Large-Amplitude Vibration of Imperfect Rectangular, Circular and Laminated Plate with Viscous Damping

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He Huang

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NOMENCLATURE

\( A \) = Initial Vibration Amplitude

\( C \) = \( \frac{Eh}{(1 - v^2)} \)

\( D \) = \( \frac{Eh^3}{[12(1 - v^2)]} \), Flexural rigidity of the skin

\( E \) = Young’s modulus of the skin

\( F \) = Stress function of the plate

\( h \) = Skin thickness

\( M \) = Number of half-waves in x direction

\((N_t, N_r)\) = Membrane stress resultants

\( P \) = Lateral pressure

\( Q \) = Forcing lateral pressure

\((U, V, W)\) = Axial, circumferential and out-of-plane displacements

\( W_0 \) = Initial out-of-plane imperfection

\( X \) = In-plane coordinate

\( Y \) = In-plane coordinate

\( a \) = Length of rectangular plate

\( a_1 \) = Radius of circular plate

\( b \) = Width of rectangular plate

\( c \) = \([3(1 - v^2)]^{0.5}\)

\( c_1, c_2 \) = Boundary condition parameters

\( f \) = \( \frac{2cF}{(Eh^3)} \); Non-dimensional stress function

\( h \) = Thickness of rectangular plate
\( k \) = Duffing equation parameter

\( n \) = Number of half-waves in y direction

\( \bar{r} \) = Radial coordinate of the circular plate

\( r \) = Nonlinearity indicator

\( t \) = Non-dimensional time

\( \bar{t} \) = Time

\( w \) = \( W/h \); Non-dimensional out-of-plane displacement

\( (x, y) \) = In-plane coordinates

\([A_{ij}], [B_{ij}], [D_{ij}]\) = Parameter matrix

\(L_A(\cdot), L_B(\cdot), L_D(\cdot)\) = Linear differential operators

**GREEK NOMENCLATURE**

\( \delta \) = Damping ratio

\( \bar{\nu}^2 \) = Differential operator

\( \epsilon \) = Duffing equation parameter

\( \mu \) = Imperfection amplitude normalized with the thickness

\( \nu \) = Poisson's ratio

\( \tau \) = \( \tau = tk^{0.5} \), reduced-time

\( \omega_L, \omega_{nl} \) = Linear, nonlinear vibration frequency
ABSTRACT

Large-amplitude vibration of thin plates and shells has been critical design issues for many engineering structures. The increasingly more stringent safety requirements and the discovery of new materials with amazingly superior properties have further focused the attention of research on this area. This thesis deals with the vibration problem of rectangular, circular and angle-ply composite plates. This vibration can be triggered by an initial vibration amplitude, or an initial velocity, or both. Four types of boundary conditions including simply supported and clamped combined with in-plane movable/immovable are considered.

To solve the differential equation generated from the vibration problem, Lindstedt's perturbation technique and Runge-Kutta method are applied. In previous works, this problem was solved by Lindstedt’s Perturbation Technique. This technique can lead to a quick approximate solution. Yet based on mathematical assumptions, the solution will no longer be accurate for large amplitude vibration, especially when a significant amount of imperfection is considered. Thus Runge-Kutta method is introduced to solve this problem numerically. The comparison between both methods has shown the validity of the Lindstedt's Perturbation Technique is generally within half plate thickness. For a structure with a sufficiently large geometric imperfection, the vibration can be represented as a well-known backbone curve transforming from soften-spring to harden-spring. By parameter variation, the effects of imperfection,
damping ratio, boundary conditions, wave numbers, young’s modulus and a dozen more related properties are studied. Other interesting research results such as the dynamic failure caused by out-of-bound vibration and the change of vibration mode due to damping are also revealed.

Keywords: Large-amplitude Vibration; Damping; Backbone Curve; Dynamic Failure; Plate and Shell
CHAPTER 1 INTRODUCTION

1.1 Motivation and Scope

1.1.1 History

Large amplitude vibration of geometrically imperfect plates [1,2] and shells [3] has been a subject of intense investigation over the past several decades. This topic is of significant importance when the engineering safety of vehicles and aircrafts [4] is considered. Perfect plate and spherical shell vibration was first studied [5]. The effect of initial geometric imperfection on finite amplitude vibration of rectangular plates was then studied by Celep [6,7]. Simultaneously, the effects of large amplitude axisymmetric vibration of perfect circular plates was examined by Yamaki et al. [8]. For perfect rectangular and perfect circular plates, the large amplitude vibrations were found to always be the hardening type (frequencies increase with an increase in vibration amplitudes) with simply supported, in-plane immovable edges. Yet this is not the case when geometric imperfections and other types of boundary conditions are considered.

Later, Hui studied the effects of geometric imperfections on the linear and non-linear axisymmetric vibrations of rectangular plates [9,10] and circular plates...
The presence of geometric imperfection raises the infinitesimal vibration frequency and cause the usual hard-spring nonlinear vibration to exhibit soft-spring behavior. Crandall [13] was the first to mention: “... the dynamic nonlinearity exhibited by contact vibration is of a strongly softening nature; the mean dynamic location shifts to the left (right) of the static equilibrium position for a nonlinear force-displacement law which curves upward (downward)”. Geometric imperfection has been established to cause the strongly softening nature in normally hardening vibration.

In the broader area of vibration, the influence of geometric imperfections and in-plane boundary conditions on finite-amplitude vibrations of simply supported cylindrical panels was examined by Hui [14]. This showed that for an imperfect cylindrical panel, the linear vibration frequency and the finite amplitude vibration behavior for the following two cases are quite different: (a) vibration mode with one axial half-wave and length-to-width ratio being one and (b) two axial half-waves and length-to-width ratio being two. Hui began investigating the effects of geometric imperfection of finite-amplitude vibrations of shallow spherical shells was performed by Hui [15]. In this study, the spherical shells were assumed sufficiently shallow so the boundary conditions were neglected and replaced by the periodicity requirements of the displacements. The above work was followed [16] and extended to laminated angle-ply and cross-ply rectangular plates with various in-plane and out-of-plane boundary conditions [17] and composite plates with curvilinear fibers [18]. Based on
previous papers on finite amplitude vibrations of plates and shells: an increase (decrease) in a linear vibration frequency due to the presence of geometric imperfection is accompanied by a more pronounced soft-spring (hard-spring) finite-amplitude vibration behavior. However, note that this conclusion is only valid within the relatively small range of applicability of the vibration amplitude using the perturbation technique.

1.1.2 Motivation

All theoretical works mentioned above concerning the effects of geometric imperfections on finite-amplitude vibrations of thin-walled structures involved solving the modified-Duffing equation: using the Lindstedt's perturbation technique [9,10] or the method of averages [11]. Therefore, the results are only valid for a sufficiently small range of vibration, which is not enough for a thin-plate large amplitude vibration problems. The author evaluates the validity of Lindstedt's perturbation technique is evaluated by the author here and in two other papers [19,20]. Through comparison to other published works [21], it has been proven that the backbone curves from Runge-Kutta Method offer a better solution for large vibration amplitude problems.

Furthermore, vibration amplitude can be extended to as large as four times the plate thickness, which has not been previously attempted [22], and requires a
thin-plate structure. The imperfection is also considered as large as two times the plate thickness and this large imperfection has a significant impact on the vibration results [23]. Also studied here is the effect of viscous damping [24] on the finite-amplitude response of plates, which previously had not drawn attention. The existence of damping leads the vibration mode to become more linear; most importantly noting that the more damping present in the system, the more linear the resulting vibration.

This research will focus more on extending previous works and verifying current works through the use of finite element analysis methods. These methods become popular in more recent works concerning vibration analysis [25,26]. If the effects of the forcing function are considered, then the resulting vibration frequency is largely affected by the forcing frequency [27,28]. Dynamic buckling analysis performed on plates [29], columns and shells will also be of great interest, followed by composite plate, cross-ply composite plate, and infinite layers next. The effects of hysterical damping [30] and viscous damping on the vibration mode, the changes to the vibration mode, and the relationship between them will also be of interest.
1.2 Plate Vibration

1.2.1 Rectangular Plate Vibration

Considering rectangular plate vibration, the effect of initial geometric imperfection on finite amplitude vibration of rectangular plates was first studied by Celep [6,7]. Theoretical [31,32] and experimental [33,34] analysis on rectangular plates and circular plates [35,36] can both be found in literature: these works were focusing on small amplitude vibration of perfect rectangular plates with boundary conditions and initial conditions set for simply supported and initial vibration amplitude. Free periodical [37] and transverse vibration [38] of plates are also considered as free vibration serve as the most basic type of plate vibration problems.

Hui further studied the finite amplitude vibration of plates within small ranges of imperfection [39]. The imperfection is proved to lead the usual hardening vibration mode to exhibit soft spring vibration mode at small vibration amplitude, then bending back to hard spring again at larger vibration amplitude. Chapter 3 of this thesis studies the vibration problem of rectangular plates and compares it with several previous works [40,41]. The solutions of the modified Duffing ordinary differential equation for large-amplitude vibrations of imperfect rectangular plate with viscous damping are generated using Lindstedt's perturbation technique [42] and the Runge-Kutta method. The solutions are then compared for a validity check. The expected outcome is that
the backbone curve will appear when the data is plotted and that it will be better
developed. The effects of viscous damping [43] is also of great interest.

Note that the vibration discussed here is a free vibration. There is no
compression or extension on the boundary conditions [44] on any of the four edges.
The vibration will be started by initial amplitude, or an initial velocity, or both, and
the usage of Matlab has guaranteed accurate mathematical solutions of all three
different conditions. In future works, consistent impact may be taken into account for
this type of vibration problems. These new parameters, along with potential boundary
pressures will have an interesting effect on the vibration of rectangular, circular, and
composite plate problems.

1.2.2 Circular Plate vibration

The effects on circular plate vibration, in particular, large amplitude
axisymmetric vibration of perfect circular plates were first examined by Yamaki et al.
[8]. Yamaki discovered that the large amplitude vibrations of circular plates were
always of the hardening type (frequencies increase with vibration amplitudes) with
simply supported, in-plane immovable edges, except when an initial imperfection is
taken into account. For a forced circular plate vibration problem [45], a constant load
result is quite different from a periodical load result; this has been proved both in
theory [46] and by experimentation [47,48]. In this paper, the difference between the
3D loading and 3D vibration mode [49] and the free circular plate vibration [50] is explored.

Later Hui [11,12] revealed the effects of geometric imperfections on the linear and non-linear axisymmetric vibrations of circular plates. The vibration mode, the initial geometric imperfection and the forcing function are assumed to have the same spatial shape. Shown in prior works [51,52]: geometric imperfections of the order of a fraction of the plate thickness may significantly raise the free vibration frequencies and even cause the plates to exhibit non-linear soft-spring behavior instead of the inherent non-linear hard-spring characteristics of plates. These pervious works used Lindstedt's perturbation technique [53] to determine whether it's soft-spring or hard-spring. Lindstedt's perturbation technique, on the other hand, is based on mathematical assumptions and simplifications. Thus the validity of prior solutions should be further analyzed.

The modified-Duffing equation by Runge-Kutta method is studied in Chapter 4 and leads to a numerical solution. In addition to the two main boundary conditions (i) simply supported or (ii) clamped boundary conditions [54] explored, two additional types of in-plane boundary conditions are considered: (iii) zero in-plane radial stress (in-plane movable) and (iv) zero in-plane radial displacement (in-plane immovable) at the circular edge. Under each of these four boundary conditions, the solutions are compared between Lindstedt's and numerical with respect to different geometric
imperfection. Since Lindstedt's perturbation technique is confirmed accurate only for small amplitude vibration, and further proved that the effects of certain large imperfection, cause a softening-hardening process, which should ultimately result in a backbone shape curve [55].

In addition to the results mentioned earlier, the boundary condition of clamped and in-plane movable is a special case: the plate is always softening, but only under such boundary condition. When the initial vibration amplitude exceeds a certain limit, the vibration is no longer a stable sine wave and a dynamic failure [56,57] occurs immediately. If a sufficiently large imperfection exists, the plate can fail with a very small initial vibration amplitude as only half plate thickness. The results are related to the buckling failure [58,59], which has the same interesting failure mode. While this is an interesting failure topic for circular plate vibration, it has unfortunately received little attention in literature on this subject.

1.2.3 Laminated Plate, Cylindrical Panel and Spherical Shell Vibration

After the first investigations on the effects of bending-stretching coupling of composite plates [60] and shells [61], large-amplitude vibration of anti-symmetric angle-ply plates were studied at similar time period by Bennett [62]. Bert [63], Chandra and Basava Raju [64] further examined this theory and completed a full-type analysis. In that analysis, instead of angle-ply, large amplitude vibration of cross-ply
anti-symmetric plates were analyzed [65] and compared with different boundary conditions [66] including new additional parameters of shear and rotary inertia [67]. The effects of these new parameters were examined by Sathyamoorthy and Chia [68], Wu and Vinson [69]. Celep [70] further introduced larger vibration amplitude and relatively large imperfection while independently Bert [71] conducted comparisons between the new and old composite plate vibration theory. Then, with the wide acceptance of finite element methods, these previous results were checked by Reddy and Chao [72]. Excellent review papers on these abundant literatures of composite plate vibration problem were written by Leissa [73] and Chia [74]. Bert specially included dynamics of composite and sandwich plates [75] in his review.

