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Finding the minimum input impedance of a second-order unity-gain Sallen-Key low-pass filter without calculus

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Abstract

We derive an expression for the input complex impedance of a Sallen-Key second-order low-pass filter of unity gain as a function of the natural frequency ω_o, quality factor Q and the ratio of the resistors of the filter. From this expression, it is shown that the filter behaves like a single capacitor for low frequencies and as a single resistor at high frequencies. Furthermore, the minimum input impedance magnitude is found without using calculus. We discovered that the minimum input impedance magnitude is inversely proportional to Q and can be substantially less than its high-frequency value. Approximations to the minimum input impedance and the frequency at which it occurs are also provided. Additionally, PSpice simulations are presented which verify the theoretical derivations.

Keywords: Sallen-Key low-pass filter, Minimum without calculus, Input impedance.

Resumen

Derivamos una expresión para la impedancia de entrada compleja de un filtro Sallen-Key de paso bajo de segundo orden en función de la frecuencia natural ω_o, el factor de calidad Q, y el cociente de los resistores del filtro. Comenzando con esta expresión, mostramos que el filtro se comporta como un sólo capacitor para frecuencias bajas y como un sólo resistor para altas frecuencias. Es más, encontramos la magnitud de la impedancia mínima sin usar cálculo. Descubrimos que la magnitud de la impedancia mínima es inversamente proporcional a Q y puede ser significativamente menor que a frecuencias altas. Proveemos aproximaciones para la impedancia de entrada mínima y para la frecuencia a la que ocurre. Presentamos también simulaciones en PSpice que verifican las derivaciones teoréticas.

Palabras Clave: Filtro Sallen-Key de paso bajo, Mínimo sin cálculo, Impedancia de entrada.

PACS: 84.30.Vn, 02.90.+p, 01.40.Ha and 01.40.Fk. **ISSN 1870-9095**

I. INTRODUCTION

Fig. 1 shows the circuit diagram of an active second-order unity-gain Sallen-Key low-pass filter, which is widely used in electronics. One important parameter of such a filter is its transfer function, which has been widely studied and which relates its output voltage to its input voltage. Another important parameter is its input impedance. Unfortunately, very little has been written about this input impedance, even though designers need to know its minimum value to ensure that the filter does not load down the source or a previous stage. Inspection of Fig. 1 would suggest to the naive designer that the minimum input impedance is R_1 as it is in series with the rest of the circuit. Unfortunately, this is not the case: the input impedance can be very much lower than R_1 as we show below. Indeed, the authors' recent design of a second-order unity-gain Sallen-Key filter failed to achieve its design criteria because the filter did not interface well with its source, as we initially used the naive assumption given above. This experience prompted us to fully investigate the input impedance and its minimum value. Furthermore, we find the minimum value of the input impedance without using calculus, which should be of benefit to the student who has not yet had the opportunity to study math at this level. In this paper, we report our theoretical findings, which are also verified by PSpice simulations. (PSpice is a popular electrical and electronic circuits simulation software package that is widely used by electrical engineers and some physicists. The latest demo version can be freely obtained from [1]).

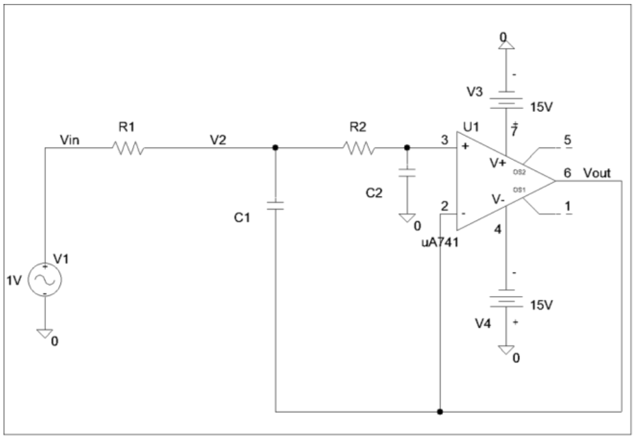

FIGURE 1. Circuit diagram of second-order unity-gain Sallen-Key low-pass filter.

II. TRANSFER FUNCTION FOR THE SALLEN-KEY LPF OF FIGURE 1

In this section, the transfer function for the circuit of Fig. 1 will be given, so that the key parameters such as natural frequency ω_o and quality factor Q can be defined. Indeed, it is straightforward to show, as demonstrated in the Appendix, that the transfer function is given by Eq. (9) of [2] (remembering that the gain of the op-amp is unity as it is in a voltage follower configuration), *i.e.*,

$$\frac{V_{out}}{V_{in}} = \frac{1}{s^2 R_1 R_2 C_1 C_2 + \left(R_1 + R_2\right) C_2 s + 1}, \tag{1}$$

where $s = i\omega$, with $i = \sqrt{-1}$ and ω (rad/sec) is the angular frequency of the applied sine-wave.

(The reader should be aware that Fig. 5 of [2] differs from our Fig. 1: C_1 and C_2 are interchanged. This means that Eq. (9) of [2] will also require that C_1 and C_2 be interchanged in Eq. (1). Many authors, *e.g.*, [3], also use our Fig. 1, rather than Fig. 5 of [2]).

The denominator of Eq. (1) can be written as

$$\begin{aligned} & s^2 R_1 R_2 C_1 C_2 + \left[\frac{\left(R_1 + R_2\right) C_2}{\sqrt{R_1 R_2 C_1 C_2}}\right] s\sqrt{R_1 R_2 C_1 C_2} + 1 \\ & = \left(\frac{s}{\omega_o}\right)^2 + \frac{1}{Q}\frac{s}{\omega_o} + 1, \end{aligned} \tag{2}$$

where the natural frequency is $\omega_o = 1/\sqrt{R_1 R_2 C_1 C_2}$ and Q, the quality factor of the filter, is $\dfrac{\sqrt{R_1 R_2 C_1 C_2}}{\left(R_1 + R_2\right) C_2}$.

