7-15-1987

# Angular sensitivity of Brewster-angle reflection polarizers: an analytical treatment

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The angular sensitivity of Brewster-angle reflection polarizers (BARP) is first studied approximately by determining the Taylor series expansion of the parallel reflectance $R_p$ as a function of angle of incidence $\phi$ near the Brewster angle $\phi_B$. Subsequently, exact and explicit equations are derived that determine the lower and upper limits, $\phi_l$ and $\phi_u$, of the range of $\phi$, that includes $\phi_B$, over which $R_p$ or the extinction ratio $ER$ is below a stated limit $L$. Examples are given of Ge and Si IR BARP for which $\phi_l$ and $\phi_u$ are calculated for $L$ from $10^{-6}$ to $10^{-1}$ in ascending multiplicative steps of 10.

## I. Introduction

Perhaps the simplest way to polarize a collimated monochromatic beam of light is by reflection from an uncoated planar surface of a dielectric medium at the Brewster angle $\phi_B = \tan^{-1}n$, where $n$ is the medium's index of refraction.[1] At $\phi_B$ the $p$ component of the electric vector (in the plane of incidence) is suppressed in the reflected wave, whereas the $s$ component (perpendicular to the plane of incidence) experiences a finite (power) reflectance given by

$$R_s = [(n^2 - 1)/(n^2 + 1)]^2. \tag{1}$$

The efficiency of the polarizer is its throughput for the unextinguished polarization (i.e., $R_s$), which must be high. To reach and exceed a marginal efficiency of 50% ($R_s = 0.5$), Eq. (1) indicates that we must have $n \geq \sqrt{2} + 1 = 2.414$. Transparent materials with this high refractive index are not readily available for visible light. However, in the IR, semiconductors become transparent and possess the requisite high $n$. A good choice is Ge,[2] which is transparent from 3 to 13 $\mu$m with $n \approx 4$, which gives $R_s = 0.78$. This reflectance is sufficiently high to make Brewster-angle reflection polarizers (BARP) using Ge surfaces practical.[3] (Other semiconductors, such as Si and Se, are also useful.) When the direction or axis of the beam is to be maintained, additional reflections from highly reflective metallic mirrors can be used.[4,5]

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Received 2 February 1987.
0003-6935/87/142847-04$02.00/0.


Our objective in this paper is to present an analytical treatment of the angular sensitivity of BARP. Specifically we answer the following questions. What is the range of incidence angles, inclusive of the Brewster angle, over which the parallel reflectance $R_p$ is below a prescribed level (e.g., $10^{-2}$)? What is the range of incidence angles, inclusive of the Brewster angle, over which the extinction ratio ($ER = R_p/R_s$) is below a prescribed level (e.g., $10^{-2}$)? Graphic results that illustrate the angular sensitivity of BARP are in Ref. 3.

In Sec. II we develop a limited Taylor series expansion of the $p$-reflection coefficient valid around the Brewster angle $\phi_B$ that may be used for an approximate estimation of angular sensitivity. In Sec. III we provide an exact solution for the lower and upper limits $\phi_l$ and $\phi_u$ of the angular range that includes $\phi_B$ ($\phi_l < \phi_B < \phi_u$) over which $R_p$ does not exceed a given level, and in Sec. IV we do the same but for the $ER$. Section V includes detailed results for Ge and Si BARP.

## II. Approximate Analysis

The amplitude reflection coefficient for the $p$ polarization is given by[6]

$$r_p = \frac{\epsilon \cos\phi - (\epsilon - \sin^2\phi)^{1/2}}{\epsilon \cos\phi + (\epsilon - \sin^2\phi)^{1/2}}, \tag{2}$$

where $\phi$ is the angle of incidence, and $\epsilon = n^2$ is the dielectric constant of the transparent reflecting medium. Incidence from vacuum or air is assumed. An example of the variation of $r_p$ with $\phi$ is given for the air–Ge interface ($\epsilon = 16$) in Fig. 1. The $r_p$ vs $-\phi$ curve intersects the $\phi$ axis at the Brewster angle, $\phi_B = 75.964°$. $r_p > 0$ for $\phi < \phi_B$ and $r_p < 0$ for $\phi > \phi_B$, i.e., $r_p$ changes sign as it goes to zero at $\phi_B$. We assume the $\exp(j\omega t)$ time dependence and the Nebraska (Muller) conventions.[7]