As the effects of geometric imperfections and in-plane boundary conditions on the linear/nonlinear vibration behavior of angle-ply [76] and cross-ply [77] rectangular thin plates received more attention, frequency-load interactions [78] on plates focused on the effect of the loading conditions. Considering free vibrations, based on the dynamic analogue of Von Karman differential equations valid for large deflections, more analyses were carried out. The effects of fiber angles and number of layers for angle-ply rectangular plates were examined specially for graphite-epoxy, glass-epoxy and boron-epoxy [79].

Chapter 6 examines the solution of modified-Duffing ordinary differential equation for large-amplitude vibrations of imperfect angle-ply composite rectangular
plates. Two in-plane and two out-of-plane boundary conditions are considered as well as viscous damping conditions. Graphite-epoxy, glass-epoxy and boron-epoxy are chosen, with respect to different fiber angles, for simplified comparisons with prior works [80,81]. Solving this angle-ply composite rectangular plate vibration problem, required Lindstedt’s perturbation technique and Runge-Kutta’s method. The solution from the two methods are plotted and compared for a validity check. Lindstedt’s perturbation technique is proven to be accurate for sufficiently small vibration amplitude and imperfection. The results from Runge-Kutta method are plotted to form backbone curves. One of the most important factors considered here is the viscous damping. Much like the parabolic umbilic catastrophe [82], viscous damping may use polynomial approach, rather than just linear. Also studied in this chapter is the effect of fiber volume ratio, imperfection amplitude and several other parameters.

When considering cylindrical panels and spherical shells, it is interesting to note that, although the derivation of the problem is quite different, the governing equation generated is still a modified Duffing equation. Specifically, a three dimensional solution of the free vibration problem of homogeneous isotropic cylindrical shells and panels is given by Soldatos [83]: other parameters, like the effect of functionally graded materials [84], or given temperature gradients [85] are separately considered. A refined formulation of an approach suitable for 3D vibration analyses of homogeneous and cross-ply laminated cylinders and cylindrical panels is then presented [86].
While spherical shell vibration is not studied in detail this thesis, there have been intense investigations and recent publications concerning this field of research. Depending on the vibration atmosphere, spherical shell vibration is studied in the acoustic medium [87] and in a fluid filled condition [88] where the effect of bending [89], plus the vibration and different kinds of boundary conditions [90] are explored later to further extend the practical meaning of this research.
CHAPTER 2

LINDSTEDT'S PERTURBATION TECHNIQUE AND NUMERICAL METHOD

2.1 Introduction

Abundant in the (large-amplitude) vibration of plates and shells literature [2-4], it is not difficult to realize that many of these problems finally become the solution of modified Duffing ordinary differential equation. Lindstedt and Poincare first carried out a perturbation technique to solve this equation through several mathematical assumptions and simplifications. After Lindstedt and Poincare's research, this perturbation technique became widely used and accepted in related research. The vibration problem also benefits from this method and provides relatively accurate solution as long as the vibration amplitude is small and there is no large imperfection.

Later, large-amplitude vibration of thin-walled structures received further attention: where the vibration amplitude of thin-walled structure can reach to three or four times plate thickness within the plastic deformation zone. The solution from Lindstedt's perturbation technique, proved in Chapters 3 and 4, was accurate only for vibration amplitude smaller than about 0.8 plate thickness. Thus, the large-amplitude
vibration problem solved in some of the early literature can be partially incorrect.

The effect of the geometric imperfection on small amplitude rectangular plate vibration was then studied. Both experimental and theoretical analysis on rectangular plates and circular plates can be found in literature [26-31]. Hui further studied the plate vibration with finite amplitude and small imperfection [11,12]. Then perfect circular plate was studied to be the hardening type while only simply supported and in-plane immovable is considered. Thus for the perfect condition, which is no imperfection and fixed boundary condition, both rectangular and circular plate can still be solved by Lindstedt’s perturbation technique when vibration amplitude is about one plate thickness.

Yet the presence of geometric imperfection further shrinks the validity of Lindstedt’s perturbation technique. Only the solution for vibration amplitude under half plate thickness can be trusted. Thus we need a new method to solve this modified Duffing equation accurately. In this thesis, Runge-Kutta Method is utilized to solve this equation numerically. The solution is proved to be accurate as long as the Duffing equation is valid. The comparison of the solution is then carried out between Lindstedt’s and the numerical method for a validity check. The details are shown in Chapters 3 and 4.
2.2 Lindstedt's Perturbation Technique

2.2.1 Derivation of Lindstedt's Perturbation Technique

This section covers the details of applying Lindstedt's perturbation technique. This method is introduced using the Donnell type equilibrium equation and the compatibility equation for the plates,

The second order non-linear Duffing ordinary differential equation:

$$w(t)_{tt} + k[w(t) + \epsilon a_2 w(t)^2 + \epsilon w(t)^3] = 0$$  \hspace{1cm} (2.1)

By changing the variable, $\tau = \Omega t$, equation 2.1 becomes

$$\Omega^2 w(\tau)_{\tau\tau} + k[w(\tau) + \epsilon a_2 w(\tau)^2 + \epsilon w(\tau)^3] = 0$$  \hspace{1cm} (2.2)

The periodic solution $w(\tau)$ and the associated frequency are assumed to be of the forms

$$w(\tau) = w_0(\tau) + \epsilon w_1(\tau) + \epsilon^2 w_2(\tau) + \cdots,$$  \hspace{1cm} (2.3)

$$\Omega = \Omega_0 + \epsilon \Omega_1 + \epsilon^2 \Omega_2 + \cdots.$$  \hspace{1cm} (2.4)

Furthermore, the non-linear terms in equation 2.2 are also expanded in a power series in $\epsilon$. Equating terms involving $\epsilon^0$, $\epsilon$, and $\epsilon^2$ and dividing through by $k$ (defined to be $\Omega_0^2$), gives:

$$w_0(\tau)_{\tau\tau} + w_0(\tau) = 0$$  \hspace{1cm} (2.5)

$$w_1(\tau)_{\tau\tau} + w_1(\tau) = -w_0(\tau) - w_0(\tau)^3 - a_2 w_0(\tau)^2 - 2(\Omega_1/\Omega_0)w_0(\tau)_{\tau\tau}$$  \hspace{1cm} (2.6)
\[ w_2(\tau)_{\tau\tau} + w_2(\tau) = \left[ -2a_2 w_0(\tau) - 3w_0(\tau)^2 \right] w_1(\tau) - \left[ 2(\Omega_1/\Omega_0) + (\Omega_1/\Omega_0)^2 \right] w_0(\tau)_{\tau\tau} - 2(\Omega_1/\Omega_0) w_1(\tau)_{\tau\tau} \quad \ldots \quad (2.7) \]

The initial conditions are:

\[ w_0(\tau)_{\tau\tau} = 0, w_1(\tau)_{\tau\tau} = 0, w_2(\tau)_{\tau\tau} = 0, \quad \ldots \quad (2.8) \]

The solution of equation 2.5 is:

\[ w_0(\tau) = A \cos(\tau) \quad (2.9) \]

Substituting \( w_0(\tau) \) into equation 2.6, one obtains:

\[ w_1(\tau)_{\tau\tau} + w_1(\tau) = -\left( a_2 A^2 / 2 \right) \left[ 1 + \cos(2\tau) \right] + \left[ 2(\Omega_1/\Omega_0) A - (3A^3 / 4) \right] \cos(\tau) - (A^3 / 4) \cos(3\tau) \quad (2.10) \]

In order to avoid secular terms, the coefficient of \( \cos(\tau) \) is set to zero so that:

\[ \Omega_1 = (3/8)\Omega_0 A^2 \quad (2.11) \]

Accordingly, the solution to the differential equation for \( w_1(\tau) \) is:

\[ w_1(\tau) = -\left( a_2 A^2 / 2 \right) + (a_2 A^2 / 6) \cos(2\tau) + (A^3 / 32) \cos(3\tau) \quad (2.12) \]

Substituting the known forms for \( w_0(\tau), w_1(\tau) \) and \( \Omega_1 \) into equation 2.6, one obtains:

\[ w_2(\tau)_{\tau\tau} + w_2(\tau) = \left[ 2(\Omega_2/\Omega_0) A + (5/6)a_2^2 A^3 + (15/128)A^5 \right] \cos(\tau) - \left[ (4/3)(\Omega_1/\Omega_0)a_2 A^2 - (a_2/32)A^4 \right] \cos(2\tau) + \left[ (21A^5 / 128) - (a_2^2 A^3 / 6) \right] \cos(3\tau) - (a_2 A^4 / 32) \cos(4\tau) - (3A^5 / 128) \cos(5\tau) \quad (2.13) \]

Again, setting the coefficient of \( \cos(\tau) \) to zero yields

\[ \Omega_2 / \Omega_0 = -\left[ (15A^4 / 256) + (5/12)a_2^2 A^2 \right] \quad (2.14) \]

Furthermore, the solution to the differential equation for \( w_2(\tau) \) is:

\[ w_2(\tau) = (-1/3) \left[ (4/3)(\Omega_1/\Omega_0)(a_2 A^2) - (a_2 A^4 / 32) \right] \cos(2\tau) - (1/8) \left[ (21/
\[ 128A^5 - (a_2^2A^3/6)\cos(3\tau) + (1/15)(a_2A^4/32)\cos(4\tau) + (A^5/10)\cos(5\tau) \]

(2.15)

The solution is obtained by assembling \(w_0(\tau), w_0(\tau)\) and \(w_0(\tau)\) in equation 2.3 and replacing \(\tau\) by \(\Omega t + \varphi\). Thus, for a given set of initial conditions \(w(t = 0)\) and \(w_r(t = 0)\), the value of the amplitude \(A\) and the phase angle \(\varphi\) can be found. Finally, substituting \(\Omega_1\) and \(\Omega_2\) into equation 2.4 shows that the ratio of the non-linear frequency to the linear frequency is related to the vibration amplitude by

\[ \frac{\Omega}{\Omega_0} = 1 + [(3\varepsilon/8) - (5/12)a_2^2\varepsilon^2]A^2 - (15\varepsilon^2/256)A^4 \]

(2.16)

### 2.2.2 Example for Lindstedt's Perturbation Technique

Here an example problem is given solved by Lindstedt's Perturbation Technique.

If there is a simply supported imperfect rectangular plate going to start vibration by an initial vibration amplitude. Then the modified-Duffing equation with additional quadratic term can be achieved by applying the Galerkin procedure followed by integration on both sides,

\[ w(t)_{tt} + 2\delta k^0.5w(t)_{t} + k[w(t) + \epsilon a_2w(t)^2 + \epsilon w(t)^3] = q(t) \]

(2.17)

Where \(k, \epsilon\) and \(a_2\) are found to be,

\[ k = (1/4)[(M^2 + n^2)^2 + (M^4 + n^4)(\mu c^2/2)] \]

\[ \epsilon = (M^4 + n^4)[c^2/(16k)] \]

\[ a_2 = 3\mu \]

(2.18)
In the above, $w(t)$ is the out-of-plane deflection normalized to the thickness, $k^{0.5}$ is the linear vibration frequency, $k\varepsilon a_2$ is the quadratic coefficient, $k\varepsilon$ is the cubic coefficient, $q(t)$ is the forcing function, $t$ is the non-dimensional time, and $\delta$ is the damping ratio. The normalized cubic coefficient $\epsilon$ is defined as the ratio of the cubic coefficient to the linear coefficient. Further details and explanations will be shown in Chapter 3.

Lindstedt's perturbation technique can be utilized to solve the equation 2.17. Figure 2.1 shows the nonlinearity indicator $r$ versus the normalized-cubic coefficient $\epsilon$ for various imperfection amplitudes $\mu$.

Figure 2.1 Nonlinearity indicator versus normalized-cubic coefficient for various imperfection magnitudes
For Lindstedt’s perturbation technique, at least within sufficiently small range of the vibration amplitude, the vibration behavior can be classified as hard-spring or soft-spring depending on whether the nonlinearity indicator is greater than or less than zero, respectively. Considering a perfect system with no imperfection, the nonlinearity indicator is positive and increases linearly with the normalized-cubic coefficient. However, for an imperfect system, as the normalized-cubic coefficient increases, the nonlinearity indicator increases from zero and then decreases to assume a negative value.