III. INPUT IMPEDANCE FOR THE SALLEN-KEY LPF OF FIGURE 1

As we show in the Appendix, the normalized complex input impedance $Z(s)/R_1$ for the circuit of Fig. 1 is given by

$$\begin{aligned} \frac{Z(s)}{R_1} &= \frac{s^2 R_1 R_2 C_1 C_2 + \left(R_1 + R_2\right) C_2 s + 1}{s^2 R_1 R_2 C_1 C_2 + R_1 C_2 s} \\ &= \frac{\left(\frac{s}{\omega_o}\right)^2 + \frac{1}{Q}\frac{s}{\omega_o} + 1}{\left(\frac{s}{\omega_o}\right)^2 + \left[\frac{R_1}{R_1 + R_2}\frac{\left(R_1 + R_2\right) C_2}{\sqrt{R_1 C_1 R_2 C_2}}\right]\frac{s}{\omega_o}} \\ &= \frac{\left(\frac{s}{\omega_o}\right)^2 + \frac{1}{Q}\frac{s}{\omega_o} + 1}{\left(\frac{s}{\omega_o}\right)^2 + \frac{k}{Q}\frac{s}{\omega_o}}, \end{aligned} \tag{3}$$

where $0 < k = \dfrac{R_1}{R_1 + R_2} = \dfrac{1}{1 + \frac{R_2}{R_1}} < 1.$

Note that the input impedance depends upon the ratio of the resistors, $r = R_2 / R_1$: in order to have low sensitivity filters, this ratio should be unity ($i.e., k = \left(1 + r\right)^{-1} = 1/2$) as pointed out in [4]. Nevertheless, for various reasons, designers might not heed to this choice of k: hence, we will give results for general k values.

Interestingly, Eq. (3) becomes unity for large frequencies $\left(\omega \to \infty\right)$, *i.e.*, the input impedance looks simply as R_1, and the phase is 0^o. On the other hand, for low frequencies (as ω approaches zero), Eq. (3) becomes, approximately, $1\Big/\left(\frac{k}{Q}\frac{s}{\omega_o}\right)$, *i.e.*, the input impedance looks like a capacitor whose value is given by $C_{e1} = \dfrac{k}{\omega_o Q R_1}$. Using the expressions for ω_o, Q and k just given, it is clear that $C_{e1} = C_2$. Hence, for low frequencies, the magnitude of the input impedance is just the reactance of this capacitor, *i.e.*, $\dfrac{1}{\omega C_{e1}} = \dfrac{1}{\omega C_2} = \dfrac{\omega_o Q R_1}{\omega k}$, and the phase approaches -90^o.

A. Magnitude of the Input Impedance

From Eq. (3), the magnitude of the normalized impedance becomes

$$\left|\frac{Z(s)}{R_1}\right| = \frac{\left|-p^2 + ip/Q + 1\right|}{\left|-p^2 + \frac{k}{Q} ip\right|}, \tag{4}$$

where $p = \omega / \omega_o$ is the normalized frequency.

Actually, it will be more convenient to work with the magnitude squared of the normalized impedance. Hence, Eq. (4) becomes

$$\left|\frac{Z(s)}{R_1}\right|^2 = \frac{\left(-p^2+1\right)^2 + p^2/Q^2}{p^4+\left(\frac{k}{Q}\right)^2 p^2} = \frac{p^4-\left(2-1/Q^2\right)p^2+1}{p^4+\left(\frac{k}{Q}\right)^2 p^2}. \quad (5)$$

Furthermore, Eq. (5) can be rewritten as

$$\left|\frac{Z(s)}{R_1}\right|^2 = \frac{x^2-Ax+1}{x^2+Bx}, \quad (6)$$

where $x=p^2$, $A=2-1/Q^2$ and $B=\left(\frac{k}{Q}\right)^2$.

Taking the square root of Eq. (6) allows us to make a plot of the normalized impedance in dB (*i.e.*, $20\log\left(\left|Z(s)/R_1\right|\right)$ as a function of the normalized frequency, as shown in Fig. 2 for $k=0.5$ and various Q values. Also shown are straight-line plots of the reactance of the equivalent capacitor $C_{e1}=C_2$, confirming our earlier statement that the magnitude of the input impedance for low frequencies is simply the reactance of C_2. Clearly, Fig. 2 also verifies the high-frequency value of the input impedance noted earlier.

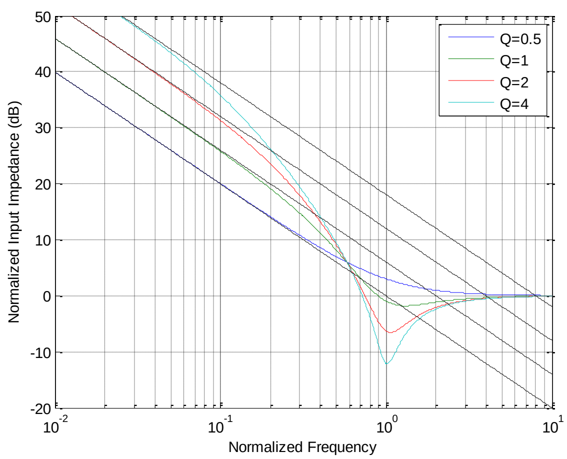

FIGURE 2. Magnitude of the normalized input impedance (in dB) as a function of the normalized frequency for $k=0.5$. The straight-lines are plots of the reactance (in dB) of $C_{e1}=C_2$.

B. Phase of the Input Impedance

The phase of the input impedance is easily found from Eq. (3) to be

$$\phi = \tan^{-1}\left(\frac{p}{Q\left(1-p^2\right)}\right) - 90^o - \tan^{-1}\left(\frac{Qp}{k}\right). \quad (7)$$

For $p=1$, the phase becomes $\phi=-\tan^{-1}\left(\frac{Q}{k}\right)$ and for $p>1$, Eq. (7) can be written as

$$\phi = 90^o - \tan^{-1}\left(\frac{p}{Q\left(p^2-1\right)}\right) - \tan^{-1}\left(\frac{Qp}{k}\right). \quad (8)$$

Note from Eq. (7) that for $\omega\to 0$, $\phi\to -90^\circ$, and from Eq. (8) that as $\omega\to\infty$, $\phi\to 0$. These facts are also confirmed by a plot of Eq. (7) and Eq. (8) shown in Fig. 3.

