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The constraint on the Jones matrix of an optical system such that there exist two linear polarization states at its input that are mapped onto two corresponding linear states at its output is derived. These principal linear polarization (PLP) states, which characterize a broad range of systems, are also found in terms of the Jones matrix elements. Special cases when the PLP states are orthogonal, collapse onto one state, or become infinite in number are indicated. For a deterministic or nondeterministic optical system described by a Mueller matrix, the existence of two PLP states places a constraint on only 3 of the 16 matrix elements, namely, the first 3 elements of the last row. In general, the output light is partially linearly polarized. Several examples are given for demonstration.

INTRODUCTION

The state of polarization of the zeroth-order (specular) beam reflected by an Al-coated 1200 G/mm diffraction grating, which is set at oblique incidence and with the grooves inclined with respect to the plane of incidence, has been measured recently[1] as a function of the azimuth angle P of incident linearly polarized light of 633-nm wavelength. In general the reflected light is elliptically polarized for an arbitrary orientation or azimuth P of the incident electric vector. However, for two specific incident linear polarizations, described by P_1 and P_2, the reflected light becomes likewise linearly polarized.

In another context, it is known[2–5] that two distinct pairs of nulling angles (P, A) and (P', A') can always be obtained by adjusting the polarizer and the analyzer in a polarizer–compensator–sample–analyzer, or the equivalent polarizer–sample–compensator–analyzer, ellipsometer regardless of the characteristics (ψ, Δ) of the isotropic reflecting sample or the retardance, diattenuation, or fast-axis orientation of the compensator. Therefore the two-element system composed of the compensator and the sample is also characterized by two input linear polarization states that are mapped onto two corresponding output linear polarization states.

The foregoing examples suggest that a general optical system of the kind considered by Jones[6] may be characterized by two and only two principal linear polarization (PLP) states at the system input that are mapped onto two linear states at its output at a given wavelength and for a given direction of incidence. In this paper it is proved that this is indeed the case, provided that the Jones matrix of the system satisfies a specific constraint. Furthermore, the PLP states are determined in terms of the Jones matrix elements. Finally, we extend the discussion to an optical system described by a Mueller matrix and show that the existence of PLP states depends on a simple constraint on the elements of the last row of the matrix.

The PLP states obviously differ from, and are not to be confused with, the two eigenpolarizations[5] of the system that are in general elliptical states that propagate through the system unchanged.

PRINCIPAL LINEAR POLARIZATION STATES OF AN OPTICAL SYSTEM DESCRIBED BY A JONES MATRIX

Consider the elastic scattering of a monochromatic or quasi-monochromatic plane wave of light by a linear optical system S (Fig. 1). The change of polarization that accompanies scattering is described by the Jones-vector transformation[5,6]

$$\begin{bmatrix} E_{ox} \\ E_{oy} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} E_{ix} \\ E_{iy} \end{bmatrix}, \tag{1}$$

where xy is a reference transverse coordinate system with the x axis in the scattering plane (the plane of the incident and scattered wave vectors) and (T_{ij}) is the complex Jones matrix. The ellipses of polarization at the input and the output of the system are succinctly represented by the complex polarization variables[5]

$$\chi_i = E_{iy}/E_{ix}, \qquad \chi_o = E_{oy}/E_{ox}, \tag{2}$$

which are interrelated by the bilinear transformation[5,7]

$$\chi_o = (T_{22}\chi_i + T_{21})/(T_{12}\chi_i + T_{11}). \tag{3}$$

We are interested in the case when the incident light is linearly polarized so that

$$\chi_i = r, \tag{4}$$

where r is a real number. Substitution of Eq. (4) into Eq. (3) gives

$$\chi_o = (rT_{22} + T_{21})/(rT_{12} + T_{11}). \tag{5}$$

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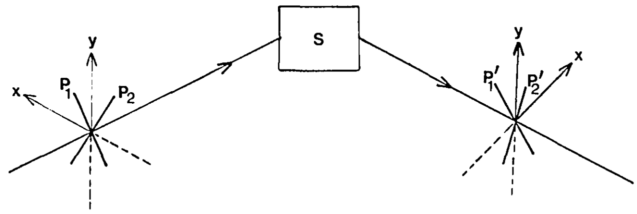

Fig. 1. Scattering of a plane wave of light by an optical system S. P_1 and P_2 identify two linear polarization states at the input of the system that are mapped onto two corresponding linear output polarization states P_1' and P_2'.