Figure 2.2  Initial vibration amplitude versus nonlinear/linear vibration frequency

Figure 2.2 gives us the graph of the initial vibration amplitude versus the ratio of the nonlinear to linear vibration frequency with respect to a series of different initial imperfection values. It is obvious from Figure 2.2 that all the curves are symmetric to
the x axis no matter the initial geometric imperfection exists or not. Physically this symmetry is not accurate since initial geometric imperfection will affect the vibration mode, which leads to an asymmetry in the result. This inaccuracy is caused by the assumption within the Lindstedt's perturbation technique when solving the modified-Duffing equation.
2.3 Runge-Kutta Method

2.3.1 Matlab Programming for Runge-Kutta Method

Furthermore, Matlab programming is introduced to utilize the Runge-Kutta method. The following is a short example for the same rectangular plate vibration problem considered in section 2.2.

Example Matlab Program:

(1)

```matlab
N=-4;m=1;
[k,e,a,del]=const(1,1,0.3,0,1);
r=3*e/8-5*a*a*e*e/12;
while (N<4),
    options=odeset; options.RelTol=0.00000001; options.AbsTol=0.0000001;
    options.MaxStep=0.00005;
    clear t;
    clear x;
    [t,x]=ode45('Duffing',[0,1000],[N,0]);
    q=1;
    clear P;
    for i=20:500,
        if x(i)*10000000*x(i+1)*1000000000>0,i=i+1;
        else P(q)=t(i); q=q+1; i=i+1;
    end
    end

Y2(m)=1+r*N*N-15*e*e/256*N^4;
T2(m)=20*pi/(-P(1)+P(21));
X2(m)=N;
U2(m)=Y2(m)-T2(m);
m=m+1; N=N+0.03;
end
```
function xp=Duffing(t,x)
[k,e,a,del]=const(1,1,0.3,0,0);
xp=zeros(2,1);
xp(1)=x(2);
xp(2)=-2*del*x(2)-x(1)-e*a*x(1)^2-e*x(1)^3;
end

function [k,e,a,del]=const(M,n,r,del,imp)
c=(3*(1-r^2))^0.5;
k=((M*M+n*n)^2+(M^4+n^4)*(imp^2*c^2/2))/4;
a=3*imp;
e=(M^4+n^4)*(c*c/(16*k));
del=del;
end

In this case, the data can be generated for the vibration thus plotting the data to get our accurate backbone curve. This is especially meaningful when there is a large-amplitude vibration with imperfection and viscous damping.

2.3.2 Example for Runge-Kutta Method

In this section the data from 2.3.1 will be plotted to generate the accurate backbone curve. Then a reasonable comparison will be made between the solution of Lindstedt's and the numerical solution. This methodology will be further utilized in Chapter 3, 4 and 5.

It is clear from Figure 2.3 that when the imperfection amplitude is larger than 0.5, there occurs two turning points in one curve, which is the reason why it's called a
backbone curve. This proves that a backbone curve will generate for any large amplitude vibration as long as there is no failure.

Figure 2.3 Initial vibration amplitude versus the ratio of nonlinear to linear vibration frequency

Since the solution from Lindstedt's Perturbation Technique has been plotted in section 2.2, there is a necessity to make a validity check. According to figure 2.4, it is obvious to find out that the two results coincident with each other only with small values of initial vibration amplitude $A$. By careful calculation and comparison, a 1% difference between Lindstedt's results and our numerical results happens at $-0.73 \sim 0.60$ times of plate thickness when the imperfection is two.
In this chapter, two methods are adopted to solve the modified Duffing equation. Details of both methods are explained. An example is carried out, analyzed, then compared. These two methods will appear frequently in later chapters.

Figure 2.4 A Comparison at Imperfection=2.0
CHAPTER 3

ACCURATE BACKBONE CURVE FOR
LARGE-AMPLITUDE VIBRATIONS OF IMPERFECT
RECTANGULAR PLATE WITH VISCOS DAMPING

3.1 Introduction

This chapter deals with the solution of modified-Duffing ordinary differential equation for large-amplitude vibrations of imperfect rectangular plate with viscous damping. Lindstedt's perturbation technique and Runge-Kutta method are applied. The results for both methods are presented and compared for a validity check. It is proved that Lindstedt’s perturbation technique only works accurately for a small range of vibration amplitude. For a structure with a sufficiently large geometric imperfection, the well-known soften-spring to harden-spring transforming backbone curve is confirmed and better developed. Although the softening to hardening behavior occurs twice in one backbone curve, the turning points share the same vibration frequency. Yet the amplitude for those two points varies due to the existence of imperfection. Moreover, the effects of damping ratio on vibration mode and vibration amplitude are studied. The usual nonlinear vibration tends to behave more linearly under the effects of large damping.
3.2 Governing Equations

3.2.1 Strain-Displacement Relationship

Based on a series of papers on single mode finite amplitude vibration of plates and shells by Hui, it can be assumed that the geometric imperfection is stress-free. Thus the nonlinear strain-displacement relationship is,

\[ \begin{align*}
\varepsilon_x &= U_x + (W/R_1) + 0.5(W_x)^2 + W_{0x} W_x \\
\varepsilon_y &= V_y + (W/R) + 0.5(W_y)^2 + W_{0y} W_y \\
\gamma_x &= U_y + V_x + (W_0 + W) \dot{w_x} + W_{0y} \dot{w_x} 
\end{align*} \]

(3.1) \hspace{2cm} (3.2) \hspace{2cm} (3.3)

Where \( U \) and \( V \) are the in-plane displacements, \( W \) is the out-of-plane displacement, \( W_0 \) is the initial geometric imperfection and \( \varepsilon_x \), \( \varepsilon_y \) and \( \gamma_x \) are the strains. Further, the above relations are applicable to flat plates \((R_1 = \infty)\), open cylindrical panels or closed cylindrical shells \((R = \text{constant}, R_1 = \infty)\) and spherical shells \((R = R_1 = \text{constant})\).

3.2.2 Non-Dimensionalization

The dynamic analogue of von Karman equilibrium and compatibility differential equations for large amplitude vibrations of plates, written in terms of the out-of-plane displacement \( w \) and the Airy stress function \( f \), incorporating the possibility of a geometric imperfection \( w_0 \) are, respectively, in non-dimensional form,
\[(1 + i\eta)(w_{xxxx} + w_{yyyy} + 2w_{xxyy}) = 4\pi^4q(x,y,t) - 4\pi^4w_{tt} + (2c)[f_{xx}(w + w_0)_{yy} + f_{yy}(w + w_0)_{xx} - 2f_{xy}(w + w_0)_{xy}]
\]

\[1/(1 + i\eta)(f_{xxxx} + f_{yyyy} + 2f_{xxyy}) = (2c)[(w_{xy})^2 + 2w_{xy}w_{0,xy} - w_{yy}(w + w_0)_{xx} - w_{xx}w_{0,yy}]\]

Where,
\[(w, w_0) = (W, W_0)/h,\]
\[f = 2cF/(Eh^3),\]
\[(x, y) = (X, Y)/b, t = \bar{t}w_r,\]
\[c = [3(1 - \nu^2)]^{0.5},\]
\[q(x,y,t) = [c^2b^4/(Eh^4\pi^4)]Q(X,Y,\bar{t}),\]
\[(w_r)^2 = 4\pi^4D/(\rho b^4) = \pi^4Eh^3/(\rho c^2b^4)\]  

(3.4)

In the foregoing, \(a, b\) and \(h\) are the length, width and thickness of the plate, \(E\) is the young's modulus, \(\nu\) is the Poisson's ratio, \(\rho\) is the plate mass per unit area, \(\eta\) is the loss factor associated with the complex-modulus model for structural damping, \(i = (-1)^{0.5}\), \(X\) and \(Y\) are the in-plane coordinates, \(W_0\) is the initial geometric imperfection, \(Q(X,Y,\bar{t})\) is the forcing function, \(\bar{t}\) is the time, and \(w_r\) is the reference frequency.

The boundary conditions are simply supported, which is
\[w(x = 0 \text{ or } a/b) = 0\]
\[w_{xx}(x = 0 \text{ or } a/b) = 0\]
\[w(y = 0 \text{ or } 1) = 0\]
\[w_{yy}(y = 0 \text{ or } 1) = 0\]  

(3.5)

(3.6)

(3.7)
Furthermore, there is no in-plane shear stress along all the plane edges and the in-plane displacements normal to the edges are constant, which leads to

\[ f_{xy}(x = 0 \text{ or } a/b) = 0; \]
\[ f_{xy}(y = 0 \text{ or } 1) = 0; \]  

(3.8)

Since the fundamental vibration for rectangular plate is considered, the vibration mode will be taken as half sine-waves in both x and y directions. By the rule of not affecting the presenting problem dealing primarily with free vibration of plates, it can be assumed the forcing pressure distribution to be of the same half sine-waves shape as the vibration mode. Then it is concluded that the vibration mode, the initial geometric imperfection and the forcing pressure distribution to be the same specific shape as following,

\[ [w(x, y, t), w_0(x, y), q(x, y, t)] = [w(t), \mu, q_1\cos(wt/w_0)]\sin(M\pi x)\sin(n\pi y); \]  

(3.9)

Where \( \mu \) is the amplitude of the geometric imperfection normalized with respect to the plate thickness, \( M = mb/a \), where m and n are the number of half-waves in the x and y directions respectively. Substitute \( w(x, y, t), w_0(x, y), \) the boundary conditions and the nonlinear compatibility equation into (2), which leads to the stress function as following,

\[ f(x, y, t) = (1 + in)[w(t)^2 + 2\mu w(t)](c/16) * [(n/M)^2 \cos(2M\pi x) + (M/n)^2 \cos(2n\pi y)]; \]  

(3.10)
3.2.3 Governing Equations

Finally, substitute equation 3.7 and 3.8 into the nonlinear equilibrium equation.

3.2. A modified-Duffing equation with additional quadratic term can be achieved by applying the Galerkin procedure followed by integration on both sides,

\[ w(t)_{tt} + 2\delta k^{0.5}w(t)_t + k[w(t) + \epsilon a_2 w(t)^2 + \epsilon w(t)^3] = q(t) \]  \hspace{1cm} (3.11)

Where \( k, \epsilon \) and \( a_2 \) are found to be,

\[ k = (1/4)[(M^2 + n^2)^2 + (M^4 + n^4)(\mu^2c^2/2)] \]

\[ \epsilon = (M^4 + n^4)[c^2/(16k)] \]

\[ a_2 = 3\mu \]  \hspace{1cm} (3.12)

In the above, \( w(t) \) is the out of plane deflection normalized to the thickness, \( k^{0.5} \) is the linear vibration frequency, \( k\epsilon a_2 \) is the quadratic coefficient, \( k\epsilon \) is the cubic coefficient, \( q(t) \) is the forcing function, \( t \) is the non-dimensional time, and \( \delta \) is the damping ratio. The normalized cubic coefficient \( \epsilon \) is defined as the ratio of the cubic coefficient to the linear coefficient.

Furthermore, the velocity term \( w(t)_t \) is due to the possible presence of viscous damping. This term will be zero if damping is not considered. The quadratic term \( w(t)^2 \), which does not exist in normal Duffing equation, is due to the initial geometric imperfection with spatial distribution the same as the assumed vibration mode shape, and the cubic term \( w(t)^3 \) arises from the usual finite strain displacement relations.
Introducing the reduced-time $\tau$ defined by

$$\tau = tk^{0.5}, \quad (r)^0 = k^{0.5}$$  \hspace{1cm} (3.13)

The above nonlinear ODE for free vibration becomes

$$w(\tau)_{r\tau} + 2\delta k^{0.5} w(\tau)_{\tau} + w(\tau) + \epsilon a_2 w(\tau)^2 + \epsilon w(\tau)^3 = 0$$  \hspace{1cm} (3.14)

The initial conditions for the initial deflection problem are:

$$w(\tau = 0) = A, \quad w_{\tau}(\tau = 0) = 0$$  \hspace{1cm} (3.15)

And the boundary condition is simply supported, in-plane movable.
3.3 Methodology for solving the ODE

To solve this modified-Duffing ordinary differential equation, two methods can be utilized. The first method will be Lindstedt's perturbation technique, which has been used in a lot of previous literature. The result for this method, however, will be invalid when the vibration amplitude is large since this perturbation is just a close approximation. The other method will be Runge-Kutta Method to solve this equation numerically.

Detailed procedure of Lindstedt's perturbation technique has been shown in Chapter 2. The perturbation results are offered below

\[ r = \left(\frac{3\epsilon}{8}\right)(1 - 10\mu^2\epsilon) \]  \hspace{1cm} (3.16)

\[ \frac{w_{\text{nonlinear}}}{w_{\text{linear}}} = 1 + rA^2 - \left(15\epsilon^2/256\right)A^4 \]  \hspace{1cm} (3.17)

Where \( r \) is the nonlinearity indicator, \( \epsilon \) is the normalized cubic coefficient and \( A \) is the initial vibration amplitude at time is 0.