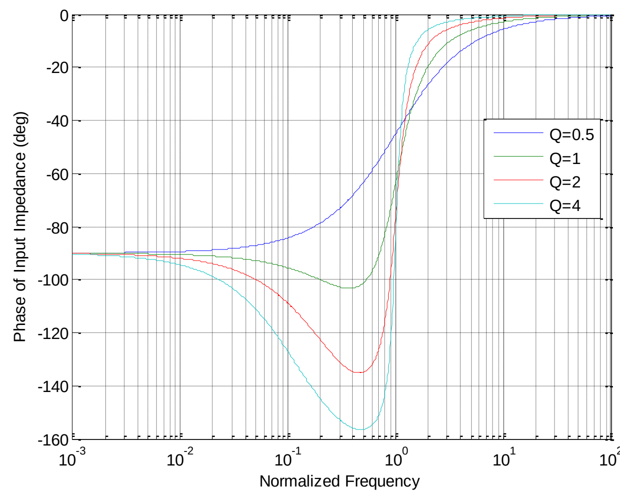

FIGURE 3. Phase of input impedance (deg) as a function of the normalized frequency for $k=0.5$.

Clearly, Fig. 3 verifies the low-frequency and high-frequency values of the phase of the complex input impedance noted earlier.

IV. FINDING THE MINIMUM INPUT IMPEDANCE MAGNITUDE WITHOUT CALCULUS

Now that the normalized input impedance has been found, it can be shown how its minimum value can be found without calculus.

Using long division, Eq. (6) can be written as a proper rational function, *i.e.*,

$$\left|\frac{Z(s)}{R_1}\right|^2 = 1-\frac{(A+B)x-1}{x^2+Bx} = 1-Zm. \quad (9)$$

Clearly, for Eq. (9) to be a minimum, $Zm=\frac{(A+B)x-1}{x^2+Bx}$ must be a maximum.

Let $y=(A+B)x-1$. Hence,

$$Zm=\frac{y(A+B)^2}{y^2+\left[2+B(A+B)\right]y+1+B(A+B)}. \qquad (10)$$

Fortunately, it is straightforward to find the maximum value of Eq. (10) without using calculus. Indeed, the method we used in a previous paper [5] will be utilized here as well. To begin, first divide the numerator and denominator of Eq. (10) by y to get

$$Zm=\frac{(A+B)^2}{y+\left[1+B(A+B)\right]y^{-1}+2+B(A+B)}$$
$$=\frac{(A+B)^2}{\left(\sqrt{y}-\frac{\sqrt{1+B(A+B)}}{\sqrt{y}}\right)^2+2+B(A+B)+2\sqrt{1+B(A+B)}}. \qquad (11)$$

Clearly, Eq. (11) is maximized when $\left(\sqrt{y}-\frac{\sqrt{1+B(A+B)}}{\sqrt{y}}\right)^2$ is minimized, *i.e.*, made to be zero, so that $y=\sqrt{1+B(A+B)}$.

Recall that $x=\frac{y+1}{A+B}$ and $x=p^2=(\omega/\omega_o)^2$. Hence, the frequency at which Zm is a maximum and Eq. (9) is a minimum is given by

$$\omega_{\min}=\omega_o\sqrt{\frac{\sqrt{1+B(A+B)}+1}{A+B}}. \qquad (12)$$

Notice that when $y=\sqrt{1+B(A+B)}$, the maximum value of Eq. (11) becomes

$$Zm_{\max}=\frac{(A+B)^2}{2+B(A+B)+2\sqrt{1+B(A+B)}}. \qquad (13)$$

Hence, from Eq. (9), the minimum value of the input impedance becomes

$$\left|Z(s)\right|_{\min}=R_1\sqrt{1-\frac{(A+B)^2}{2+B(A+B)+2\sqrt{1+B(A+B)}}}. \qquad (14)$$

Recall that $A=2-1/Q^2$ and $B=\left(\frac{k}{Q}\right)^2$. Hence, the normalized frequency at which the minimum input impedance occurs is, from Eq. (12),

$$\frac{\omega_{\min}}{\omega_o}=\sqrt{\frac{1+\sqrt{1+\frac{2k^2}{Q^2}+\frac{k^4-k^2}{Q^4}}}{2+\frac{k^2-1}{Q^2}}}$$
$$=\sqrt{\frac{1+\sqrt{\frac{Q^4+2k^2Q^2+k^4-k^2}{Q^4}}}{2+\frac{k^2-1}{Q^2}}}$$
$$=\sqrt{\frac{Q^2+\sqrt{Q^4+k^2\left(2Q^2+k^2-1\right)}}{2Q^2+k^2-1}}, \qquad (15)$$

and, from Eq. (14), the normalized minimum input impedance magnitude is

$$\left|Z(s)\right|_{\min}=R_1\sqrt{1-\frac{\left(2+\frac{k^2-1}{Q^2}\right)^2}{2+\frac{2k^2}{Q^2}+\frac{k^4-k^2}{Q^4}+2\sqrt{1+\frac{2k^2}{Q^2}+\frac{k^4-k^2}{Q^4}}}}$$
$$=R_1\sqrt{1-\frac{\left(2Q^2+k^2-1\right)^2}{2Q^4+k^2\left(2Q^2+k^2-1\right)+2Q^2\sqrt{Q^4+k^2\left(2Q^2+k^2-1\right)}}}. \qquad (16)$$

To illustrate how the normalized minimum frequency depends upon the quality factor of the filter, Eq. (15) is plotted in Fig. 4 for three values of k.

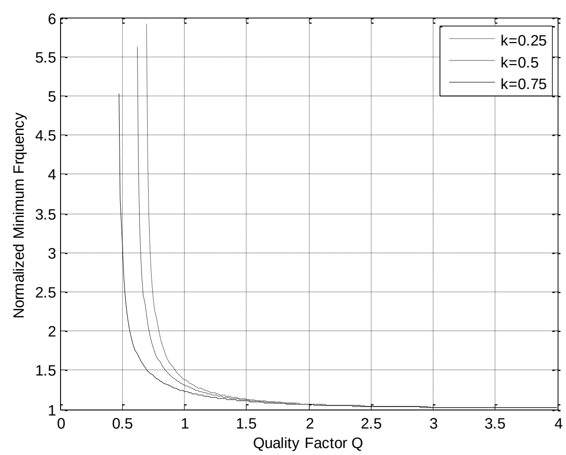

FIGURE 4. Normalized minimum frequency as a function of the quality factor of the filter.