The angular sensitivity of BARP is determined by the behavior of $r_p$ with $\phi$ near its zero at $\phi_B$. This is well described by the Taylor series expansion of $r_p$ around $\phi_B$, which is given by

$$r_p = 0 + a(\Delta\phi) + b(\Delta\phi)^2 + \ldots, \tag{3}$$

where

$$\Delta\phi = \phi - \phi_B, \tag{4}$$

$$a = (\partial r_p/\partial\phi)_{\phi_B} \tag{5a}$$

$$b = \tfrac{1}{2}(\partial^2 r_p/\partial\phi^2)_{\phi_B}. \tag{5b}$$

Higher-order terms (cubic and above) will heretofore be ignored in Eq. (3).

The partial derivatives $\partial r_p/\partial\phi$ and $\partial^2 r_p/\partial\phi^2$ at any $\phi$ are given elsewhere[8] and are not repeated here. Evaluating these derivatives at $\phi_B$ gives the following results:

$$a = -\tfrac{1}{2}\epsilon^{-3/2}(\epsilon^2 - 1) = -(n^4 - 1)/2n^3, \tag{6a}$$

$$b = -\tfrac{1}{4}\epsilon^{-3}(\epsilon^4 + \epsilon^3 + \epsilon^2 - \epsilon + 2) = -(n^8 + n^6 + n^4 - n^2 + 2)/4n^6. \tag{6b}$$

The intensity reflectance $R_p$ is given by

$$R_p = r_p^2 = a^2(\Delta\phi)^2 + 2ab(\Delta\phi)^3 + \ldots, \tag{7}$$

where terms of power of >3 are dropped. For very small angular excursions $\Delta\phi$ around $\phi_B$, the first term describes the parabolic rise in the intensity of the reflected $p$ component. If $\Delta\phi$ increases further, asymmetry of the $R_p$-vs $-\phi$ curve around $\phi_B$ appears as represented by the second term of Eq. (7). The angular range $2\Delta\phi$ for very low levels of $R_p$ (e.g., $\leq 10^{-3}$) is adequately obtained by keeping only the first term of Eq. (7); this gives

$$2\Delta\phi = 4R_p^{1/2}/n\left(1 - \frac{1}{n^4}\right), \tag{8}$$

when Eq. (6a) is used. For an efficient BARP $n$ is large, and Eq. (8) further simplifies to

$$2\Delta\phi = 4R_p^{1/2}/n. \tag{9}$$

The angular range in Eq. (8) or (9) is in radians. Conversion of Eq. (9) into degrees gives

$$2\Delta\phi = 720R_p^{1/2}/\pi n \text{ (deg)}. \tag{10}$$

If we take $n = 4$ (Ge) and $R_p = 10^{-3}$, Eq. (10) gives $2\Delta\phi = 1.81°$.

In general it is easier analytically to work with the amplitude reflectance $r_p$ instead of the intensity reflectance $R_p$. More accurate but still approximate limits on $\phi$, so that $R_p$ is less than a specified level, can be obtained by setting the left-hand side (LHS) of Eq. (3) equal to $+R_p^{1/2}$ and calculating the negative root $\Delta\phi_-$ of the resulting quadratic equation; next the LHS is set equal to $-R_p^{1/2}$, and the positive root $\Delta\phi_+$ of the corresponding quadratic is obtained. The lower and upper limits of $\phi$ are then given as $\phi_l = \phi_B + \Delta\phi_-$ and $\phi_u = \phi_B + \Delta\phi_+$, respectively.

As an example of the application of this quadratic approximation of Eq. (3) (which corresponds to the approximation of a segment of the $r_p$-vs-$\phi$ curve in the neighborhood of $\phi_B$ by a parabola), let $n = 4$, $R_p = 10^{-2}$. In this case, $r_p = \pm 0.1$, $a = -1.9922$, $b = -4.2648$, and quadratic Eq. (3) gives for $r_p = +0.1$ and $r_p = -0.1$, respectively, $\Delta\phi_- = -3.277°$ and $\Delta\phi_+ = 2.620°$. With $\phi_B = 75.964°$, the approximate limits of the range of $\phi$ over which $R_p \leq 10^{-2}$ are $\phi_l = 72.687°$ and $\phi_u = 78.583°$. These angles differ by <0.05° from the exact angles obtained in the following section.