The scattered light is also linearly polarized if

$$\operatorname{Im} \chi_o = 0 \tag{6}$$

or, from Eq. (5),

$$\operatorname{Im}[(rT_{22} + T_{21})(rT_{12}^* + T_{11}^*)] = 0, \tag{7}$$

where * indicates the complex conjugate as usual. Equation (7) can be cast in the form of a quadratic equation in r:

$$g_2 r^2 + g_1 r + g_0 = 0, \tag{8}$$

in which

$$g_0 = \operatorname{Im}(T_{11}^* T_{21}),$$
$$g_1 = \operatorname{Im}(T_{11}^* T_{22} + T_{12}^* T_{21}),$$
$$g_2 = \operatorname{Im}(T_{12}^* T_{22}). \tag{9}$$

Equation (8) has two solutions:

$$r_\pm = [-g_1 \pm (g_1^2 - 4g_0 g_2)^{1/2}]/2g_2. \tag{10}$$

All real values of r are acceptable. Consequently, the condition for the existence of two input PLP states that are mapped onto two linear output states is that the quantity under the square root in Eq. (10) is positive or

$$g_1^2 > 4g_0 g_2. \tag{11}$$

Inequality (11), in which g_0, g_1, and g_2 are given by Eqs. (9), represents the desired constraint on the Jones matrix of a linear optical system such that two PLP states do exist. Equation (10) determines those characteristic input states. Their associated azimuth angles are given by

$$P_{1,2} = \arctan(r_\pm), \tag{12}$$

and the corresponding output linear states are obtained by substituting $r_\pm$ into Eq. (5). Three special cases warrant attention:

1. The two PLP states collapse onto one state in the limiting case when

$$g_1^2 = 4g_0 g_2. \tag{13}$$

2. The two PLP states exist and are orthogonal ($r_+ r_- = -1$) if

$$g_0 = -g_2, \tag{14}$$

or, from Eqs. (9),

$$\operatorname{Im}(T_{11}^* T_{21} + T_{22} T_{12}^*) = 0. \tag{15}$$

3. All linear polarizations at the input of the system are mapped to linear states at its output; i.e., the number of PLP states is infinite if

$$g_0 = g_1 = g_2 = 0, \tag{16}$$

which reduces Eq. (8) to an identity regardless of r. From Eqs. (9) it is apparent that Eqs. (16) are satisfied when the Jones matrix is real. Important physical examples that produce real Jones matrices include the external reflection of light at a dielectric–dielectric interface, half-wave retarders, and optical rotation caused by natural optical activity or the Faraday effect. All the cases considered can be elegantly clarified by invoking the circle-to-circle mapping property of the bilinear transformation[7,8] of Eq. (3). In particular, we note that the locus of all incident linear polarization states χ_i is the real axis (which is a degenerate circle) and that the locus of the corresponding output states χ_o must in general be a circle. Figure 2 shows the four different possibilities in which the circle of χ_o intersects the real axis indicating the existence of two PLP states (case a), touches the real axis indicating the collapse of the two PLP states onto one (case b), degenerates into the real axis itself indicating the mapping of all linear states onto linear states (case c), and lies entirely on one side of the real axis (in the lower or upper half-plane) so that PLP states do not exist (case d). It is interesting to note that, in case d, the optical system maps all input elliptical polarization states of one handedness onto output elliptical states that are also all of one handedness, be it the same or opposite to that of the input states.

