1-1-2011

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Dongming Wei
University of New Orleans, dwei@uno.edu

Harry Borden
University of New Orleans

Recommended Citation

Wei, Dongming and Borden, Harry. Traveling Wave Solutions of Burgers' Equation for Power-Law Non- Newtonian Flows, co-author Harry Borden , Appl. Math. E-Notes, 11(2011), 133-138.

Applied Mathematics E-Notes, 11(2011), 133-138 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Traveling Wave Solutions of Burgers' Equation for Power-Law Non-Newtonian Flows*

Dongming Wei[†], Harry Borden[‡]

Received 30 June 2010

Abstract

In this work we present some analytic and semi-analytic traveling wave solutions of a generalized Burgers' equation for unidirectional flow of power-law non-Newtonian fluids. The solutions include the corresponding well-known traveling wave solution of the Burgers' equation for Newtonian flows. We also derive estimates of shock thickness for the power-law flows.

1 Introduction

In this work, we are interested in finding traveling wave solutions to the following generalized Burgers' equation for power-law fluid flows

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)=\mu\frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{n-1}\frac{\partial u}{\partial x}\right) \tag{1}$$

where ρ is the density, μ the viscosity, u the velocity of the fluid in $x-$direction, and $n \neq 1$ the power-law index. For $n = 1$, (1) reduces to the famous Burgers' equation for Newtonian flows

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)=\mu\frac{\partial^2 u}{\partial x^2}. \tag{2}$$

It is well-known that if we impose boundary conditions

$$\lim_{\xi\to-\infty}u(\xi)=u_2,\ \lim_{\xi\to+\infty}u(\xi))=u_1,\ \lim_{|\xi|\to+\infty}u'(\xi)=0,$$

where $u_2 > u_1$, then (2) has the following celebrated traveling wave solution

$$u(\xi)=\frac{u_1+u_2\exp[-\frac{\xi}{2\nu}(u_1-u_2)]}{1+\exp[-\frac{\xi}{2\nu}(u_1-u_2)]} \tag{3}$$

where $\xi = x - \lambda t$, u_1 and u_2 are downstream and upstream fluid velocities respectively. It can be shown that there exists of a thin transition layer of thickness δ in the order

*Mathematics Subject Classifications: 35L67, 76A10, 35Q53.

†Department of Mathematics, University of New Orleans, New Orleans, Louisiana 70148, USA
‡Department of Mathematics, University of New Orleans, New Orleans, Louisiana 70148, USA

of $\frac{2\nu}{u_2-u_1}$ for flows defined by (3). This thickness δ can be referred to as the shock thickness, which tends to zero as $\nu \to 0$, and for fixed ν, $\delta \to \infty$ as $(u_2 - u_1) \to 0$. See, for example, [6] or [7] for derivation of (3) and analysis of (2). In this work, we find analytic and semi-analytic solutions to (1) for various values of n, and we derive the corresponding order of thickness for the transition layers in the power-law Non-Newtonian flows. Applications power-law flows are abundant in studying of flows in glacier, blood, food, oil, polymer etc., see e.g. [3]. There are numerous papers denoted to study equation (2) in the literature for understanding shock formation and traveling waves in Newtonian flows dating back to the original papers of Burgers, Hopf, and Cole, see [6], [8], and [9]. A generalized Burgers' equation for Non-Newtonian flows based on the Maxwell model has recently been studied in [7] recently. In this work, we will show that the corresponding traveling wave solutions to (1) with the same boundary conditions can be implicitly defined by

$$\left(\frac{1}{2\nu}\right)^{\frac{1}{n}} \xi = \frac{(u_2-u)\left(\frac{u-u_1}{u_2-u_1}\right)^{1/n} {}_2F_1\left(1-\frac{1}{n}, \frac{1}{n}, 2-\frac{1}{n}, \frac{u_2-u}{u_2-u_1}\right)}{(1-\frac{1}{n})[(u-u_1)(u_2-u)]^{\frac{1}{n}}}$$

and the first order approximation of the thickness of the transition layer is

$$(u_2-u_1)\left[\frac{8\nu}{(u_2-u_1)^2}\right]^{1/n}.$$

This result extends the classical result from $n = 1$ to $n \neq 1$. We have not found any paper in the literature which deals with the power-law Burges's equation (1) for $n \neq 1$.