In order for Eq. (15) to be valid, $2Q^2+k^2-1>0$. Hence, $Q>\sqrt{\frac{1-k^2}{2}}$ in order for there to be a minimum in the input impedance magnitude. For $Q \le \sqrt{\frac{1-k^2}{2}}$, the input impedance decreases monotonically from infinity to R_1 as the frequency increases from zero to infinity. This is illustrated in Fig. 5 for $k = 0.5$.

Notice from Fig. 5 that if there is a minimum, then there is also a frequency at which the normalized impedance is unity. From Eq. (6), this frequency is determined to be

$$\frac{\omega_{unity}}{\omega_o} = \sqrt{\frac{Q^2}{2Q^2 + k^2 - 1}}. \qquad (17)$$

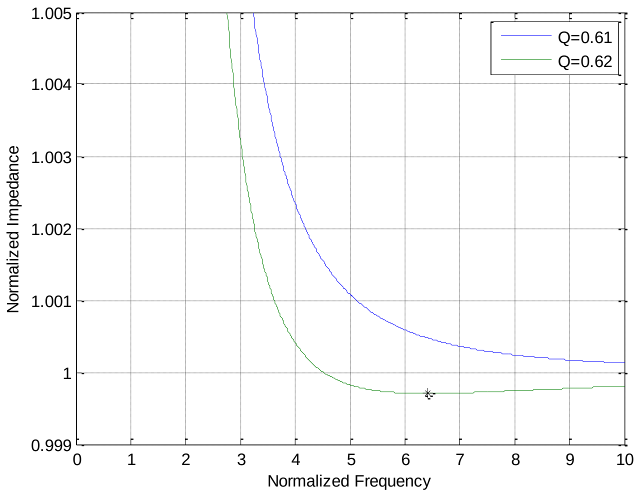

FIGURE 5. Normalized impedance as a function of the normalized frequency, for $k = 0.5$, for Q values on either side of $Q_{\min} = \sqrt{(1-k^2)/2} = \sqrt{3/8} = 0.6124$. In order for a minimum impedance to exist, $Q > Q_{\min}$.

Note also, from Eqs. (17) and (15), that ω_{unity} behaves as a lower bound on $\omega_{\min}$.

A. Approximations to the Minimum Frequency

Using the first line of Eq. (15) and the fact that $(1+z)^r \approx 1 + rz + \frac{r(r-1)}{2} z^2$ for small z [6], it is straightforward to show that for large Q,

$$\frac{\omega_{\min}}{\omega_o} \approx 1 + \frac{1}{4Q^2}, \qquad (18)$$

or more accurately, as obtained with Matlab's Taylor function operating on Eq. (15):

$$\frac{\omega_{\min}}{\omega_o} \approx 1 + \frac{1}{4Q^2} + \frac{-8k^2 + 3}{32Q^4}. \qquad (19)$$

Also, for large Q, $\sqrt{Q^4 + k^2\left(2Q^2 + k^2 - 1\right)} \approx Q^2$. Hence, Eq. (15) becomes

$$\frac{\omega_{\min}}{\omega_o} \approx \sqrt{\frac{2Q^2}{2Q^2 + k^2 - 1}}. \qquad (20)$$

By the way, comparing Eq. (17) with Eq. (20) reveals that

$$\omega_{\min} \approx \sqrt{2}\,\omega_{unity}. \qquad (21)$$

A plot of the percentage error, 100 (1-Aproximate Value/True Value), of Eq. (18) is shown in Fig. 6, where it is clearly seen that the percentage error depends upon the value of k, for small values of Q; however, as Q increases this dependence becomes smaller and smaller.

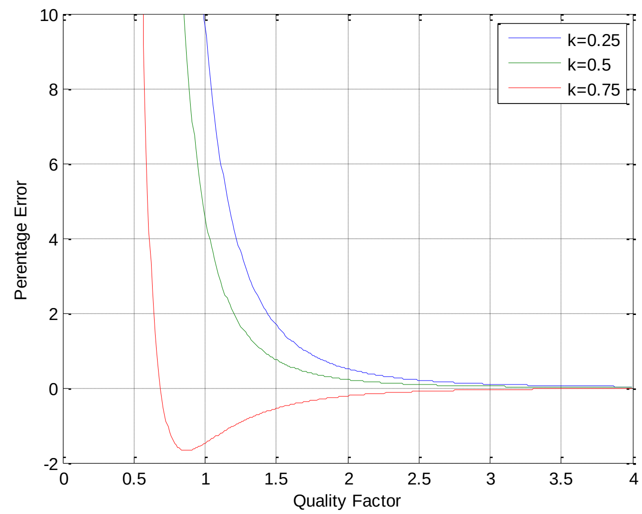

FIGURE 6. Percentage error of Eq. (18) approximation to the minimum frequency for various values of k.

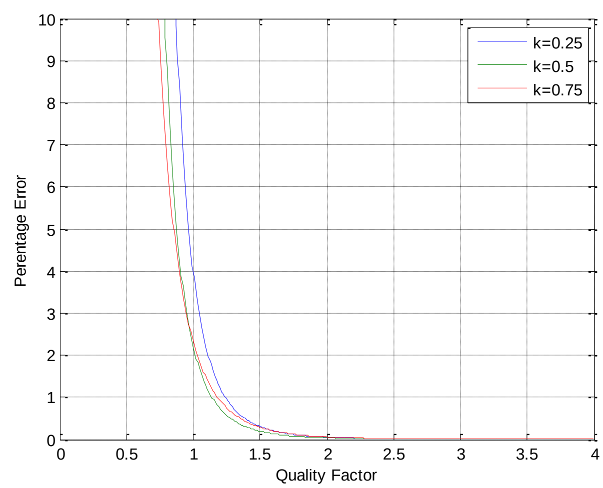

FIGURE 7. Percentage error of Eq. (19) approximation to the minimum frequency for various values of k.

On the other hand, the percentage error of Eq. (19), shown in Fig. 7, depends upon k only in a minor way. Surprisingly, Eq. (18) is more accurate than Eq. (19) for some Q and k values, *e.g.*, $Q = .7$ and $k = 0.75$.