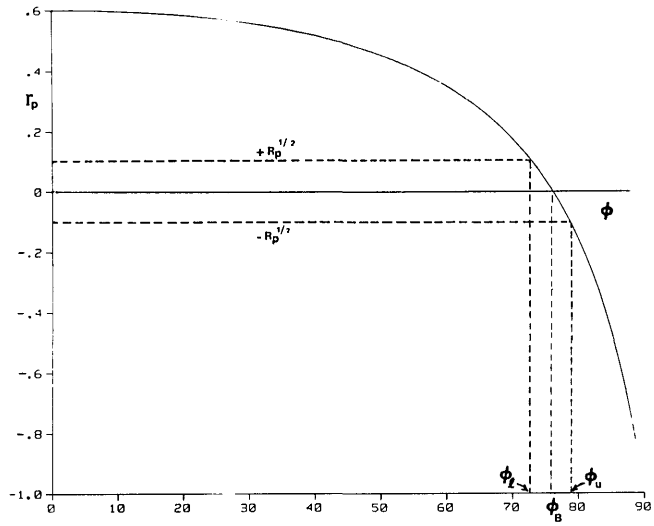


Fig. 1. Amplitude reflectance $r_p$ for $p$-polarized light vs angle of incidence $\phi$ for an air–Ge interface ($\epsilon = 16$). $\phi_B$ is the Brewster angle (where $r_p = 0$), and $\phi_l$, $\phi_u$ are the lower and upper limits of the range of $\phi$ over which the intensity reflectance stays below a specified level $R_p$.

## III. Exact Analysis: Angular Range for Specified Parallel Reflectance

The exact analysis is based on the finding that Eq. (2) has an explicit solution for the angle of incidence $\phi$ required to attain a prespecified amplitude reflectance $r_p$. To see this Eq. (2) is rearranged to read

$$(1 - r_p)/(1 + r_p) = (\epsilon - \sin^2\phi)^{1/2}/\epsilon\cos\phi. \tag{11}$$

If both sides of Eq. (11) are squared, and the substitutions

$$U = \sin^2\phi, \tag{12}$$

$$P = [(1 - r_p)/(1 + r_p)]^2, \tag{13}$$

are made, the resulting equation can be readily solved for $U$ to give

$$U = (P\epsilon^2 - \epsilon)/(P\epsilon^2 - 1). \tag{14}$$

For equal positive and negative values of $r_p$, below and above the Brewster angle, Eq. (13) gives

$$P_+ = [(1 - R_p^{1/2})/(1 + R_p^{1/2})]^2, \tag{15}$$

$$P_- = 1/P_+, \tag{16}$$

respectively, where $R_p$ is the specified intensity reflectance level. Equation (14) subsequently gives

$$U_l = (P_+\epsilon^2 - \epsilon)/(P_+\epsilon^2 - 1), \quad (17a)$$

$$U_u = (\epsilon^2 - P_+\epsilon)/(\epsilon^2 - P_+), \quad (17b)$$

from which one obtains

$$\phi_l = \sin^{-1}U_l^{1/2}, \qquad \phi_u = \sin^{-1}U_u^{1/2}. \quad (18)$$

Equations (15), (17), and (18) determine exactly and explicitly the limits of the angular range $\phi_l < \phi < \phi_u$, which includes the Brewster angle, over which the parallel reflectance remains less than a specified level $R_p$.

Let us take the same example that was considered near the end of Sec. II of a Ge substrate with $n = 4$, $\epsilon = 16$, and a specified reflectance level $R_p = 10^{-2}$. In this case $P_+ = 0.66942$ from Eq. (15), and Eqs. (17) and (18) give the exact limits $\phi_l = 72.739°$ and $\phi_u = 78.562°$. Additional results, corresponding to other reflectance levels, are given in Sec. V.

## IV. Exact Analysis: Angular Range for Specified Extinction Ratio

An essential parameter that describes the performance of a polarizer is its extinction ratio, which is the ratio of throughputs for the nominally extinct and passed orthogonal polarizations. For BARP, the extinction ratio is

$$ER = R_p/R_s. \quad (19)$$

Here we determine the range of $\phi$ over which $ER$ is less than a specified level.