2 The Generalized Burgers' Equation for Power-Law Fluid Flows

The general Navier-Stokes equation for incompressible viscous flows is given by

$$\rho \frac{D\vec{\mathbf{u}}}{Dt} = \nabla \cdot \vec{\boldsymbol{\sigma}} - \nabla p + \vec{\mathbf{g}} \tag{4}$$

where $\vec{\mathbf{u}} = (u_1, u_2, u_3)$ is the fluid velocity,

$$\boldsymbol{\sigma} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} \text{ and } D\vec{\mathbf{u}} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix}$$

are the stress and strain tensor, ρ the density, p the scaler pressure, and $\vec{\mathbf{g}}$ the external force, $\epsilon = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}), 1 \leq i, j \leq 3$. For unidirectional flows, we assume that $\vec{\mathbf{u}} = (u_1, 0, 0), \tau_{ij} = 0$ for $i \neq 1$ or $j \neq 1, \vec{\mathbf{g}} = (g_1, 0, 0)$, and $\nabla p = (\frac{\partial p}{\partial x_1}, 0, 0)$. The Navier-Stokes equation (4), in this case, takes the following simple form

$$\rho \frac{Du_1}{Dt} = \frac{\partial \tau_{11}}{\partial x_1} - \frac{\partial p}{\partial x_1} + g_1 \tag{5}$$

where $\frac{Du_1}{Dt} = \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1}$.

Rheological relationships between $\boldsymbol{\sigma}$ and $D\vec{\mathbf{u}}$ are frequently used to determine the type of fluids. It is well-known that for power-law fluids, the rheological relation is given by

$$\tau_{ij} = 2K|2\epsilon_{kl}\epsilon_{kl}|^{(n-1)/2}\epsilon_{ij}, 1 \le i, j \le 3 \tag{6}$$

where n is called the power-law index, see e.g., [1] or [9]. If $n = 1$, then the fluid is said to be a Newtonian fluid, and it is non-Newtonian if $n \neq 1$. For many important industrial polymer fluids, the value of n is between 0 and 1. Table1 provides values of K and n for some important industrial power-law fluids:

Polymer	Temperature (Kelvin)	K(Pas^n)	n	
Nylon	493	2.62×10^3	0.63	
Polystyrene	463	4.47×10^3	0.22	
Polyethylene	453	4.47×10^3	0.56	

Table 1: Values of K and n

For the unidirectional power-law flows, (6) reduces to $\tau_{11} = 2^n K|\epsilon_{11}|^{(n-1)}\epsilon_{11}$. Let $u_1, x_1, g_1,$ and $2^n K$ be denoted by u, x, g and μ respectively. Then from (5), we have the equation for the unidirectional power-law flows

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = \mu\frac{\partial}{\partial x}\Phi\left(\frac{\partial u}{\partial x}\right) - \frac{\partial p}{\partial x} + g \tag{7}$$

where $\Phi(t) = |t|^{n-1}t, 0 < n < \infty$. In (7), let $\nu = \frac{\mu}{\rho}$, and let $-\frac{\partial p}{\partial x} + g = 0$, we then have the generalized Burgers' equation for power-law fluids

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial}{\partial x}\Phi(\frac{\partial u}{\partial x}) \tag{8}$$

We are interested in finding solutions of (8) for $n \neq 1$.

3 Traveling Wave Solutions

Let $u(x,t) = u(\xi)$, with $\xi = x - \lambda t$. Then $\frac{\partial u}{\partial t} = -\lambda\frac{\partial u}{\partial \xi}$, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}$, and equation (8) becomes

$$-\lambda\frac{\partial u}{\partial \xi} + u\frac{\partial u}{\partial \xi} = \nu\frac{\partial}{\partial \xi}\Phi\left(\frac{\partial u}{\partial \xi}\right) \tag{9}$$

Therefore,

$$\frac{\partial}{\partial \xi}\left[\frac{1}{2}u^2 - \lambda u - \nu\Phi\left(\frac{du}{d\xi}\right)\right] = 0$$

which gives

$$\frac{1}{2}u^2 - \lambda u - \nu\Phi\left(\frac{du}{d\xi}\right) = A$$

in which A is the integration constant. Applying the downstream and upstream boundary conditions: $\lim_{\xi\to+\infty} u(\xi) = u_1$, $\lim_{\xi\to-\infty} u(\xi) = u_2$, $\lim_{|\xi|\to+\infty} u'(\xi) = 0$ we get

$$\Phi\left(\frac{du}{d\xi}\right) = \frac{1}{2\nu}(u^2 - 2\lambda u - 2A) = \frac{1}{2\nu}(u-u_1)(u-u_2)$$

where $\lambda = \frac{1}{2}(u_1+u_2)$, and $A = -\frac{1}{2}u_1u_2$. Since $\Phi^{-1}(t) = |t|^{(\frac{1}{n}-1)}t$, we have

$$\frac{du}{d\xi} = \Phi^{-1}\left(\frac{1}{2\nu}(u-u_1)(u-u_2)\right) = \left(\frac{1}{2\nu}\right)^{\frac{1}{n}} \Phi^{-1}((u-u_1)(u-u_2))$$