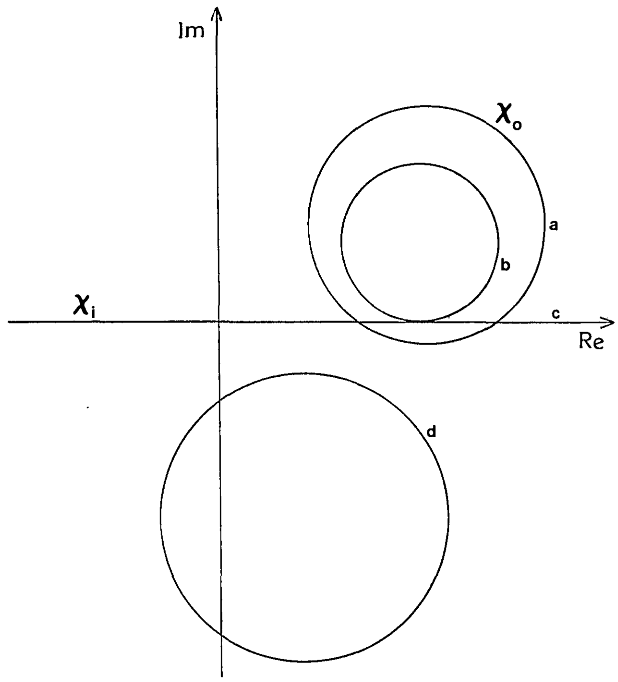

Fig. 2. Complex plane representation of the four different possibilities of the mapping of all incident linear polarization states χ_i onto output states χ_o. In case a two PLP states exist that correspond to the points of intersection of the circle locus of χ_o with the real axis, in case b the two PLP states collapse onto one state, in case c all incident linear polarization states are mapped onto linear states, and in case d all incident linear polarization states are mapped onto elliptical states of the same handedness (left handedness for the case shown).

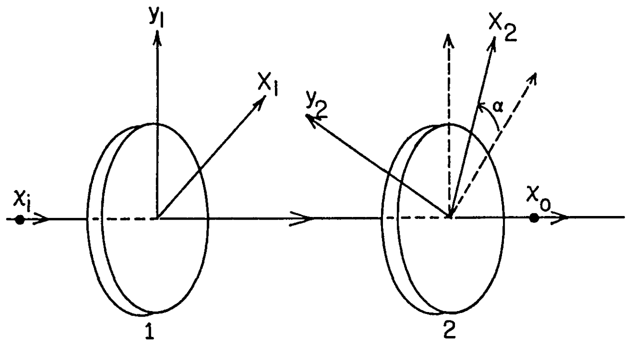

Fig. 3. Optical system composed of two optical elements 1 and 2 in series each of which is characterized by linear eigenpolarizations with axes x_1, y_1 and x_2, y_2, respectively. (After Ref. 5, Fig. 3.6.)

EXAMPLES

Consider a system composed of two optical elements in series each with orthogonal linear eigenpolarizations denoted by x_1, y_1 and x_2, y_2, with a rotation angle α between the x_2 and x_1 axes (Fig. 3). Let the ratio of y to x complex eigenvalues be denoted by $\tau_1 \exp(j\delta_1)$ and $\tau_2 \exp(j\delta_2)$ for the two elements, where τ and δ represent the relative attenuation and retardance along the y and x axes, respectively. Such a system is already referred to in the Introduction in relation to the polarizer–compensator–sample-analyzer or the polarizer–sample–compensator-analyzer null ellipsometer and has been analyzed.[5,7] Its Jones matrix, referred to the x_1y_1 and x_2y_2 coordinate axes at its input and output, respectively, is given by

$$\mathbf{T} = \begin{bmatrix} \cos\alpha & \tau_1 \sin\alpha \exp(j\delta_1) \\ -\tau_2 \sin\alpha \exp(j\delta_2) & \tau_1\tau_2 \cos\alpha \exp[j(\delta_1 + \delta_2)] \end{bmatrix}. \tag{17}$$

From Eqs. (9) and (17), we find that

$$4g_0g_2 = -\tau_1^2\tau_2^2 \sin^2 2\alpha \sin^2 \delta_2, \tag{18}$$

which is always negative (or 0) for all possible values of the system parameters; hence inequality (11) is always satisfied, and the existence of two PLP states is guaranteed.

For an optical system composed of any number of retardation plates and rotators, the Jones matrix is unitary and can be cast in the form[9]

$$\mathbf{T} = \begin{bmatrix} \cos\theta \exp(j\phi) & -\sin\theta \exp(-j\psi) \\ \sin\theta \exp(j\psi) & \cos\theta \exp(-j\phi) \end{bmatrix}, \tag{19}$$

where θ, ϕ, and ψ are three real angles. From Eqs. (9) and (19), we calculate

$$4g_0g_2 = -\sin^2 2\theta \sin^2(\psi - \phi), \tag{20}$$

which is always negative (or 0); hence inequality (11) is satisfied. Therefore a system that consists of any number of retardation plates and rotators always has two PLP states at its input that are propagated as linear states at its output.