On the other hand, by employing Runge-Kutta method in Matlab, the numerical solution for this modified-Duffing ordinary differential equation can be achieved. The numerical solution is relatively more accurate than Lindstedt's perturbation technique when considering large amplitude vibration. Both results will be presented and compared in the section of Results and Discussion.
3.4 Results and Discussion

3.4.1 Lindstedt's Solution

Lindstedt's perturbation technique is first utilized. Figure 3.1 shows the nonlinearity indicator $r$ versus the normalized-cubic coefficient $\epsilon$ for various imperfection amplitudes $\mu$ based on equation 3.14 and 3.15.

![Figure 3.1 Nonlinearity indicator versus normalized-cubic coefficient for various imperfection magnitudes](image)

For Lindstedt's perturbation technique, at least within sufficiently small range of the vibration amplitude, the vibration behavior can be classified as hard-spring or
soft-spring depending on whether the nonlinearity indicator is greater than or less than zero, respectively. Considering a perfect system with no imperfection, the nonlinearity indicator is positive and increases linearly with the normalized-cubic coefficient. However, for an imperfect system, as the normalized-cubic coefficient increases, the nonlinearity indicator increases from zero and then decreases to assume a negative value.

![Figure 3.2 Initial vibration amplitude versus the ratio of nonlinear to linear vibration frequency](image)

Figure 3.2 gives us the graph of the initial vibration amplitude versus the ratio of the nonlinear to linear vibration frequency with respect to a series of different initial imperfection values. It is obvious from Figure 3.2 that all the curves are symmetric to the x axis no matter the initial geometric imperfection exists or not. Physically this symmetry is not accurate since initial geometric imperfection will affect the vibration
mode, which leads to an asymmetry in the result. This inaccuracy is caused by the assumption within the Lindstedt's perturbation technique when solving the modified-Duffing equation.

### Numerical Solution

By employing Runge-Kutta method in Matlab, the numerical solution for this modified-Duffing ordinary differential equation can be obtained. Figure 3.3 displays a diagram of the out-of-plane displacement versus the reduced-time for various values of the imperfection amplitude. The relating parameters are $M=1$, $n=1$, $\delta=0$, $\nu=0.3$ and $c=1.237$. By plugging different imperfection and other related parameters into equation 3.10, different vibration conditions can be solved.

Considering a perfect system in Figure 3.3, with the initial vibration amplitude being the plate thickness ($A=\pm 1$), the structure presents a perfect sine-wave vibration between -1 and 1. For imperfect systems with $\mu = \pm 1, \pm 2$, the inward deflections are bigger than the outward deflections, which confirms the speculation that the plate tends to deflect more in the inward direction. The periods and amplitude for the vibration curves change dramatically as the imperfection increases.
A graph of the out-of-plane displacement versus the reduced-time for various values of the imperfection amplitude with viscous damping ratio to be 0.2 is given in Figure 3.4. (M=1, n=1, Poisson ratio=0.3) It is apparent to notice the peak values for deflection response curves decrease as time increases due to the existence of damping. This effect becomes tremendous when the damping ratio further arises. It is also important to note that the period for both the damped and the undamped system (with all other factors fixed) remains relatively unchanged. Besides, for a viscously damped system, the deflection is no longer necessarily predominantly inward. When facing large value of damping ratio, the effects of geometric imperfection become less apparent.
Based on the numerical calculation results shown in Figure 3.3 and 3.4, Figure 3.5 presents the backbone curves of the initial vibration amplitude versus the ratio of the nonlinear to linear vibration frequency. The system being considered is a rectangular plate simply supported along all four edges where $M$ is 1 and $n$ is 1. For a perfect system, the finite amplitude response is of the hardening type for all values of the initial vibration amplitude $A$. However, for an imperfect system, the system behaves as soft spring for small values of $A$ and hard spring for large values of $A$, which leads to a backbone curve. It is important to note that given a non-zero imperfection, the backbone curves are not symmetric to the x axis. But the peak values share the same frequency. For example, when geometric imperfection equals 2, the peak values are 0.676 for both positive and negative $A$. Yet the corresponding
vibration amplitude at the peak differs significantly. This asymmetry is mainly caused by the existence of initial imperfection. It is easier for the plate to reach the turning point when there is already a relatively big imperfection added to the initial vibration amplitude.

![Graph](image)

Figure 3.4(a) Initial vibration amplitude versus the ratio of nonlinear to linear vibration frequency

It is clear from Figure 3.5(a) that when the imperfection amplitude is larger than 0.5, there occurs two turning points in one curve like a human's backbone, which is the reason why it's called a backbone curve. Previous work on flat plate vibration, shown in Figure 3.5(b), can be used to verify the accuracy of this plot. By checking the values of both figures, it shows a great consistency between the old work and our
new work. Yet previous work only gives the results with initial vibration amplitude between -2 to 2. Thus the second turn for the backbone curve is missing. This proves that a backbone curve will be generated with respect to any large amplitude vibration as long as there is no failure.

Figure 3.5(b) New work compared with previous work from Hui [39]

Figure 3.5 has a significant value in engineering design. For a plate with initial imperfection, its vibration amplitude can be obtained as long as the vibration frequency is given. In other words, if the structure can be designed to vibrate within certain range of frequency, the vibration amplitude can be controlled to avoid structural failure.
3.4.3 Comparison between two solutions

By comparing Lindstedt's perturbation technique and numerical methods, the validity can be checked. Figure 6 shows a series of backbone curves with respect to different initial imperfections. It is not hard to point out that when there is no imperfection, the two solutions agree with each other much better than the situation when a huge imperfection exists.

Figure 3.6(a) A Comparison at Imperfection =0
Figure 3.7(b) A Comparison at Imperfection = 1.0

Figure 3.8(c) A Comparison at Imperfection = 2.0
According to the results of two calculation methods, it is obvious to find out that the two results coincident with each other only with small values of initial vibration amplitude A. By careful calculation and comparison, a 1% difference between Lindstedt's results and our numerical results is considered to be accurate, the corresponding region is listed in Table 3.1.

<table>
<thead>
<tr>
<th>Imperfection</th>
<th>Accurate region</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-2.00~2.00</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.87~0.63</td>
</tr>
<tr>
<td>1.0</td>
<td>-0.62~0.88</td>
</tr>
<tr>
<td>1.5</td>
<td>-0.67~0.60</td>
</tr>
<tr>
<td>2.0</td>
<td>-0.73~0.60</td>
</tr>
</tbody>
</table>

Table 3.1 serves a direct proof for our previous statement that the result of Lindstedt's technique is accurate only when the initial vibration amplitude A is small enough. It is important to note that the region for the Lindstedt's results to coincide with numerical results decreases as the imperfection grows. The above comparison is based on the situation of M=1, n=1, Poisson ratio=0.3 and no damping exists. The following analysis will focus on the effects of damping ratio on the vibration mode and vibration amplitude.
3.4.4 Effect of Damping Ratio

Figure 3.7 shows a series of backbone curves considering the effects of damping ratio on vibration mode and vibration amplitude. Other factors are $M=1$, $n=1$, Poisson’s ratio=0.3 and geometric imperfection to be one time of plate thickness ($\mu=1$). Note that the peak value of nonlinear vibration frequency moves closer to the linear vibration frequency as the damping ratio increases. The absolute value of $A$ at the peak also increases while the damping ratio increasing. It can be estimated that the backbone curve will be almost a straight vertical line of $\omega_{nl}/\omega_L=1$ at large damping ratio, which means in large damping situation, vibration will only be linear.

![Figure 3.7 Backbone curves considering the effects of damping ratio.](image)

Figure 3.9 Initial vibration amplitude versus ratio of vibration with respect to damping ratio
The vibration amplitude and frequency at the two peaks for a backbone curve are compared in Table 3.2. With the increasing of damping ratio, the nonlinear vibrating frequency at the peaks comes much closer to the linear frequency from 94.8% to 98.6%. Thus the vibration becomes more linearly at higher damping ratio.

Table 3.2 Values of $A$ and $w_{nL}/w_L$ at backbone curves' peak with respect to different damping ratio

<table>
<thead>
<tr>
<th>Damping</th>
<th>$A_1(&lt;0)$</th>
<th>$w_{nL}/w_L$</th>
<th>$A_2(&gt;0)$</th>
<th>$w_{nL}/w_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>-1.60</td>
<td>0.948</td>
<td>0.89</td>
<td>0.948</td>
</tr>
<tr>
<td>0.01</td>
<td>-2.08</td>
<td>0.955</td>
<td>1.12</td>
<td>0.951</td>
</tr>
<tr>
<td>0.02</td>
<td>-2.38</td>
<td>0.967</td>
<td>1.21</td>
<td>0.959</td>
</tr>
<tr>
<td>0.05</td>
<td>-2.56</td>
<td>0.986</td>
<td>1.49</td>
<td>0.974</td>
</tr>
</tbody>
</table>

All those solutions and analysis on vibration of plates can be beneficial for the designers in civil engineering and mechanical engineering. If it is possible to simulate earthquake as vibration source, the ceiling and wall structure may be simplified as rectangular plates. An estimate vibration amplitude will be generated to help prevent structural failure.
3.5 Conclusions

The solution of Lindstedt's perturbation technique and Runge-Kutta method of the modified Duffing equation has been studied. By Runge-Kutta numerical method, the backbone curve solution has been proved and better revealed when the vibration amplitude is larger than 1. A comparison between the solutions has been carried out. As a result, the validity of Lindstedt's perturbation technique has been calculated to be about half plate thickness. The effects of damping ratio on the vibration mode and vibration amplitude are then studied. The usual nonlinear vibration tends to behave more linearly under the effects of large damping. The present work confirms the predominantly inward deflections of an imperfect thin-walled structure undergoing free finite-amplitude vibrations without damping. However, this behavior becomes unobvious and finally disappears with the increasing of damping ratio for a viscously damped structure.
CHAPTER 4

IMPERFECTION SENSITIVITY OF ANTISYMMETRIC CROSS-PLY CYLINDER UNDER COMPRESSION USING HUI'S POSTBUCKLING METHOD

4.1 Introduction

This chapter deals with the solution of modified-Duffing ordinary differential equation for large-amplitude vibrations of imperfect circular plate. Four types of boundary conditions are considered as well as viscous damping. Lindstedt's perturbation technique and Runge-Kutta method are applied. The solution from two methods are plotted and compared for a validity check. Lindstedt's perturbation technique is proved to be accurate for sufficiently small vibration amplitude and imperfection. The results from Runge-Kutta method is plotted to form a backbone curve except for the case with clamped and zero radial stress boundary condition. Instead of expected softening-hardening process, the curve is only softening thus no longer backbone. More importantly, a dynamic failure is noticed when initial vibration amplitude grows under this boundary condition. A geometric imperfection will further help to trigger this failure at a smaller amplitude. This finding has served a new way to judge dynamic failure mode and is valuable for structure design concerning vibration.
4.2 Governing Equations

4.2.1 Governing Equations

The dynamic analogues of the Von Karman equilibrium and compatibility equations for axisymmetric vibrations of circular plates can be written in terms of the out-of-plane displacement \( W \) and the Airy stress function \( F \) as shown below,

\[
(1 + i\eta)D\bar{\nabla}^2(\bar{\nabla}^2 W) + \rho W_{tt} - (1/\bar{r})(F_{r}(W + W_{0})_{r} )_{r} = Q(\bar{r}, \bar{t}) \quad (4.1)
\]

\[
\bar{\nabla}^2(\bar{\nabla}^2 F) = (1 + i\eta)(-Eh/\bar{r})\{(W + W_{0})_{r} W_{tt} + W_{0tt} W_{r}\} \quad (4.2)
\]

Where the differential operator is defined to be,

\[
\bar{\nabla}^2(\ ) = (1/\bar{r})\{F(\ )_{r}\}_{r} \quad (4.3)
\]

Here \( \bar{r} \) is the radial coordinate of the circular plate, \( D \) is the flexural rigidity, \( E \) is the young's modulus, \( \rho \) is the mass per unit area, \( W_{0} \) is the radius of the circular plate, \( h \) is the thickness, \( \bar{t} \) is the time, \( W_{0} \) is the initial geometric imperfection, \( Q \) is the forcing lateral pressure and \( \eta \) is the damping ratio.

