\mathbf{U}$ and $+\infty > |\mathbf{U}| =$ measure of $\mathbf{U}$.

Proof. Since f is convex, we have that for each $p \in \mathbb{R}$ there exists $r \in \mathbb{R}$ such that

$$f(q) \ge f(p) + r(q-p) \qquad \forall \;\; q \in \mathbb{R}.$$

Let $p = ⨍_{\mathbf{U}} u\,dx$ and $q = u$. The inequality follows after integrating the above in x over $\mathbf{U}$. $\square$

Vita

Alejandro Sarria was born in March 20 1982 in Tocaima, Colombia. He obtained his Bachelor and Master of science degrees in Mathematics in May 2008 and December 2009, respectively, at the University of New Orleans. At this institution, he is currently pursuing a Doctor of Philosophy in applied Mathematics (engineering and applied sciences program) under the guidance of Professor Ralph Saxton. The anticipated date for culmination of the program is December 2012.