It is easier to work with the ratio of amplitude reflectances

$$\rho = r_p/r_s, \quad (20)$$

which determines $ER$ simply by

$$ER = \rho^2. \quad (21)$$

The expression for $\rho$ for a planar interface separating air or vacuum (the medium of incidence) and a transparent medium with dielectric constant $\epsilon$ is[6]

$$\rho = \frac{\sin^2\phi - \cos\phi(\epsilon - \sin^2\phi)^{1/2}}{\sin^2\phi + \cos\phi(\epsilon - \sin^2\phi)^{1/2}}, \quad (22)$$

where $\phi$ is the angle of incidence as before. As an example, Fig. 2 shows $\rho$ vs $\phi$ for the air–Ge interface ($\epsilon = 16$). Following a procedure similar to that used in Sec. III, let

$$Q = [(1 - \rho)/(1 + \rho)]^2 \quad (23)$$

and $U = \sin^2\phi$. This change of variables transforms Eq. (22) into the following quadratic equation:

$$(1 - Q)U^2 - (1 + \epsilon)U + \epsilon = 0, \quad (24)$$

with solution

$$U = \{(1 + \epsilon) \pm [(1 + \epsilon)^2 - 4\epsilon(1 - Q)]^{1/2}\}/2(1 - Q). \quad (25)$$

Equation (25), where $Q$ is given by Eq. (23), determines the angle of incidence at which any prespecified value of $\rho$ between $-1$ and $+1$ can be attained on reflection at the air–medium interface.

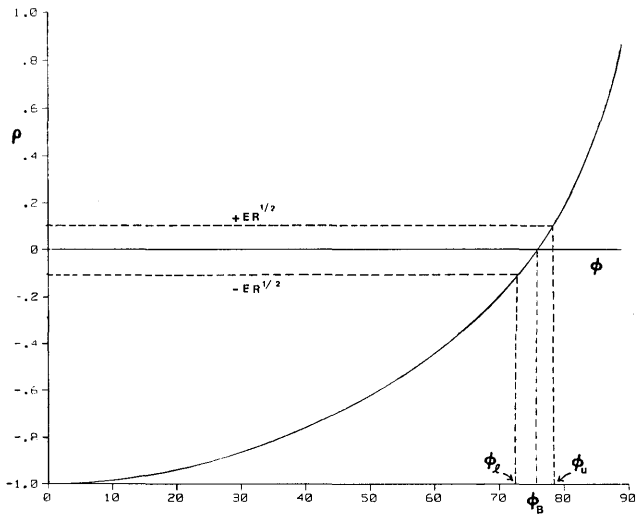


Fig. 2. Ratio of amplitude reflectances $\rho = r_p/r_s$ vs angle of incidence $\phi$ for an air–Ge interface ($\epsilon = 16$). $\phi_B$ is the Brewster angle (where $\rho = 0$), and $\phi_l$, $\phi_u$ are the lower and upper limits of the range of $\phi$ over which the extinction ratio stays below a specified level $ER$.

For a specified $ER$, the lower limit on $\phi$, $\phi_l$, is obtained by setting

$$Q = Q_l = [(1 - ER^{1/2})/(1 + ER^{1/2})]^2 \quad (26)$$

in Eq. (25). To get the upper limit $\phi_u$, put

$$Q = Q_u = 1/Q_l \quad (27)$$

in Eq. (25). Thus the angular range over which $ER$ remains below a specified level is found exactly and explicitly.

For a Ge BARP ($\epsilon = 16$) we calculate (using the foregoing steps) $\phi_l = 72.231°$ and $\phi_u = 78.327°$ as the lower and upper limits of the range of incidence angles over which $ER < 10^{-2}$. These angles are $\sim 0.5$ and 0.2° downshifted from the correspondings limits (see the end of Sec. III) of the range over which $R_p < 10^{-2}$. Other results, corresponding to other extinction ratio levels, are given in the following section.

## V. Additional Results for Ge and Si BARP

For a further demonstration the exact explicit equations of Secs. III and IV were used to determine the lower and upper limits, $\phi_l$ and $\phi_u$, of the range of incidence angles over which $R_p$ and $ER$ stay below a specified limit $L$, where $L$ is from $10^{-6}$ to $10^{-1}$ in ascending multiplicative steps of 10. The results for Ge, $n = 4$ ($\lambda = 3$–$13\ \mu$m), and for Si, $n = 3.42$ ($\lambda = 4$–$11\ \mu$m)[2], appear in Tables I and II, respectively. These data should be of reference value to users of BARP made of these substrates; of course, similar tables can be compiled for other materials as needed.