Nonetheless, Eq. (19) is clearly more accurate than Eq. (18) for Q greater than 1.25 approximately, for all k values studied here.

Furthermore, recall that Eq. (20) was derived assuming large Q; hence, it is somewhat surprising that the percentage error is quite good for all Q values, as shown in Fig. 8.

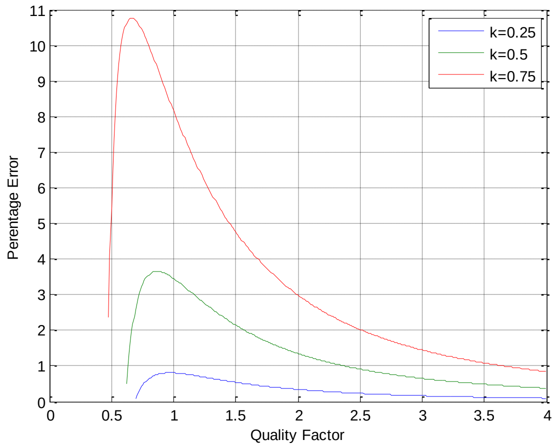

FIGURE 8. Percentage error of Eq. (20) approximation to the minimum frequency for various values of k.

B. Approximations to the Minimum Input Impedance

Working with the second line of Eq. (16), it is straightforward to obtain the following approximation to the normalized minimum input impedance magnitude for large Q:

$$|Z(s)|_{\min} \approx \frac{R_1}{Q}, \tag{22}$$

or more accurately (as found with Matlab software):

$$|Z(s)|_{\min} \approx \frac{R_1}{Q}\left(1 - \frac{4k^2+1}{8Q^2}\right). \tag{23}$$

A plot of the percentage error of Eq. (22) is shown in Fig. 9, where it is clearly seen that the percentage error depends upon the value of k. However, as Q increases the percentage error decreases for all values of k.

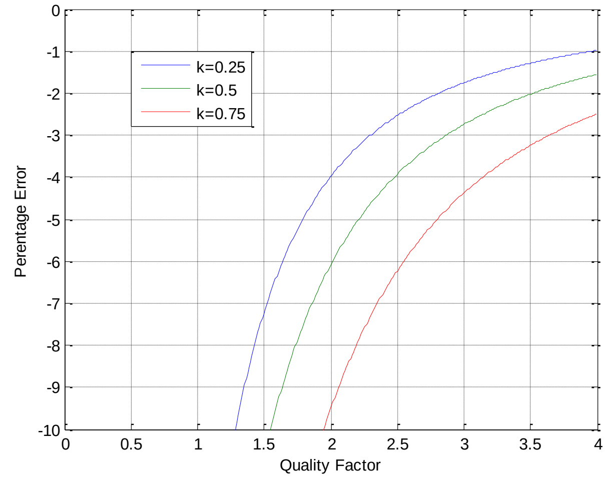

FIGURE 9. Percentage error of Eq. (22) approximation to the minimum magnitude of the impedance for various values of k.

Plotting the percentage error of Eq. (23) in Fig. 10 shows that it provides a more accurate estimation for the minimum input impedance than Eq. (22) does.

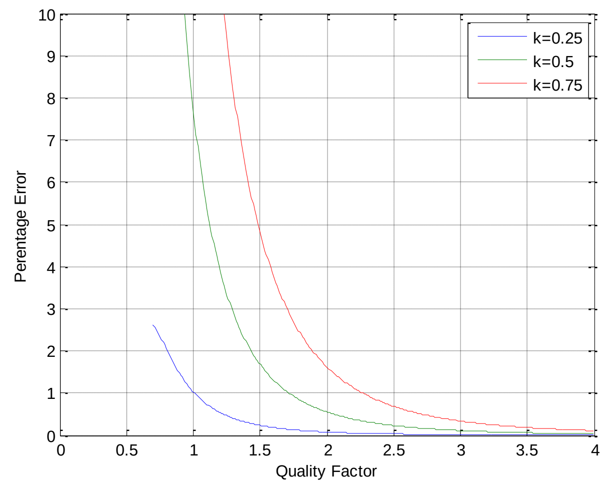

FIGURE 10. Percentage error of Eq. (23) approximation to the minimum magnitude of the impedance for various values of k.

From Fig. 2, it is apparent that the minimum input impedance occurs approximately when $p = 1$, for large Q values. Hence, substituting $p = 1$ into Eq. (5) gives

$$|Z(s)|_{\min} \approx R_1\sqrt{\frac{1}{Q^2+k^2}}. \tag{24}$$

The percentage error for Eq. (24), shown in Fig. 11, is clearly much better than that of Eq. (22) and is more easily derived than Eq. (23).

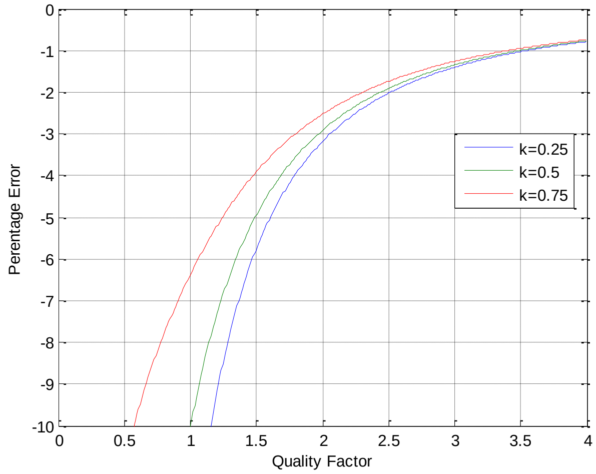

FIGURE 11. Percentage error of Eq. (24) approximation to the minimum magnitude of the impedance for various values of k.

Surprisingly, by trial and error, we found that Eq. (24) can be improved significantly by simply adding a constant in its denominator to get

$$\left|Z(s)\right|_{\min} \approx R_1\sqrt{\frac{1}{Q^2+k^2+.34}}. \qquad (25)$$

As shown in Fig. 12, the percentage of error for the approximation of Eq. (25) is less than $\pm 6.2\%$ for all Q values. The constant was chosen so that the positive and negative peak percentage errors are approximately the same magnitude for $0.25 \le k \le 0.75$. If k values outside this range are used, the constant will have to be modified for best results.