To offer an example of a system for which PLP states do not exist, consider a slab of a circularly dichroic medium with a Jones matrix[10]

$$\mathbf{T} = \begin{bmatrix} \cosh(a/2) & j\sinh(a/2) \\ -j\sinh(a/2) & \cosh(a/2) \end{bmatrix}, \tag{21}$$

where $\exp(-a)$ represents the relative amplitude attenuation of the right- and left-handed circular polarization states. Evaluation of the g parameters using Eqs. (9) and (21) gives

$$g_1 = 0, \qquad g_0 = g_2 = -(1/2)\sinh a, \tag{22}$$

so that inequality (11) is violated for all a. However, when $a = 0$, $\mathbf{T}$ reduces to the identity matrix, Eq. (16) is satisfied, and the number of PLP states becomes infinite.

PRINCIPAL LINEAR POLARIZATION STATES OF AN OPTICAL SYSTEM DESCRIBED BY A MUELLER MATRIX

A nondeterministic (random) optical system scatters incident totally polarized light into partially polarized light in general. In this situation the Jones matrix description becomes inadequate. Instead, light is represented by its 4×1 real Stokes vector $\mathbf{S}$, the system by a 4×4 real Mueller matrix $\mathbf{M}$, and the scattering process by the linear transformation[11]

$$\mathbf{S}_o = \mathbf{M}\mathbf{S}_i. \tag{23}$$

Incident totally polarized light (of unit intensity) is represented by the Stokes vector

$$\mathbf{S}_i = [1 \quad \cos 2P \sin 2P \quad 0]^t, \tag{24}$$

where P is the linear polarization azimuth and t indicates the transpose. The degree of circular polarization of the scattered light is specified by the fourth element of the Stokes vector $\mathbf{S}_o$, which, from Eqs. (23) and (24), is given by

$$S_{o3} = m_{30} + m_{31} \cos 2P + m_{32} \sin 2P. \tag{25}$$

Obviously, the scattered light is (partially or totally) linearly polarized if

$$S_{o3} = 0. \tag{26}$$

Equations (25) and (26) lead to

$$\cos(2P - \beta) = -m_{30}/(m_{31}^2 + m_{32}^2)^{1/2}, \tag{27}$$

where β is defined by

$$\cos\beta = m_{31}/(m_{31}^2 + m_{32}^2)^{1/2},$$
$$\sin\beta = m_{32}/(m_{31}^2 + m_{32}^2)^{1/2}. \tag{28}$$

Equation (27) has two independent real (physically acceptable) solutions for P in the range $-90° < P \le 90°$ if and only if

$$|m_{30}| < +(m_{31}^2 + m_{32}^2)^{1/2}. \tag{29}$$

Inequality (29) is the desired constraint on the elements of the Mueller matrix such that two PLP states exist. Only three elements appear in that constraint, namely, the first three elements of the last row of $\mathbf{M}$. The scattered states of partial or complete linear polarization can be calculated directly from Eq. (23) if need be.

For an optical system with a known Jones matrix, the equivalent Mueller matrix can be determined using a known transformation.[11] Although that exercise is not carried out here, it should be possible to prove that, for nonscattering systems, the two constraints stated by inequalities (11) and (29) are equivalent.

For the example of a diffraction grating mentioned in the Introduction the zeroth-order Mueller matrix was measured using the four-detector photopolarimeter[1] and gave

$$m_{30} = -0.164, \qquad m_{31} = 0.054, \qquad m_{32} = -0.610, \quad (30)$$

at 65° angle of incidence and 45° inclination of the grating lines with respect to the plane of incidence. These elements satisfy inequality (29), give $\beta = -84.94°$ using Eqs. (28), and lead to the two solutions $P_1 = 100.30°$ and $P_2 = -5.245°$ for Eq. (27), in accord with experimental observations.[1]

In summary it has been shown that, for a broad range of optical systems whose Jones or Mueller matrices satisfy a specific constraint, two principal linear polarization states can be launched at the input of the system that propagate and appear as linear states at its output. This has been illustrated by several examples from the literature.

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