4.2.2 Non-Dimensionalization

To non-dimensionalize the equilibrium and compatibility equation, the following quantities are introduced,

\[
(w, w_{0}) = (W, W_{0})/h,
\]

\[
c = [3(1 - \nu^2)]^{0.5}
\]
\[ f = 2cF/(Eh^3), \]
\[ r = \bar{r}/a_1, \quad t = \bar{t}w_r, \]
\[ w_r = [D/(\rho a_1^4)]^{0.5} \]
\[ q(r, t) = [a_1^4/(hD)]Q(\bar{r}, \bar{t}) \quad (4.4) \]

Substituting equations 4 into equation 1 and 2, that will result in the non-dimensional dynamic equilibrium and compatibility equations,

\[ (1 + i\eta)\nabla^2(\nabla^2 w) + w_{tt} - (2c/r)[f_r (w + w_0)_r]_r = q(r, t) \quad (4.5) \]
\[ \nabla^2(\nabla^2 f) = (1 + i\eta)(-2c/r)[(w + w_0)_r w_{rr} + w_{0rr} w_r] \quad (4.6) \]

Where,

\[ \nabla^2( ) = (1/r)[r( )]_r \]
\[ \nabla^2(\nabla^2( )) = (1/r)[r[\nabla^2( )]_r]_r \quad (4.7) \]

**4.2.3 Four Boundary Conditions**

When considering boundary conditions, the out-of-plane displacement is taken to be zero at plate edge,

\[ w(r = 1) = 0 \quad (4.8) \]

Then the simply supported and clamped boundary conditions are defined as,

\[ w_{rr} (r = 1) + vw_r (r = 1) = 0, \quad \text{and} \quad w_r (r = 1) = 0, \quad \text{respectively.} \quad (4.9) \]

In addition to the two out-of-plane boundary conditions considered above, I also need to formulate the in-plane boundary conditions by expressing the membrane stress resultants \( N_r \) and \( N_t \) in terms of the in-plane radial displacement \( U \) and the
out-of-plane displacement $W$ as following,

\[
N_r = \left[(1 + \eta)Eh/(1 - \nu^2)\right]\left[(\nu U/\bar{r}) + U_{,rr} + 0.5(W_{,rr})^2 + W_{0,rr} W_{rr}\right]
\]

\[
N_t = \left[(1 + \eta)Eh/(1 - \nu^2)\right]\left[(U/\bar{r}) + \nu U_{,rr} + 0.5\nu(W_{,rr})^2 + \nu W_{0,rr} W_{rr}\right]
\]

(4.10)

Where,

$N_r = (1/\bar{r})F_{,rr}$, and $N_t = F_{,rr}$

(4.11)

Thus the stress function that related to $U$ is

$F_{,rr} = (\nu/\bar{r})F_r = (1 + \eta)Eh(U/\bar{r})$

(4.12)

Finally, the zero in-plane radial stress and the zero in-plane radial displacement boundary conditions are,

$f_r (r = 1) = 0$ and $f_{rr} (r = 1) = 0$, respectively.

(4.13)

The out-of-plane displacement, the initial geometric imperfection and the forcing pressure can be assumed to be of the same spatial shape as half-sine wave,

\[
[w(r,t),w_0(r),q(r,t)] = [w(t),\mu_1\cos(wt/w_r)](1 + c_1r^2 + c_2r^4)
\]

(4.14)

If simply supported boundary conditions are considered, $c_1$ and $c_2$ are calculated to be

$c_1 = -(6 + 2\nu)/(5 + \nu)$ and $c_2 = (1 + \nu)/(5 + \nu)$

(4.15)

Otherwise for clamped boundary conditions, $c_1$ and $c_2$ are taken to be

$c_1 = -2$ and $c_2 = 1$

(4.16)

Substituting equation 14 into the non-linear compatibility equation 6, one obtains,
\[ [r(\nabla^2 f)_r]_r = (1 + i\eta)(-8c)(c_1^2 r + 8c_1c_2r^3 + 12c_2^2 r^5)[w(t)^2 + 2\mu w(t)] \]

(4.17)

After integration,

\[ \nabla^2 f = (1 + i\eta)(-2c)(c_1^2 r^2 + 2c_1c_2 r^4 + 4c_2^2 r^6/3 + d_1)[w(t)^2 + 2\mu w(t)] \]

(4.18)

By similar calculation,

\[ f_r = (1 + i\eta)(-2c)(c_1^2 r^3/4 + c_1c_2 r^5/3 + c_2^2 r^7/6 + d_1 r/2)[w(t)^2 + 2\mu w(t)] \]

(4.19)

From equation 13, the zero radial stress boundary condition at the circular edge implies that

\[ d_1 = -(c_1^2/2 + 2c_1c_2/3 + c_2^2/3) \]

(4.20)

And the in-plane immovable boundary condition yields

\[ d_1 = \frac{-2}{(1 - \nu)}[(3 - \nu)(c_1^2/4) + (5 - \nu)(c_1c_2/3) + (7 - \nu)(c_2^2/6)] \]

(4.21)

4.2.4 Governing Equation

Finally by applying the Galerkin procedure, one obtains the Duffing equation with additional quadratic terms,

\[ (1 + i\eta)64c_2 l_0 w(t) + G_0 w(t)_{tt} + G_1[w(t)^3 + 3\mu w(t)^2 + 2\mu^2 w(t)](1 + i\eta)(4c^2) = q_1 G_0 \cos[\omega(w/w_r)t] \]

(4.22)
Where,

\[ I_0 = \int_0^1 (r + c_1 r^3 + c_2 r^5) \, dr = 0.5 + c_1/4 + c_2/6 \]  \hspace{1cm} (4.23)

\[ I_j = \int_0^1 (r^j)(r + c_1 r^3 + c_2 r^5) \, dr = 1/(j + 2) + c_1/(j + 4) + c_2/(j + 6) \]

\[ j = 2, 4, 6, 8 \ldots \]  \hspace{1cm} (4.24)

\[ G_0 = I_0 + c_1 I_2 + c_2 I_4 \]
\[ G_1 = (2c_1 d_1 I_0) + (8c_2 d_1 + 2c_1^3) I_2 + 10c_1^2 c_2 I_4 + (80c_1 c_2^2/3) I_6 + (20c_2^3/3) I_8 \]  \hspace{1cm} (4.25)

This modified-Duffing equation written in standard form is,

\[ w_{tt} + (1 + i\eta)\{w(t)^3 + a_2 w(t)^2\} + kw(t) = q_1 \cos[(w/w_r)t] \]  \hspace{1cm} (4.26)

Where,

\[ k = 64c_2 I_0/G_0 + 8c_2 \mu^2 G_1/G_0, \]
\[ \epsilon k = 4c_2^2 G_1/G_0, \]
\[ a_2 = 3\mu \]  \hspace{1cm} (4.27)

This Duffing equation can be solved numerically. At the same time, by using Lindstedt’s perturbation technique, one can also achieve the approximate solution in the following form. The difference between the numerical solution and Lindstedt’s solution will be discussed in a later chapter.

\[ w_{\text{nonlinear}}/w_{\text{linear}} = 1 + r^* A^2 - (15\epsilon^2/256)A^4 \]

\[ r^* = (3\epsilon/8)(1 - 10\mu^2 \epsilon) \]  \hspace{1cm} (4.28)

For numerical method, two supporting boundary conditions are needed to solve
the equation. By combining the two types of in-plane boundary conditions and the
two out-of-plane boundary conditions, four types of boundary conditions are
produced at the circular edge.

(i) Simply supported and zero in-plane radial stress

(ii) Simply supported and zero in-plane radial displacement

(iii) Clamped and zero in-plane radial stress

(iv) Clamped and zero in-plane radial displacement

In this chapter the damping ratio is taken to be zero. A non-zero damping will
lead to a gradual decrease in vibration amplitude. Detailed solution can also be solved
by equation 4.26. With no damping as assumed, the present analysis yields Duffing
type differential equations for each of the above four types of boundary conditions, as
follows (Poisson ratio=0.3):

\[ w(t)_{i_0} + (4c^2)[0.58811w(t)^3 + 1.76433\mu w(t)^2 + (2.24144 + 1.17762\mu^2)] \times \]
\[ w(t) = q_1\cos[(w/w_r)t] \]  
\[ w(t)_{i_0} + (4c^2)[4.14812w(t)^3 + 12.4444\mu w(t)^2 + (2.24144 + 8.29624\mu^2)] \times \]
\[ w(t) = q_1\cos[(w/w_r)t] \]  
\[ w(t)_{i_0} + (4c^2)[0.58811w(t)^3 + 1.76433\mu w(t)^2 + (2.24144 + 1.17762\mu^2)] \times \]
\[ w(t) = q_1\cos[(w/w_r)t] \]  
\[ w(t)_{i_0} + (4c^2)[0.58811w(t)^3 + 1.76433\mu w(t)^2 + (2.24144 + 1.17762\mu^2)] \times \]
\[ w(t) = q_1\cos[(w/w_r)t] \]  
\[ (4.29) \]
4.3 Results and Discussion

4.3.1 Lindstedt's Solution

The solution from Lindstedt's perturbation technique in the beginning of this section is examined. The nonlinearity indicator $r$ can be determined by the normalized-cubic coefficient $\epsilon$ and the imperfection amplitudes $\mu$. Depending on whether the nonlinearity indicator is greater than or less than zero, the finite amplitude vibration behavior can be classified as hard-spring or soft-spring, respectively. This result is valid for sufficiently small values of vibration amplitude.

Figure 4.1 gives us a graph of the backbone curves of the initial vibration amplitude versus the ratio of the nonlinear to linear vibration frequency with respect to five initial imperfection values. Figure 4.1 is based on boundary condition 1. Other boundary conditions will result in similar solution. It is obvious that all the backbone curves are symmetric to the x axis ($y=0$) no matter the initial geometric imperfection exists or not. Physically this symmetry is not accurate since initial geometric imperfection will affect the vibration mode, which leads to an asymmetry in the result. This inaccuracy is caused by the mathematical assumption within the Lindstedt's perturbation technique when solving the modified-Duffing equation.
4.3.2 Numerical Solution

In this work, Runge-Kutta method is used in Matlab to achieve the numerical solution. The imputing parameters are $M=1$, $n=1$, $\delta=0$ and $\nu=0.3$. Then the vibration result is plotted in Figure 4.2, which is a graph of the out-of-plane displacement versus the reduced-time for various values of the imperfection amplitude.
As shown in Figure 4.2, when the initial imperfection is zero, the vibration mode is perfect sine wave symmetric to the axis of \( w=0 \). Yet when initial imperfection occurs, the inward deflections are bigger than the outward deflections. And the vibration period is influenced dramatically. For each different initial vibration amplitude, the vibration frequency is calculated from similar procedure as Figure 4.2. Based on all these points, the backbone curves can be generated of the initial vibration amplitude at time is zero versus the ratio of the nonlinear to linear vibration frequency. In the following calculations, it is assumed \( M \) and \( n \) to be one.
Figure 4.3(a) Numerical Solution for Boundary Condition 1

Figure 4.4(b) Numerical Solution for Boundary Condition 2
Figure 4.5(c) Numerical Solution for Boundary Condition 3

Figure 4.6(d) Numerical Solution for Boundary Condition 4
Figure 4.3 (a) to (d) represent four different solutions based on four boundary conditions in equation 4.28. For Figure 4.3 (a), (b) and (d), there are similar backbone curves with vibration starting amplitude from -4 to 4. This is the well-known softening-hardening process.

4.3.3 Dynamic Failure

Yet Figure 4.3 (c), quite different from the other three, only has the softening process. If taking a perfect system with no imperfection, the curve is symmetric to the x axis (y=0). The vibration is stable when the starting amplitude is 3.5, which is 3.5 times plate thickness. However, when the initial amplitude grows to 3.51, the vibration amplitude starts to increase during every period and goes to impossibly huge deformation within five cycles. This vibration amplitude going out of bound is a sign of dynamic failure. A small increase in the starting amplitude leads to a huge change in vibration mode and process.

Similarly for imperfect systems, the dynamic failure can be noticed for the plate under boundary condition 3. Table 4.1 shows the smallest initial vibrating amplitude that will result in dynamic failure with respect to 5 different geometric imperfections. It is apparent to find out that the critical initial vibration amplitude decreases dramatically while increasing the imperfection. And the curve is no longer symmetric due to the existence of imperfection. For any fixed geometric imperfection (for
example μ=1), it is interesting to note that even though the failure happens at quite different critical initial amplitude (-1.21 and 2.05), the corresponding vibration frequency is almost the same. The same phenomenon can be noticed for all five curves. This may help the designers to stop the dynamic failure by controlling the vibration frequency.

Table 4.1 Critical Amplitude Versus Geometric Imperfection

<table>
<thead>
<tr>
<th>Imperfection</th>
<th>Critical Amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-3.51~3.51</td>
</tr>
<tr>
<td>0.5</td>
<td>-1.83~2.77</td>
</tr>
<tr>
<td>1.0</td>
<td>-1.21~2.05</td>
</tr>
<tr>
<td>1.5</td>
<td>-0.73~1.29</td>
</tr>
<tr>
<td>2.0</td>
<td>-0.34~0.60</td>
</tr>
</tbody>
</table>

The reason why dynamic failure only occurs when the boundary condition is clamped and zero radial stress can be explained physically. Clamped makes the plate edge to be horizontally flat. Zero radial stress, unlike zero radial displacement, "encourages" out-of-plane displacement to occur. Thus by combining those two boundary conditions, there is a large deformation of the plate and a very large radiant near the clamped tip. This radiant is out of tolerance when the vibration amplitude is too big, which leads to the final dynamic failure.
4.3.4 Comparison Between two solutions

Since the Duffing equation is solved by two different methods, it is necessary to check the validity of the Lindstedt's perturbation technique. Figure 4.4 (a) and (b) compares the solution from those two methods by adapting boundary condition 1 and 3 respectively.