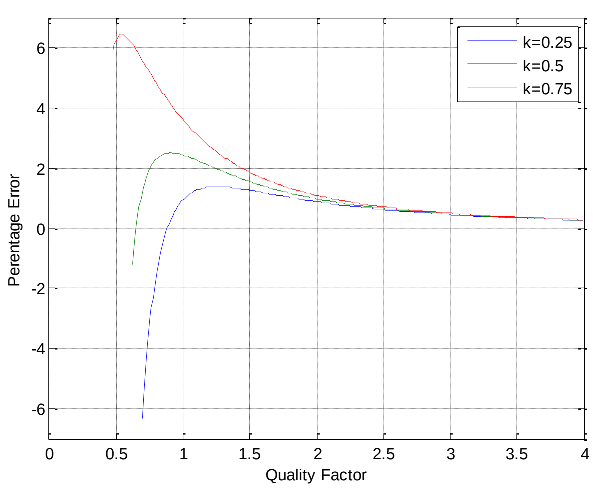

FIGURE 12. Percentage error of Eq. (25) approximation to the minimum magnitude of the impedance for various values of k.

V. PSPICE SIMULATIONS

In order to verify the theoretical derivations, we performed PSpice simulations of the filter in Fig. 1. For all the simulations, we set $\omega_o = 1000$ rad/s or $f_o = 159.15$ Hz, $Q = 1$, $k = 0.5$ and $R_1 = 1000\Omega$.

A. Design of the Filter

To simulate the filter, we must select values for the capacitors.

Recall that $R_2 = rR_1 = (1-k)R_1/k$ and that $\omega_o = 1/\sqrt{R_1R_2C_1C_2}$; hence, $R_2 = 1000\Omega$, and

$$\omega_o = \frac{1}{R_1\sqrt{\left(\frac{1-k}{k}\right)C_1C_2}}. \qquad (26)$$

Also, recall that $Q = \frac{\sqrt{R_1R_2C_1C_2}}{(R_1+R_2)C_2}$; therefore,

$$Q = \frac{R_1\sqrt{\left(\frac{1-k}{k}\right)C_1C_2}}{\left(R_1+\left(\frac{1-k}{k}\right)R_1\right)C_2} = \sqrt{k(1-k)\frac{C_1}{C_2}}. \qquad (27)$$

Solving Eq. (26) and Eq. (27) simultaneously gives $C_1 = 2\mu F$ and $C_2 = 0.5\mu F$.

Furthermore, recall that $C_{e1} = \frac{k}{\omega_o QR_1}$. Using Eq. (26) and Eq. (27), it is straightforward to show that $C_{e1} = C_2$, as we claimed earlier.

B. Verification of the Design of the Filter

The first thing we want to do with our simulations is to verify that our design has met our specifications for ω_o and Q. To do this, we find the maximum gain of the filter, $\left|T(\omega)\right|_{\max}$, by plotting the magnitude response of the simulated filter, as shown in Fig. 13. From this graph, it is clear that the maximum gain is 1.2496 dB or 1.1547.

However, from Eq. (12) of [5], $\left|T(\omega)\right|_{\max} = Q/\sqrt{1-1/(4Q^2)} = 1.1547$; hence, the simulated $Q = \sqrt{\frac{1.1547^2+\sqrt{1.1547^4-1.1547^2}}{2}} = 1.0000.$

Also, from Fig. 13, the frequency at which the maximum gain occurs is found to be $\omega_{\max} = 2\pi 112.695$ $= 708.08$ rad/s. Hence, the simulated natural frequency is

$\omega_o = \omega_{\max} / \sqrt{1-\left(2Q^2\right)^{-1}} = \sqrt{2}\omega_{\max} = 1001.4$ rad/s. (See Eq. (11) of [5]).

Clearly, the parameters of the simulation match the theoretical design quite well.

C. Verification of the Magnitude of the Input Impedance

PSpice measures the magnitude of the input impedance by dividing V_{in} by $I(R_1)$, the current through R_1. Furthermore, the PSpice command $dB(V_{in} / I(R_1))$ measures $20\log(|Z|)$, *i.e.*, the magnitude of the input impedance in dB, a plot of which is shown in Fig. 14. Also shown in this figure is the straight-line plot of the reactance of the equivalent capacitor for low-frequencies, which is $20\log\left(\frac{1}{\omega C_2}\right) = 20\log\left(\frac{10^6}{\pi f}\right)$.

Clearly, the simulated plot coincides with the reactance straight-line at low frequencies, as expected. Furthermore, the simulated plot shows that the magnitude of the input impedance approaches 60 dB or $1000\,\Omega$, as expected for high-frequencies.

Additionally, from Eq. (17), the theoretical value of $\omega_{unity} = \sqrt{0.8}\omega_o = 894.427$ rad/s or 142.35 Hz. From Fig. 14, PSpice simulates this as 142.21 Hz or 893.51 rad/s.