Therefore

$$\int \frac{du}{\Phi^{-1}((u-u_1)(u-u_2))} = \int \left(\frac{1}{2\nu}\right)^{\frac{1}{n}} d\xi \tag{10}$$

Without loss of generality, in the following, we assume that $u_2 > u > u_1$. For $n = 1$, (10) gives

$$\frac{1}{2\nu}\xi = \int \frac{du}{(u-u_1)(u-u_2)} = \frac{1}{u_1-u_2}\ln\left(\frac{u-u_1}{u_2-u}\right)$$

which gives the celebrated traveling wave solution

$$u(\xi) = \frac{u_1 + u_2 \exp\left[-\frac{\xi}{2\nu}(u_2-u_1)\right]}{1+\exp\left[-\frac{\xi}{2\nu}(u_2-u_1)\right]}$$

to the Burgers' equation for Newtonian flows. In the following, we are interested in finding solutions to (10) for $n \neq 1$. By using Mathematica, we find that

$$\int \frac{du}{\Phi^{-1}((u-u_1)(u-u_2))} = \frac{(u_2-u)\left(\frac{u-u_1}{u_2-u_1}\right)^{1/n} {}_2F_1\left(1-\frac{1}{n}, \frac{1}{n}, 2-\frac{1}{n}, \frac{u_2-u}{u_2-u_1}\right)}{(1-\frac{1}{n})[(u-u_1)(u_2-u)]^{\frac{1}{n}}} \tag{11}$$

where ${}_2F_1$ is the well-known Gauss hypogeometric function defined by the series

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a,k)(b.k)}{(c,k)} x^k, |x| < 1$$

where $(a,k) = a(a+1)...(a+k-1)$ is the Appel symbol, see e.g. [2]. Therefore, from (11), we have the following power series solution to the generalized power-law Burgers' equation

$$\left(\frac{1}{2\nu}\right)^{\frac{1}{n}} \xi = -\frac{n}{n-1}\sqrt[n]{\frac{(u_2-u)^{n-1}}{u_2-u_1}} \sum_{k=0}^{\infty} \frac{(1-\frac{1}{n},k)(\frac{1}{n},k)}{(2-\frac{1}{n},k)k!} \left(\frac{u_2-u}{u_2-u_1}\right)^k,$$

where $n \neq 1, |\frac{u_2-u}{u_2-u_1}| < 1$.

It is interesting to note that for some special values of n, the integral (10) can be expressed in terms of elementary functions. In particular, for $n = \frac{1}{2}$, we have

$$\begin{aligned} -\frac{\xi}{(2\nu)^2} &= \int \frac{du}{((u-u_1)(u_2-u))^2} \\ &= \frac{1}{(u_2-u_1)^2}\left(\frac{1}{u-u_2}+\frac{1}{u-u_1}\right)+\frac{2}{(u_1-u_2)^3}\ln\left(\frac{u_2-u}{u-u_1}\right) \end{aligned}$$

for $n = \frac{1}{3}$, we have

$$\begin{aligned} -\frac{\xi}{(2\nu)^3} &= \int \frac{du}{((u-u_1)(u_2-u))^3} = \frac{3}{(u_2-u_1)^4}\left(\frac{1}{u-u_2}-\frac{1}{u-u_1}\right) \\ &+\frac{1}{2(u_1-u_2)^3}\left[\frac{1}{(u-u_2)^2}-\frac{1}{(u-u_1)^2}\right]+\frac{6}{(u_2-u_1)^5}\ln\left(\frac{u_2-u}{u-u_1}\right) \end{aligned}$$

for $n = 2$, $u_2 > u > u_1$, by using the identity $z_2F_1(1/2, 1/2/; 3/2; z^2) = a \arcsin z$ and (11), we also get

$$\begin{aligned} \left(\frac{1}{2\nu}\right)^{1/2}\xi &= \int \frac{du}{((u-u_1)(u_2-u))^{1/2}} = 2\left(\frac{u_2-u}{u-u_1}\right)^{1/2} {}_2F_1\left(1/2, 1/2/; 3/2; \frac{u_2-u}{u-u_1}\right) \\ &= 2\arcsin\left(\frac{u_2-u}{u-u_1}\right)^{1/2} \end{aligned}$$

respectively. Therefore, we have the following three traveling wave solutions for the three special values of n:

$$\frac{\xi}{(2\nu)^2} = \frac{1}{(u_2-u_1)^2}\left(\frac{1}{u_2-u}-\frac{1}{u-u_1}\right)-\frac{2}{(u_1-u_2)^3}\ln\left(\frac{u_2-u}{u-u_1}\right), \text{ for } n=\frac{1}{2},$$

$$\begin{aligned} \frac{\xi}{(2\nu)^3} &= \frac{3}{(u_2-u_1)^4}\left(\frac{1}{u_2-u}+\frac{1}{u-u_1}\right)+\frac{1}{2(u_2-u_1)^3}\left[\frac{1}{(u-u_2)^2}-\frac{1}{(u-u_1)^2}\right] \\ &-\frac{6}{(u_2-u_1)^5}\ln\left(\frac{u_2-u}{u-u_1}\right) \end{aligned}$$

for $n = \frac{1}{3}$, and

$$\left(\frac{1}{2\nu}\right)^{1/2}\xi = 2\arcsin\left(\frac{u_2-u}{u-u_1}\right)^{1/2} \quad \text{for } n=2$$

More analytic solutions in terms of elementary functions for special values of n can be derived and they are not listed here due to limited spaces. We have omitted the integration constants in the above solutions. For simplicity, let $u_1 = -1, u_2 = 1$, profiles of the transition layers for three special cases are given in Figure 1, with u versus $(\frac{1}{2\nu})^{1/n}\xi$. These three cases represent Newtonian ($n = 1$), Non-Newtonian shear-thinning ($n < 1$), and Non-Newtonian shear-thickening fluid flows ($n > 1$) respectively.

In Figure 1, solid thick line represents the Newtonian fluids, the solid thin line represents the shear thickening fluids, and the dash line represents the shear thinning fluids.

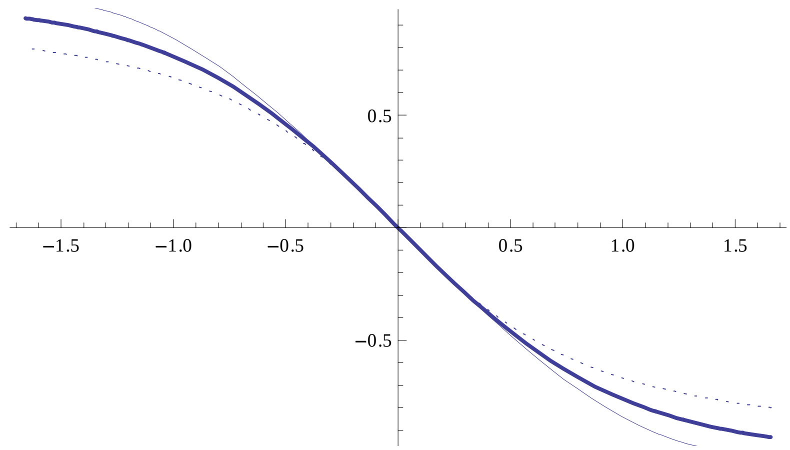

Figure 1: Transition layers

4 The Order of Thickness of the Transition Layers

The transition layer thickness or the shock thickness can be estimated by using the first order derivative $\frac{du}{d\xi}(0)$. From

$$\frac{du}{d\xi} = \Phi^{-1}\left(\frac{1}{2\nu}(u-u_1)(u-u_2)\right) = \left(\frac{1}{2\nu}\right)^{\frac{1}{n}} \Phi^{-1}((u-u_1)(u-u_2))$$

and $u(0) = \frac{u_1+u_2}{2}$ we get

$$\frac{du}{d\xi}(0) = -\left[\frac{(u_2-u_1)^2}{8\nu}\right]^{1/n}$$

Let δ denote the thickness of the transition layer. Using the Taylor expansion, we have

$$u_2 - u_1 \approx u\left(-\frac{\delta}{2}\right) - u\left(\frac{\delta}{2}\right) = -\delta\frac{du}{d\xi}(0) + O(\delta^2)$$

Therefore, we have

$$\delta = (u_2-u_1)\left[\frac{8\nu}{(u_2-u_1)^2}\right]^{1/n}$$

which is the first order approximation of the thickness of the transition layer for the power-law flows. This estimate, for $n = 1$, gives the well-known estimate for the thickness of the transition layer of the Newtonian flows.

5 Conclusions

In this work, we have derived a generalized Burgers' equation for the power-law flows, and we also derive a new general traveling wave solution of this equation in terms of the Gauss hypergeometric function. As special cases of this general solution, we show several analytic solutions in terms of elementary functions as well as the profiles of the transition layers of the solutions. We defined a first order approximation of the thickness of the transition layer or thickness of the shock for the generalized Burgers' equation. The results generalize the known solution and shock-layer estimate for the Newtonian flows.

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