![Figure 4.7(a) A Comparison at Boundary Condition 1](image)

The solution from boundary condition 1 has the same vibration mode with BC 2 and 4. Thus Figure 4.4(a) is a representative for those three boundary conditions. It is obvious that without the influence of the imperfection, Lindstedt's solution agrees with the numerical method until initial vibration goes to about 2.5 plate thickness.
However, on the other hand, a huge initial imperfection can lead to a much smaller valid range for Lindstedt's technique; the only accurate result is from -0.5 to 0.5. Lindstedt's perturbation technique, on one hand, is easy and fast to calculate; on the other hand, has restrained valid region, especially for imperfect systems.

Figure 4.8(b) A Comparison at Boundary Condition 3

The only special solution with boundary condition 3 is plotted along with Lindstedt's solution in Figure 4.4(b). Unlike previous comparison, this set of two curves is nearly the same. Yet important difference still exists in this case. The numerical method can determine the limit of vibration amplitude easily, which allows us to check the validity of the result. Lindstedt's perturbation technique will return you a solution no matter how big your input initial amplitude is. Thus the plate may have already failed physically, but we will be not able to know when did or when will
it happen.

In general, if neither the initial vibration amplitude nor the geometric imperfection is bigger than half plate thickness, Lindstedt's perturbation technique can offer relatively quick and accurate solution. Otherwise numerical method will be your best choice for solving a circular plate vibration problem.
4.4 Conclusions

Four modified-Duffing ODEs are derived with respect to four different boundary conditions for a circular plate vibration problem. Lindstedt's perturbation technique is applied to give a basic solution. Runge-Kutta method is then utilized to calculate the accurate vibration mode and parameters. By plotting the initial vibration amplitude versus vibration frequency, backbone curves are generated as expected except for the case with clamped and zero radial stress boundary condition. Instead of the softening-hardening process, the curve is only softening thus no longer backbone anymore. Due to the unacceptable deformation radiant at the plate tip for large initial vibration amplitude, dynamic failure of the plate occurs. This dynamic failure is an initial finding that should be considered as a possible failure mode in dynamic problems. Later on the solution from Lindstedt's and numerical one is compared to check the validity. The accuracy of Lindstedt's technique can be trusted solely within small range of vibration (half plate thickness).
CHAPTER 5

ACCURATE BACKBONE CURVE FOR LARGE AMPLITUDE VIBRATIONS OF IMPERFECT LAMINATED RECTANGULAR PLATE

5.1 Introduction

This chapter deals with the solution of modified-Duffing ordinary differential equation for large-amplitude vibrations of imperfect antisymmetrically rectangular plate. Two in-plane and two out-of-plane boundary conditions are considered as well as viscous damping. Graphite epoxy is chosen to be the example with respect to different fiber angles. To solve this angle-ply composite rectangular plate vibration problem, Lindstedt's perturbation technique and Runge-Kutta method are applied. The solution from two methods are plotted and compared for a validity check. Lindstedt's perturbation technique is proved to be accurate for sufficiently small vibration amplitude and imperfection. The results from Runge-Kutta method are plotted to form backbone curves. The effects of fiber volume ratio, imperfection amplitude and other parameters are studied.
5.2 Governing Equations

5.2.1 Governing Equations

The dynamic analogues of the Von Karman equilibrium and compatibility equations for a laminated thin plate can be written in terms of the out-of-plane displacement \( W \) and the stress function \( F \) as shown below,

\[
L_{D_*}(W) + L_{B_*}(F) + \rho W_{tt} - Q(X, Y, \ddot{t}) = F_{,YY} (W + W_0)_{,XX} + F_{,XX} (W + W_0)_{,YY} - 2F_{,XY} (W + W_0)_{,XY} \quad (5.1)
\]

\[
L_{A_*}(F) - L_{B_*}(W) = (W_{,XY})^2 + 2W_{0,XY} W_{,XY} - (W + W_0)_{,XX} W_{,YY} - W_{0,YY} W_{,XX} \quad (5.2)
\]

Here \( X \) and \( Y \) are the in-plane coordinates, \( \rho \) is the mass per unit area, \( \ddot{t} \) is the time, \( W_0 \) is the initial geometric imperfection, \( Q \) is the forcing function. \( L_{A_*}(\cdot), L_{B_*}(\cdot) \) and \( L_{D_*}(\cdot) \) are the linear differential operators [91]. The strains and moments are related to the stresses by

\[
[\varepsilon_x, \varepsilon_y, \gamma_{xy}, M_x, M_y, M_{xy}]^T = \begin{bmatrix} A_{ij} & B_{ij}^* \\ -B_{ij}^* & D_{ij} \end{bmatrix} \begin{bmatrix} N_x, N_y, N_{xy}, -W_{XX}, -W_{YY}, -2W_{XY} \end{bmatrix}^T \quad (5.3)
\]

Where,

\[
[A_{ij}] = [A_{ij}]^{-1}, \quad [B_{ij}]^* = -[A_{ij}]^{-1}[B_{ij}], \quad [D_{ij}] = [D_{ij}] - [B_{ij}][A_{ij}]^{-1}[B_{ij}] \quad (5.4)
\]

Generally, matrix \([A_{ij}], [B_{ij}], [D_{ij}], [A_{ij}]^*, [D_{ij}]^*\) are symmetric, while \([B_{ij}]^*\) is not.
5.2.2 Non-Dimensionalization

To non-dimensionalize the equilibrium and compatibility equation, the following quantities are introduced,

\[(w, w_0) = (W, W_0)/h,\]

\[(x, y) = (X, Y)/B,\]

\[(a_{ij}^*, b_{ij}^*, d_{ij}^*) = (EhA_{ij}^*, B_{ij}^*/h, D_{ij}^*/Eh^3),\]

\[f = F/(Eh^3),\]

\[t = \bar{t}w_r,\]

\[w_r = [Eh^3\pi^4/(\rho B^4)]^{0.5}\]

\[q(x, y, t) = [Eh^4\pi^4/B^4]Q(X, Y, \bar{t}) \quad (5.5)\]

Where \(h\) is the thickness of the plate, \(B\) is the width of the plate, \(E\) is force per square length, \(w_r\) is the reference frequency. By applying the above non-dimensional transformation, the non-dimensional equations will be,

\[L_{a^*}(w) + L_{b^*}(f) + \pi^4w_{tt} - \pi^4q(x, y, t) = f_{yy}(w + w_0)_{xx} + f_{xx}(w + w_0)_{yy} - 2f_{xy}(w + w_0)_{xy} \quad (5.6)\]

\[L_{a^*}(f) - L_{b^*}(w) = (w_{xy})^2 + 2w_{0,xy}w_{xy} - (w + w_0)_{xx}w_{yy} - w_{0,yy}w_{xx} \quad (5.7)\]

For angle-ply laminated plates, the linear operators are

\[L_{a^*}( \ ) = a_{32}^2( \ )_{xxxx} + (2a_{12}^* + a_{66}^*)( \ )_{xxyy} + a_{11}^*( \ )_{yyyy} \quad (5.8)\]

\[L_{b^*}( \ ) = (2b_{26}^* - b_{61}^*)( \ )_{xxyy} + (2b_{16}^* - b_{62}^*)( \ )_{xxyy} \quad (5.9)\]

\[L_{d^*}( \ ) = d_{11}^*( \ )_{xxxx} + 2(d_{12}^* + 2d_{66}^*)( \ )_{xxyy} + d_{22}^*( \ )_{yyyy} \quad (5.10)\]
In this chapter, only anti-symmetric angle-ply plates are considered, thus

\[ A_{16} = A_{26} = B_{11} = B_{12} = B_{22} = B_{66} = D_{16} = D_{26} = 0 \]  \hspace{1cm} (5.11)

### 5.3 Simply Supported Imperfect Angle-Ply Plates

If the out-of-plane boundary condition is taken to be simply supported along all four edges, then

\[ w = 0, M_x = 0 \text{ at } X = 0, L \]

\[ w = 0, M_y = 0 \text{ at } Y = 0, B \]  \hspace{1cm} (5.12)

A one-mode approximate analysis is made in this case. The vibration mode, the geometric imperfection and the forcing function under such condition are,

\[ [w(x, y, t), w_0(x, y, q(x, y, t))] = [w(t), \mu, q(t)] \sin(M\pi x) \sin(n\pi y) \]  \hspace{1cm} (5.13)

Where \( M, n, \mu \) are the same with previous chapters. By applying 5.13 to the compatibility equation, the stress function can be solved as,

\[ f(x, y, t) = [w(t)^2 + 2\mu w(t)] [c_0 w(t) \cos(M\pi x) \cos(n\pi y) + c_1 \cos(2M\pi x) +
\]

\[ c_2 \cos(2n\pi y) + e_1 x^2/2 + e_2 y^2/2 \]  \hspace{1cm} (5.14)

Where,

\[ c_0 = -C_{b^*}(M, n)/C_{a^*}(M, n) \]  \hspace{1cm} (5.15)

\[ c_1 = n^2/(32M^2a_{22}^*) \]  \hspace{1cm} (5.16)

\[ c_2 = M^2/(32n^2a_{11}^*) \]  \hspace{1cm} (5.17)

In the above,

\[ C_{a^*}(P, Q) = -(2b_{26}^* - b_{61}^*)(P^3Q) + (2b_{16}^* - b_{62}^*)(PQ^3) \]  \hspace{1cm} (5.18)
\[ C_{b_0}(P, Q) = a_{22}^2 P^4 + (2a_{12} + a_{66}^*) (P^2 Q^2) + a_{11}^* Q^4 \]  
(5.19)

For the in-plane boundary condition, if each of the four edges is permitted to move in-plane, there will be no in-plane shear along the edges. To satisfy \( f_{xy} = 0 \), one obtains,

\[ e_1 = e_2 = 0 \]  
(5.20)

On the other hand, if the in-plane boundary condition is in-plane immovable, one can obtain,

\[ e_1 = (\pi^2/8) (a_{21}^* M^2 + a_{22}^* n^2) \]
\[ e_2 = (\pi^2/8) (a_{11}^* M^2 + a_{12}^* n^2) \]  
(5.21)

Substituting the vibration mode into the nonlinear dynamic equilibrium equation for laminated plates, the Galerkin procedure is then applied to form the following ordinary differential equation,

\[ w(t)_{tt} + \left[ C_{d_0}(M, n) + C_{b_0}(M, n)^2/C_{a_0}(M, n) \right] w(t) - q(t) = \left[ w(t)^3 + 3\mu w(t)^2 + 2\mu^2 w(t) \right] (-e_0) \]  
(5.22)

Where,

\[ C_{d_0}(P, Q) = d_{11}^* P^4 + 2(d_{12}^* + 2d_{66}^*) (P^2 Q^2) + d_{22}^* Q^4 \]  
(5.23)
\[ e_0 = (M^2 e_2 + n^2 e_2)/\pi^2 + 2M^2 n^2 (c_1 + c_2) \]  
(5.24)

Finally, the ODE in standard Duffing equation form is written as,

\[ w(t)_{tt} + k w(t) + \varepsilon k a_2 w(t)^2 + \varepsilon k w(t)^3 = q(t) \]  
(5.25)

Where, \( k = C_{d_0}(M, n) + C_{b_0}(M, n)^2/C_{a_0}(M, n) + 2\mu^2 e_0 \)  
(5.26)
\[ \varepsilon k a_2 = 3\mu e_0 \]  
(5.27)
\[ \varepsilon k = e_0 \]  
(5.28)
5.4 Clamped Imperfect Angle-Ply Plates

Similarly with the simply supported condition, to investigate the effects of imperfection and in-plane boundary conditions on the vibration behavior more straightforwardly, a one-mode approximate analysis is made similarly as before. Thus the vibration mode, the geometric imperfection and the forcing function that suit the clamped boundary conditions are,

\[
[w(x, y), w_0(x, y), q(x, y, t)] = (1/4)[w(t), \mu, q(t)][1 - \cos(2M\pi x)][1 - \cos(2n\pi y)]
\]  

(5.29)