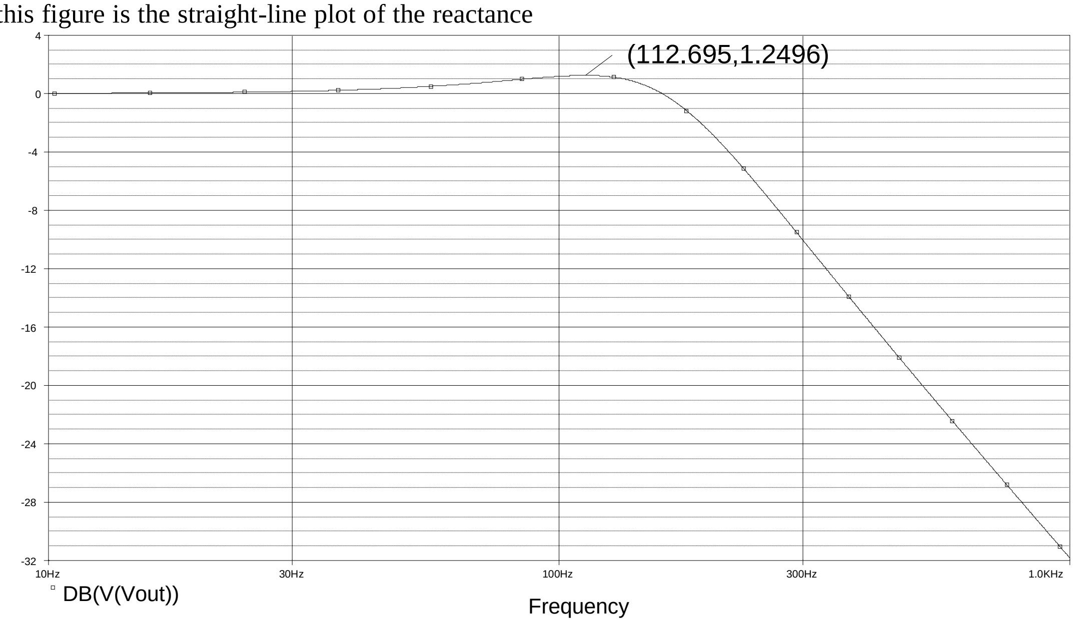

FIGURE 13. Simulated magnitude response (dB) of the filter.

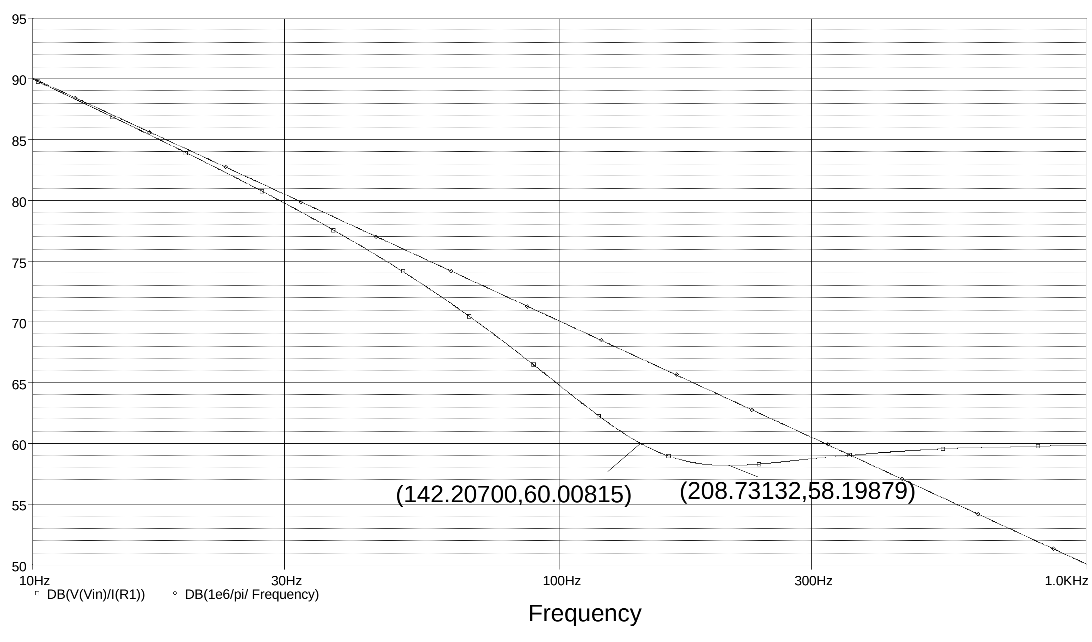

FIGURE 14. Simulated magnitude (dB) of the input impedance of the filter. The straight-line is the reactance of the equivalent capacitor at low-frequencies, *i.e.*, the reactance of C_2.

Also, from Fig. 14, $f_{\min} = 208.731$ Hz or $\omega_{\min} = 1311.50$ rad/s. The theoretical value for this is given by Eq. (15), *i.e.*, $2\sqrt{\frac{1+\sqrt{21}/4}{5}}1000 = 1310.16$ rad/s or given approximately by Eq. (20), *i.e.*, $\sqrt{8/5} = 1264.91$ rad/s.

Furthermore, from Fig. 14, the minimum magnitude of the input impedance is 58.1988 dB or 812.718 Ω. On the other hand, Eq. (16) gives the theoretical value for this as

$$1000\sqrt{1-\frac{\left(\frac{5}{4}\right)^2}{2+\frac{1}{4}\left(\frac{5}{4}\right)+2\sqrt{1+\frac{1}{4}\left(\frac{5}{4}\right)}}} = 1000\sqrt{1-\frac{25}{37+8\sqrt{21}}}$$

$= 812.776\Omega$. Alternatively, Eq. (25) gives the approximate value of $1000\sqrt{\frac{1}{1.34+0.25}} = 793.052\Omega$.

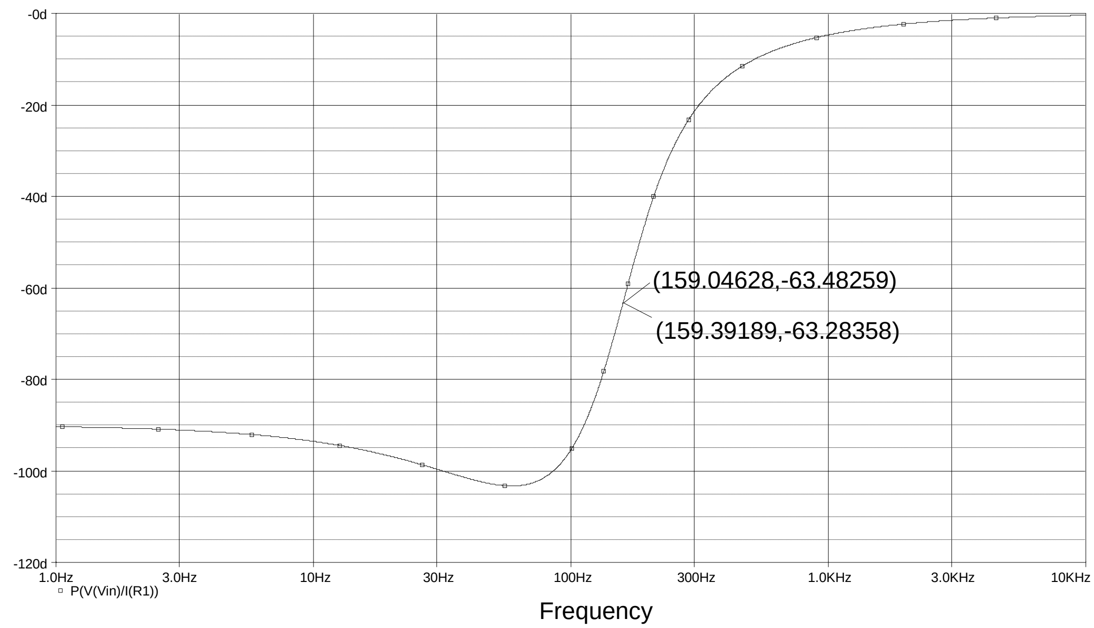

FIGURE 15. Simulated phase response (deg) of the input complex impedance of the filter.