Two conclusions are apparent from Tables I and II. First, the angular range for specified level of $ER$ is less than the angular range for an equal level of $R_p$. Second, angular sensitivity is improved (i.e., $\phi_u - \phi_l$ is increased for a given $L$) when the substrate refractive index is decreased (from 4 of Ge to 3.42 of Si). Such an

**Table I. Lower and Upper Limits, $\phi_l$ and $\phi_u$, of the Range of Incidence Angles over which the Parallel Reflectance $R_p$ or Extinction Ratio $ER$ Stays Below a Specified Level $L$ for a Ge Brewster-Angle Reflection Polarizer[a]**

| $L$ | Angular range for specified $R_p$ | | | Angular range for specified $ER$ | | |
|---|---|---|---|---|---|---|
| | $\phi_l$ | $\phi_u$ | $\phi_u - \phi_l$ | $\phi_l$ | $\phi_u$ | $\phi_u - \phi_l$ |
| $10^{-6}$ | 75.935 | 75.992 | 0.057 | 75.948 | 76.001 | 0.053 |
| $10^{-5}$ | 75.872 | 76.054 | 0.182 | 75.885 | 76.040 | 0.155 |
| $10^{-4}$ | 75.673 | 76.248 | 0.575 | 75.709 | 76.215 | 0.506 |
| $10^{-3}$ | 75.022 | 76.843 | 1.821 | 75.143 | 76.748 | 1.605 |
| $10^{-2}$ | 72.739 | 78.562 | 5.823 | 73.231 | 78.327 | 5.096 |
| $10^{-1}$ | 62.007 | 82.772 | 20.765 | 65.743 | 82.431 | 16.688 |

[a] The refractive index of Ge is taken to be $n = 4$, which corresponds to a wide IR spectral range (3–13 $\mu$m). The Brewster angle for Ge is $\phi_B = \tan^{-1} 4 = 75.964°$. All angles are in degrees.

**Table II. Lower and Upper Limits, $\phi_l$ and $\phi_u$, of the Range of Incidence Angles over which the Parallel Reflectance $R_p$ or Extinction Ratio $ER$ Stays Below a Specified Level $L$ for a Si Brewster-Angle Reflection Polarizer[a]**

| $L$ | Angular range for specified $R_p$ | | | Angular range for specified $ER$ | | |
|---|---|---|---|---|---|---|
| | $\phi_l$ | $\phi_u$ | $\phi_u - \phi_l$ | $\phi_l$ | $\phi_u$ | $\phi_u - \phi_l$ |
| $10^{-6}$ | 73.667 | 73.735 | 0.068 | 73.670 | 73.733 | 0.063 |
| $10^{-5}$ | 73.594 | 73.808 | 0.214 | 73.613 | 73.790 | 0.177 |
| $10^{-4}$ | 73.360 | 74.035 | 0.675 | 73.415 | 73.984 | 0.569 |
| $10^{-3}$ | 72.595 | 74.732 | 2.137 | 72.784 | 74.583 | 1.799 |
| $10^{-2}$ | 69.901 | 76.742 | 6.841 | 70.667 | 76.369 | 5.702 |
| $10^{-1}$ | 56.928 | 81.639 | 24.711 | 62.633 | 81.091 | 13.458 |

[a] The refractive index of Si is taken to be $n = 3.42$, which corresponds to a wide IR spectral range (4–11 $\mu$m). The Brewster angle for Si is $\phi_B = \tan^{-1} 3.42 = 73.701°$. All angles are in degrees.

improvement is, of course, at the expense of lower efficiency (or smaller $R_s$).

## VI. Summary

In this paper we examined analytically the angular sensitivity of Brewster-angle reflection polarizers. A Taylor series expansion of the parallel reflectance around the Brewster angle, Eqs. (3)–(7), leads to an approximate but satisfactory evaluation of angular sensitivity. Exact determinations are also made of the lower and upper limits of the range of angles, inclusive of the Brewster angle, over which the parallel reflectance or extinction ratio remains below a specified level. The results are applied to reflection polarizers using Ge and Si.

I am pleased to acknowledge the support of the National Science Foundation under grant ECS850035.

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