From the nonlinear compatibility equation, the forcing function for clamped angle-ply plates is,

\[
f(x, y, t) = k_0 w(t) \sin(M\pi x) \sin(n\pi y) + [w(t)^2 + 2\mu w(t)] [k_1 \cos(2M\pi x) + k_2 \cos(2n\pi y) + k_3 \cos(4M\pi x) + k_4 \cos(4n\pi y) + k_5 \cos(2M\pi x) \cos(2n\pi y) + k_6 \cos(2M\pi x) \cos(4n\pi y) + k_7 \cos(4M\pi x) \cos(2n\pi y) + e_3 x^2/2 + e_4 y^2/2]
\]  

(5.30)

Where

\[
k_0 = C_b,(2M, 2n)/[-4C_d,(2M, 2n)]
\]

\[
k_1 = n^2/(32M^2a_{22}^2)
\]

\[
k_2 = M^2/(32n^2a_{11}^2)
\]

\[
k_3 = -n^2/(16 * 32M^2a_{22}^2)
\]

\[
k_4 = -M^2/(16 * 32n^2a_{11}^2)
\]
\[ k_5 = -M^2 n^2 / C_{a_*}(2M, 2n) \]
\[ k_6 = M^2 n^2 / [2C_{a_*}(2M, 4n)] \]
\[ k_7 = M^2 n^2 / [2C_{a_*}(4M, 2n)] \]
\[ k_8 = [(k_2 - k_4 - k_5 + k_6 + k_7/2) + (k_1 - k_3 - k_5 + k_6/2 + k_7/2) + (k_5 - k_6 - k_7)] / 2 \]

Substituting 5.30 into the nonlinear dynamic equilibrium equation for angle-ply plates and applying the Galerkin procedure, the ODE in time is obtained in the same standard form with 5.25,

\[ w(t)_{tt} + kw(t) + \varepsilon k a_2 w(t)^2 + \varepsilon kw(t)^3 = q(t) \]  

(5.32)

Where,

\[ k = 2\mu^2 R + (4/9)[-k_0 C_{b_*}(2M, 2n) + (1/4)C_{d_*}(2M, 2n) + 8M^4 d_{11}^* + 8n^4 d_{22}^*] \]

(5.33)

\[ \varepsilon k a_2 = 3\mu R \]

(5.34)

\[ \varepsilon k = R \]

(5.35)

\[ R = (16/9\pi^4)(4M^2 n^2 \pi^4 k_8 + 3M^2 \pi^2 e_4 / 4 + 3n^2 \pi^2 e_3 / 4) \]

(5.36)

The constants \( e_3 \) and \( e_4 \) in R depends on the in-plane boundary conditions.

For four edges in-plane movable,

\[ e_3 = e_4 = 0 \]

(5.37)

For four edges in-plane immovable,

\[ e_3 = (3\pi^2 / 32)(a_{21} M^2 + a_{22} n^2) \]

\[ e_4 = (3\pi^2 / 32)(a_{11} M^2 + a_{12} n^2) \]

(5.38)
5.5 Result and Discussion

5.5.1 Lindstedt's Solution

The solution from Lindstedt's perturbation technique for four different BCs is examined in this section. The nonlinearity indicator \( r \) can be determined by the normalized-cubic coefficient \( \epsilon \) and the imperfection amplitudes \( \mu \). Depending on whether the nonlinearity indicator is greater than or less than zero, the finite amplitude vibration behavior can be classified as hard-spring or soft-spring, respectively. This result is valid for sufficiently small values of vibration amplitude.

Graphite epoxy is chosen to be the example for angle-ply laminated plates. The related material parameters are picked to be

\[
\frac{E_1}{E_2} = 40
\]

\[
\frac{G_{12}}{E_2} = 0.5
\]

\[
\gamma_{12} = 0.25
\]

\[
\gamma_{21} = \gamma_{12}(E_2/E_1)
\]

\[v_f = 0.5\]  \hspace{1cm} (5.39)

Figure 5.1 (a), (b) and (c) give us three graphs of the backbone curves of the initial vibration amplitude versus the ratio of the nonlinear to linear vibration frequency with respect to different fiber angles. The fiber angles considered in Figure 5.1 are 7.5°, 15° and 30° respectively. In this way the angle between the fibers of the two layers are 15°, 30° and 60° respectively. These three results are under
simply supported and in-plane movable boundary condition. The solution of (a) and (b) are almost the same since the fiber angle are small. While (c) makes a difference from the previous two solutions.

Figure 5.1(a) Initial vibration amplitude versus the ratio of nonlinear to linear vibration frequency (7.5°, SS-IM)
Figure 5.2(b) Initial vibration amplitude versus the ratio of nonlinear to linear vibration frequency (15°, SS-IM)

Figure 5.3(c) Initial vibration amplitude versus the ratio of nonlinear to linear vibration frequency (30°, SS-IM)
By setting the boundary condition to simply supported and in-plane immovable, the solution is in Figure 5.2. Similarly for clamped and in-plane movable/immovable, they are shown in Figures 5.3 and 5.4. We keep the fiber angle to maintain 7.5° thus easier to make a comparison. The change of boundary conditions has largely affected the value and even the mode of the curve. Besides, it is obvious that all the backbone curves are symmetric to the x axis (y=0) no matter the initial geometric imperfection exists or not. Physically this symmetry is not accurate since initial geometric imperfection will affect the vibration mode, which leads to an asymmetry in the result. This inaccuracy is caused by the mathematical assumption within the Lindstedt's perturbation technique when solving the modified-Duffing equation.

Figure 5.4 Initial vibration amplitude versus the ratio of nonlinear to linear vibration frequency (7.5°, SS-II)
Figure 5.5 Initial vibration amplitude versus the ratio of nonlinear to linear vibration frequency (7.5°, CL-IM)

Figure 5.6 Initial vibration amplitude versus the ratio of nonlinear to linear vibration frequency (7.5°, CL-II)
5.5.2 Numerical Solution

In this chapter, the same Runge-Kutta method is utilized from previous chapters to achieve the numerical solution in Matlab. Since this problem is the most complicated and representative, the Matlab program been used is listed in Appendix A. The imputing parameters are chosen to be the same with last section of Lindstedt's technique. It should be noted that only a ratio of young's modulus is needed instead of knowing each one. The thickness is not needed due to the non-dimensional process.

By picking the fiber angle to be $7.5^\circ$, $15^\circ$ and $30^\circ$, Figure 5.5 (a) (b) and (c) are generated respectively.

![Figure 5.7(a) Initial vibration amplitude versus the ratio of nonlinear to linear vibration frequency (7.5°, SS-IM)](image-url)
Figure 5.8(b) Initial vibration amplitude versus the ratio of nonlinear to linear vibration frequency (15°, SS-IM)

Figure 5.9(c) Initial vibration amplitude versus the ratio of nonlinear to linear vibration frequency (30°, SS-IM)
It can be noted that 5.5 (a) and (b) are almost the same, which agrees with the Lindstedt's solution. However, all imperfect systems will result in an unsymmetrical curve with respect to y=0 axis. This proves our previous statement of inaccuracy of Lindstedt's technique. 5.5 (c) has shown some difference in the vibration of plates from (a) and (b). This is the expected change since fiber angle will largely affect the young's modulus in x and y directions.

By fixing the fiber angle to be 7.5°, the effects of the change of boundary conditions can be checked. Figure 5.6, 5.7 and 5.8 correspond to the boundary condition of simply supported and in-plane immovable, clamped and in-plane movable and clamped with immovable, respectively.

![Figure 5.10 Initial vibration amplitude versus the ratio of nonlinear to linear vibration frequency (7.5°, SS-II)](image-url)
Figure 5.11 Initial vibration amplitude versus the ratio of nonlinear to linear vibration frequency (7.5°, CL-IM)

Figure 5.12 Initial vibration amplitude versus the ratio of nonlinear to linear vibration frequency (7.5°, CL-II)
Comparing all four results for four different boundary conditions, it is obvious that though the values of the curve vary significantly, the vibration mode has not changed. All four solutions generate the typical backbone curve. It is interesting to note that unlike the vibration of rectangular and circular homogeneous in Chapter 3 and 4, there are far more parameters that can be changed in this problem. Parameters such as M, n, young’s modulus, boundary condition, fiber volume, fiber angle, damping ratio and so on will all have an effect on the solution.

5.5.3 Comparison Between two solutions

Since the problem has been solved by two different methods, it is important to check the validity of the Lindstedt's perturbation technique. Figure 5.9 (a), (b), (c) and (d) compare the solution from those two methods by adapting the four different boundary conditions I have mentioned above.
Figure 5.13(a) Comparison under Simply Supported, In-plane Movable

Figure 5.14(b) Comparison under Simply Supported, In-plane Immovable
Figure 5.15(c) Comparison under Clamped, In-plane Movable

Figure 5.16(d) Comparison under Clamped, In-plane Immovable
All the four figures in Figure 5.9 have shown us the same vibration mode shape. Lindstedt's perturbation technique can generate an accurate solution when the vibration amplitude is relatively small. Especially for the clamped and in-plane movable boundary condition with no imperfection, Lindstedt's solution agrees with the accurate numerical solution in a wide range (about -3 to 3). Yet the existence of imperfection has been proved to significantly decrease the validity of this range. This conclusion can be checked throughout all the four figures. This decrease can be as big as 70% of the original valid range. In general, if neither the initial vibration amplitude nor the initial geometric imperfection is bigger than half of the plate thickness, Lindstedt's perturbation technique can offer us relatively quick and accurate solution. Otherwise numerical method will be a better choice for solving a composite plate vibration problem.


5.6 Damping Effect

Chapter 3 introduces viscous damping into the vibration of rectangular homogenous plate. And the solution has proved that the existence of damping will generally lead the non-linear vibration backbone curve to behave more linearly, especially when the damping ratio is big.

In this section of Chapter 5, the effects of viscous damping are studied. By definition, viscous damping is caused by such energy losses as occur in liquid lubrication between moving parts or in a fluid forced through a small opening by a piston, as in automobile shock absorbers. The viscous-damping force is directly proportional to the relative velocity between the two ends of the damping device. In our situation, the damping of the resin and the relative movement between resin and fiber will be the main reason of the existence of viscous damping. This damping can vary depending on the type of fiber and resin, the volume of fiber and environmental factors like temperature.

5.6.1 Governing Equations

Detailed speaking, the modified-Duffing equation for simply supported and clamped boundary condition are 5.25 and 5.32, respectively. The simply supported governing equation is
\[ w(t)_{tt} + kw(t) + \varepsilon k a_2 w(t)^2 + \varepsilon kw(t)^3 = q(t) \]  \hspace{1cm} (5.39)

Where,

\[ k = C_{d_s}(M,n) + C_{b_s}(M,n)^2/C_{a_s}(M,n) + 2\mu^2 e_0 \]

\[ \varepsilon k a_2 = 3\mu e_0 \]

\[ \varepsilon k = e_0 \]  \hspace{1cm} (5.40)

The clamped boundary condition governing equation is

\[ w(t)_{tt} + kw(t) + \varepsilon k a_2 w(t)^2 + \varepsilon kw(t)^3 = q(t) \]  \hspace{1cm} (5.41)

Where,

\[ k = 2\mu^2 R + (4/9)[-k_0 C_{b_s}(2M,2n) + (1/4) C_{d_s}(2M,2n) + 8M^4 d_{11} + 8n^4 d_{22}] \]

\[ \varepsilon k a_2 = 3\mu R \]

\[ \varepsilon k = R \]

\[ R = (16/9\pi^4)(4M^2 n^2 \pi^4 k_0 + 3M^2 \pi^2 e_4/4 + 3n^2 \pi^2 e_3/4) \]  \hspace{1cm} (5.42)

The constants \( e_3 \) and \( e_4 \) in \( R \) depends on the in-plane boundary conditions.

For four edges in-plane movable,

\[ e_3 = e_4 = 0 \]  \hspace{1cm} (5.43)

For four edges in-plane immovable,

\[ e_3 = (3\pi^2/32)(a_{21} M^2 + a_{22} n^2) \]

\[ e_4 = (3\pi^2/32)(a_{11} M^2 + a_{12} n^2) \]  \hspace{1cm} (5.44)

It is interesting to see that the governing equation for the two boundary conditions are the same type. Only the parameters change due to the difference of
boundary conditions. Thus later only one governing equation will be used for both situation.

The above two governing equations have not included any kind of damping. Then the viscous damping factor is introduced into the modified-Duffing equation. The revised governing equation for both kinds of boundary conditions can be given as the following,

\[ w(t)_{tt} + 2\delta k^{0.5}w(t)_t + kw(t) + \varepsilon k_2 w(t)^2 + \varepsilon kw(t)^3 = q(t) \]  \hspace{1cm} (5.45)

In the above equation, \( \delta \) is the viscous damping ratio. Note that the viscous damping force is directly proportional to the relative velocity; thus it is reasonable to see the additional damping term differential with respect to the time.