D. Verification of the Phase of the Input Impedance

The PSpice command $P(V_{in}/I(R_1))$ measures the phase of the input impedance in degrees, a plot of which is shown in Fig. 15. As can be seen, the phase becomes -90^o at low-frequencies and 0^o at high frequencies, as expected. Furthermore, recall (from Eq. (7) with $p=1$) that the theoretical phase at the natural frequency $(1000/(2\pi) = 159.155$ Hz) is given by $\phi = -\tan^{-1}(Q/k)$ $= -\tan^{-1}(2) = -63.4349^o$. On the other hand, the PSpice simulation gives a phase shift of -63.4826^o at a frequency of 159.046 Hz and a phase shift of -63.2836^o at a frequency of 159.392 Hz. Therefore, interpolating between these two points gives a PSpice simulated value of −63.4199^o at the natural frequency.

E. Summary of Theoretical-PSpice Comparison

For convenience, the theoretical and PSpice results given above are summarized in Table I. As can be seen, there is excellent agreement between the two.

TABLE I. Summary Theoretical-PSpice comparison.

Item	Theoretical Value	PSpice Value
ω_o	1000 rad/s	1001.4 rad/s
Q	1	1.000
ω_{unity}	894.427 rad/s	893.532 rad/s
$\omega_{\min}$	1310.16 rad/s	1311.16 rad/s
Phase at ω_o	-63.4349^o	-63.4199^o
Min. Input Impedance	812.776 Ω	812.718 Ω.

VI. CONCLUSIONS

We have derived an expression for the input complex impedance for the second-order unity-gain Sallen-Key low-pass filter, which is given in Eq. (3). From this expression, we have shown that the input complex impedance is R_1 for high-frequencies, whereas for low-frequencies it is simply $(i\omega C_2)^{-1}$. Furthermore, we have found the minimum of the magnitude of the input impedance without calculus, as given in Eq. (16) and its approximations in Eq. (22)-Eq. (25). We have also discovered an expression for $\omega_{\min}$, as given in Eq. (15) and its approximations in Eq. (18)-Eq. (20). Finally, we provided PSpice simulations which verified the theoretical results.

In future work, we intend to study the input impedance for arbitrary gain Sallen-Key low-pass and high-pass filters.

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APPENDIX

In this appendix, we derive Eqs. (1) and (3).

Let $I_{in}(s)$ be the current through the source V_1 $I_1(s)$ be the current through C_1, from top to bottom, and $I_2(s)$ be the current through R_2 from left to right. Looking at the node marked with voltage $V_2(s)$ we obtain:

$$I_2(s)=\frac{V_2(s)}{R_2+\frac{1}{sC_2}}=\frac{sC_2V_2(s)}{sC_2R_2+1}, \tag{A1}$$

and

$$I_1(s)=\frac{V_2(s)-V_{out}(s)}{\frac{1}{sC_1}}=\left(V_2(s)-V_{out}(s)\right)sC_1. \tag{A2}$$

Noticing that the voltage across C_2 is $V_{out}(s)$ because the input to a voltage follower is equal to its output voltage, and using voltage division, we obtain:

$$V_{out}(s)=\frac{\frac{1}{sC_2}}{R_2+\frac{1}{sC_2}}V_2(s), \tag{A3}$$

or

$$V_2(s)=\left(sC_2R_2+1\right)V_{out}(s). \tag{A4}$$

Using (A4) in (A1) and (A2) gives, respectively:

$$I_2(s)=sC_2V_{out}(s), \tag{A5}$$

and

$$\begin{aligned}I_1(s)&=\frac{\left(sC_2R_2+1\right)V_{out}(s)-V_{out}(s)}{1/sC_1}\\&=s^2C_1C_2R_2V_{out}(s).\end{aligned} \tag{A6}$$

Using KCL,

$$\begin{aligned}I_{in}(s)&=I_1(s)+I_2(s)\\&=sC_2\left(sC_1R_2+1\right)V_{out}.\end{aligned} \tag{A7}$$

Writing the loop equation for the leftmost loop using KVL, we obtain:

$$V_{in}(s)=I_{in}(s)R_1+I_2(s)\left(R_2+\frac{1}{sC_2}\right). \tag{A8}$$

Finally, using (A7) and (A5) in (A8), we get the expression relating the output voltage to the input voltage:

$$\begin{aligned}V_{in}(s)&=sC_2\left(sC_1R_2+1\right)R_1V_{out}(s)+sC_2\left(R_2+\frac{1}{sC_2}\right)V_{out}(s)\\&=V_{out}(s)\left[sC_2R_1\left(sC_1R_2+1\right)+sC_2R_2+1\right].\end{aligned} \tag{A9}$$

Eq. (A9) can clearly be written as the transfer function shown in Eq. (1).

In order to find the input complex impedance, we simply divide the input voltage by the input current, using (A9) and (A7), to get:

$$\begin{aligned} Z(s) &= \frac{V_{in}(s)}{I_{in}(s)} \\ &= \frac{V_{out}(s)\left[s^2 R_1 R_2 C_1 C_2 + sC_2\left(R_1 + R_2\right) + 1\right]}{sC_2\left(sC_1 R_2 + 1\right)V_{out}(s)} \\ &= \frac{s^2 R_1 R_2 C_1 C_2 + sC_2\left(R_1 + R_2\right) + 1}{s^2 C_1 C_2 R_2 + sC_2}. \end{aligned} \tag{A10}$$

Finally, dividing by R_1 we obtain the normalized complex impedance of Eq. (3).