5.6.2 Solution and Comparison

Now the new modified-Duffing equation involving viscous damping needs to be solved. Since the effects of the damping ratio on the vibration mode and amplitude are the focus, there is no need to use Lindstedt's perturbation to get the approximate solution. Directly using the numerical method will be the best way to approach the accurate solution for the best quality.
Figure 5.17(a) SS, IM, Imperfection=0

Figure 5.18(b) SS, IM, Imperfection=2
Figure 5.19(c) SS, II, Imperfection=0

Figure 5.20(d) SS, II, Imperfection=2
All the four figures in Figure 5.10 are under simply supported (SS) out-of-plane boundary condition. Figure 5.10 (a) and (b) are under in-plane movable boundary condition and (c), (d) are under in-plane immovable boundary condition. To make the comparison easier, the fiber angle is chosen to be 7.5°, the imperfection to be 0 and 2. Other factors like M, n will be the same with previous section. When there is no imperfection, the curves are symmetric with respect to y=0. And when there is imperfection existing, our familiar backbone curves will be shown. But no matter imperfection exists or not, we can see very clearly that the viscous damping is leading the ration of nonlinear/linear vibration frequency to approach 1. That means the nonlinear vibration mode of this composite plate vibration problem becomes much more linear when damping shows up. When the damping ratio reaches 0.1, which is not strange in normal materials, the backbone curves are almost a straight line. This result agrees with the homogeneous rectangular plate vibration solution and physically supports each other.
Figure 5.21(a) CL, IM, Imperfection=0

Figure 5.22(b) CL, IM, Imperfection=2
Figure 5.23(c) CL, II, Imperfection=0

Figure 5.24(d) CL, II, Imperfection=2
Similarly, the boundary condition is changed from simply supported to clamped. By keeping all other parameters unchanged, the results are shown in Figure 5.11 in the above. As expected, the four figures have shown the same linear trend for the effects of viscous damping. It can be concluded that the effects of viscous damping in the vibration of plates, include homogeneous and laminated ones, is to change the vibration mode from nonlinear to more linear.
5.7 Conclusions

The rectangular angle-ply two-layer composite plate vibration problem is studied in this chapter. Different parameters are generated for the modified-Duffing ODE due to four different boundary conditions. Lindstedt's perturbation technique is applied to give a basic solution. Runge-Kutta method is then utilized to calculate the accurate vibration mode and parameters. By plotting the initial vibration amplitude versus vibration frequency, backbone curves are generated as expected. The effects of fiber angle are studied for the situation of simply supported and in-plane movable. Later on the solution from Lindstedt's and numerical one is compared to check the validity. The accuracy of Lindstedt's technique can be trusted solely within a small range of vibration amplitude or initial velocity (half plate thickness). The viscous damping is further studied and proved to lead the nonlinear vibration to behave more linearly.
CHAPTER 6

SUMMARY AND FUTURE WORKS

6.1 Summary

This research has focused on the vibration of rectangular, circular and laminated plates. The vibration is started by an initial vibration amplitude, or an initial velocity. Four types of boundary conditions, simply supported and clamped combined with in-plane movable/immovable are considered. There is no biaxially compression in this work. But this will be an interesting future subject. Chapter 3 and 4 studied the vibration of homogeneous rectangular and circular plates. This is further extended to inhomogeneous angle-ply rectangular laminated plate in Chapter 5.

These vibration problems all deal with the solution of a modified-Duffing equation. In previous works, this equation is generally solved by Lindstedt's Perturbation Technique. This technique can give us a quick approximate solution. Yet based on mathematical assumptions, the solution will no longer be accurate for large amplitude vibration, especially when a significant amount of imperfection exists. Thus Runge-Kutta method is introduced to solve this problem numerically. This new solution is carried out in Matlab is proved to be accurate. The comparison between
both methods has shown us the validity of the Lindstedt's Perturbation Technique is generally within half plate thickness. Viscous damping is proved to lead the nonlinear vibration to behave more linearly. Other interesting research results such as the backbone shape curve, the dynamic failure due to out-of-bound vibration and the significant effects of parameter variation have also played an important role in this thesis.
6.2 Suggestion for Future Works

The future works will further study the effects of parameter variation on the vibration of rectangular, circular and composite plates. For rectangular plates, effects of geometric imperfections on frequency-load interaction of biaxially compressed antisymmetric angle ply rectangular plates can be further studied. Besides, the effects of four different boundary conditions can be checked same as the circular plate. As for composite plate, cross-ply composite plate will be the next step. How will hysterical damping and viscous damping change the vibration mode will also be interesting for angle-ply and cross-ply composite plates.
APPENDIX A

MATLAB PROGRAM FOR COMPOSITE PLATE

Throughout this thesis Matlab is utilized to conduct Runge-Kutta method to solve the Modified-Duffing equation from the vibration problem. The most complicated and representative example will be angle-ply composite plate vibration problem. In this Appendix, the Matlab program that is used in thesis is given as another way to understand and solve this problem. This program can be subject to more than 10 parameter variations. And this will be of good help to future learners in this area of research.

The main program is:

```
%Used in composite rectangular plate calculation
%Use ODE45 to solve the Duffing eqn. Then find the T for the sin curve.
%Find linstedt's result Y and initial imperfection is set to be N
%[k,e,a2]=Cd(E1,E2,G12,r12,v1,theta,h,M,n,bc,imp,E)

N=-4;m=1;
[k,e,a]=Cd(200,5,2.5,0.25,0.5,pi/6,0.005,1,1,1,1.5,5);
r=3*e/8-5*a*a*e*e/12;
```
while(N<4),

    options=odeset; options.RelTol=0.00000001; options.AbsTol=0.0000001;
    options.MaxStep=0.0005;

    clear t;
    clear x;

    [t,x]=ode45('ODE_comp',[0,80],[N,0]);

    q=1;
    clear P;

    for i=10:1000,

        if x(i,2)*x(i+1,2)*100000000>0,i=i+1;

    else P(q)=t(i); q=q+1; i=i+1;
    end;

    end;

    Y2(m)=1+r*N*N-15*e*e/256*N^4;

    T2(m)=(36)*pi/(-P(1)+P(37))/k^0.5;

    T(m)=2*pi/t(i);

    X2(m)=N;
function \([k,e,a2]=Cd(E1,E2,G12,r12,v1,theta,h,M,n,bc,imp,E)\)

% boundary condition 1: inplane movable; BC 2: inplane immovable

\([A,B,D]=\text{comp}(E1,E2,G12,r12,v1,theta,h);\)

\(AA=\text{inv}(A);\)

\(BB=-A\backslash B;\)

\(DD=D-B*(A\backslash B);\)

\(aa=E*h*AA;\)
bb = BB/h;

dd = DD/(E*h^3);

Caa = @(M,n) -((2*bb(2,3)-bb(3,1))*M^3*n+(2*bb(1,3)-bb(3,2))*n^3*M;

Cbb = @(M,n) aa(2,2)*M^4+(2*aa(1,2)+aa(3,3))^2+(M*n)^2+aa(1,1)*n^4;

Cdd = @(M,n) dd(1,1)*M^4+2*(dd(1,2)+2*dd(3,3))*M*M*n+n+dd(2,2)*n^4;

% use double a,b,d to show a*, b*, d*.

c0 = -Cbb(M,n)/Caa(M,n);

c1 = n*n/(32*M*M*a(2,2));

c2 = M*M/(32*n*n*a(1,1));

a11 = A(1,1)/E/h;

a12 = A(1,2)/E/h;

a21 = A(2,1)/E/h;

a22 = A(2,2)/E/h;

if bc == 1
    e1 = 0;
    e2 = 0;

end

if bc == 2
\[ e_1 = (a_{21}M^2 + a_{22}n^2) \pi^2 / 8; \]
\[ e_2 = (a_{11}M^2 + a_{12}n^2) \pi^2 / 8; \]

end

\[ e_0 = ((M^2 e_2 + n^2 e_1) / \pi^2) + 2M^2 n^2 (c_1 + c_2); \]
\[ k = C_{dd}(M,n) + C_{bb}(M,n) / C_{aa}(M,n) + 2 \text{imp} \times \text{imp} e_0; \]
\[ e = e_0 / k; \]
\[ a_2 = 3 \text{imp} \times e_0 / e / k; \]

And Cd2:

\% This is for clamped angle-ply

function \[ [k,e,a_2] = \text{Cd2}(E1,E2,G12,r12,v1,theta,h,M,n,bc,imp,E) \]

\% boundary condition 1: inplane movable; BC 2: inplane immovable

\[ [A,B,D] = \text{comp}(E1,E2,G12,r12,v1,theta,h); \]
\[ AA = \text{inv}(A); \]
\[ BB = -A \backslash B; \]
\[ DD = D - B \times (A \backslash B); \]
aa = E*h*AA;

bb = BB/h;

dd = DD/(E*h^3);

Caa = @(M,n) -(2*bb(2,3)-bb(3,1))*M^3*n+(2*bb(1,3)-bb(3,2))*n^3*M;
Cbb = @(M,n) aa(2,2)*M^4+(2*aa(1,2)+aa(3,3))*(M*n)^2+aa(1,1)*n^4;
Cdd = @(M,n) dd(1,1)*M^4+2*(dd(1,2)+2*dd(3,3))*M*n*n+dd(2,2)*n^4;

% use double a,b,d to show a*, b*, d*.

k0 = Cbb(2*M,2*n)/(-4*Caa(2*M,2*n));
k1 = n*n/(32*M*M*aa(2,2));
k2 = M*M/(32*n*n*aa(1,1));
k3 = -n*n/(16*32*M*M*aa(2,2));
k4 = -M*M/(16*32*n*n*aa(1,1));
k5 = -M*M*n*n/(Caa(2*M,2*n));
k6 = M*M*n*n/(2*Caa(2*M,4*n));
k7 = M*M*n*n/(2*Caa(4*M,2*n));
k8 = 1/2*((k2-k4-k5+k6+k7/2)+(k1-k3-k5+k6/2+k7/2)+k5-k6-k7);

a11 = A(1,1)/E/h;
a12 = A(1,2)/E/h;
a21 = A(2,1)/E/h;
a22=A(2,2)/E/h;

if bc==1
    e3=0;
    e4=0;
end
if bc==2
    e3=(a21*M*M+a22*n*n)*pi*pi/32*3;
    e4=(a11*M*M+a12*n*n)*pi*pi/32*3;
end

R=16/9/(pi^4)*(4*M*M*n*n*pi^4*k8+3*M*M*pi*pi*e4/4+3*n*n*pi*pi*e3/4);

k=2*imp*imp*R+4/9*(-k0*Cbb(2*M,2*n)+Cdd(2*M,2*n)/4+8*M^4*dd(1,1)+8*n^4*dd(2,2));

e=R/k;

a2=3*imp;

Both Cd and Cd2 cited the last program "comp". This is used for the parameter calculation to support upper level program.
Program Comp:

function [A,B,D]=comp(E1,E2,G12,r12,v1,theta,h)

v2=1-v1;

E11=E1*v1+E2*v2;

E22=E1*E2/(E1*v2+E2*v1);

r21=r12*E2/E1;

q11=E11/(1-r12*r21);

q22=E22/(1-r12*r21);

q12=r12*E22/(1-r12*r21);

q21=q12;

q66=G12;

U1=(3*q11+3*q22+2*q12+4*q66)/8;

U2=(q11-q22)/2;

U3=(q11+q22-2*q12-4*q66)/8;

U4=(q11+q22+6*q12-4*q66)/8;

U5=(U1-U4)/2;

Q11=U1+U2*cos(2*theta)+U3*cos(4*theta);
Q12=U4-U3*cos(4*theta);
Q21=Q12;
Q22=U1-U2*cos(2*theta)+U3*cos(4*theta);
Q16=U2/2*sin(2*theta)+U3*sin(4*theta);
Q26=U2/2*sin(2*theta)-U3*sin(4*theta);
Q66=U5-U3*cos(4*theta);

A11=h*Q11*2;
A12=2*h*Q12;
A22=2*h*Q22;
A66=2*h*Q66;
B16=-h*h*Q16;
B26=-h*h*Q26;
D11=2/3*h^3*Q11;
D12=2/3*h^3*Q12;
D22=2/3*h^3*Q22;
D66=2/3*h^3*Q66;
A=[A11,A12,0;A12,A22,0;0,0,A66];
B=[0,0,B16;0,0,B26;B16,B26,0];
D=[D11,D12,0;D12,D22,0;0,0,D66];
end
REFERENCES

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VITA

He Huang was born in DaQing, Heilongjiang, China, on March 15th, 1988. He is the only child in his family. In 2010, He graduated his bachelor degree in Mechanical Engineering in Harbin Institute of Technology. At the same year, he attended University of New Orleans as a Ph.D. student. Supervised by Dr. David Hui, he has been a research assistant and teaching assistant for several courses. During the past four years, he conducted research on the mechanical and thermal properties of composite material, the vibration of rectangular, circular and laminated plates and shells, buckling and the dynamic failure. Currently, he is the teaching assistant of the Material Science and Material Lab.