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Asymptotic Performance of the P^{th} Power-Law Phase Estimator

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***Abstract*—An expression for the true variance of the P^{th} power-law phase estimator, as the number of samples approaches infinity, is given. This expression is an extension to the linear approximation of Moeneclaey and de Jonghe [1] which is known to be inadequate in some practical systems. Our new expression covers general 2π/*P*-rotationally symmetric constellations that include those of PAM, QAM, PSK, Star M-QAM, MR-DPSK, and others. This expression also generalizes the known expressions for QAM and PSK. Additionally, our expression reduces to the Cramer-Rao bound given by Steendam and Moeneclaey [9], as *SNR* goes to zero. Monte Carlo simulations provide experimental verification of the theoretical expression for various constellations.**

***Index Terms*—Carrier phase estimation, quadrature amplitude modulation, phase-shift keying, general constellations, synchronization, blind estimation, asymptotic performance.**

I. Introduction

The non-data-aided (NDA) P^{th} power-law phase estimator, used with 2π/*P*-rotationally symmetric constellations, is known to be the maximum likelihood estimator as the signal-to-noise ratio (*SNR*) goes to zero [1]. The exact variance of error for this estimator has been obtained for the fourth power case by Serpedin et al. [2] for *M*-QAM systems, as the number of samples, *L*, goes to infinity. Graphical results for the exact variance are also provided in [1] for *M*-QAM, *M* = 4, *M* = 16, and $M \to \infty$. However, no numerical details are given for how these were obtained. More recently, Wang et al. have determined the variance of the more general Viterbi & Viterbi type phase estimator, again for *M*-QAM in [3] and for *M*-PSK in [4], both of which treat the power-law estimator as a special case. However, to date, no expression has been provided for the variance of the P^{th} power estimator for more general constellations such as Star 16-QAM [5]-[6], multi-ring DPSK (MR-DPSK) [7], or those in [8]. Moeneclaey and de Jonghe [1] have found an expression for general 2π/*P*-rotationally symmetric constellations that is the linear approximation to the exact variance and is valid for medium to large *SNR*. However, it has been noted by Serpedin et al. [2] that this linear approximation is not always adequate, especially if coding is employed (as the *SNR* available at the phase estimator will be reduced by an amount proportional to the coding gain), or if the system is operating at a relatively high probability of error.

The purpose of this paper is to provide an expression for the variance of the estimation error for the general P^{th} power-law estimator. As in [1], this expression will be valid for 2π/*P*-rotationally symmetrical constellations and for all *SNR* as the number of samples, *L*, goes to infinity. (However, its usefulness for finite *L* will be demonstrated with Monte Carlo simulations). We show that as $SNR \to 0$, the performance of the P^{th} power-law phase estimator reaches the Cramer-Rao bound (CRB), as given by Steendam and Moeneclaey [9]. Also, using our new expression, an error in [2] will be corrected.

The organization of the rest of this paper is as follows: In Section II, a review of the P^{th} power-law estimator is given, followed by the derivation of its asymptotic performance in Section III. This performance is experimentally verified using Monte Carlo computer simulations for various constellations in Section IV, followed in Section V by a clarification of these simulations. Finally, in Section VI, conclusions and possibilities for future work are stated.

II. Review of the P^{TH} Power-Law Phase Estimator

The following review borrows heavily from [10], where it is assumed that the system is already equalized and frequency-synchronized, and that timing and relative gain control have been achieved. Given these assumptions, the baud-rate samples of the output of a matched filter are given by:

$$r_k = d_k e^{j\phi} + n_k, \qquad k = 1,2,...,L \tag{1}$$

where $d_k = a_k + jb_k$ is a complex number that represents the two-dimensional symbol transmitted at time *k*T, 1/T is the signaling rate, ϕ, which is assumed to be constant over the *L* symbols, is the unknown phase offset that is to be estimated, and n_k are complex independent identically distributed (i.i.d.) zero-mean Gaussian random variables with independent real and imaginary parts having variance σ^2. The average constellation energy is given by $W = E[|d_k|^2]$. Hence, the symbol signal-to-noise ratio (*SNR*) is given by $W/2\sigma^2$.

The P^{th} power-law phase estimator produces a phase estimate from the following:

$$\hat{\phi} = \frac{1}{P}\text{angle}\left[E\left[d^{*P}\right]\sum_{k=1}^{L} r_k^P\right], \tag{2}$$

where L is the length of the observed data block.

Note that because the constellation has $2\pi/P$-rotational symmetry, it is only possible to retrieve the unknown phase within a modulo $2\pi/P$-phase ambiguity. (However, this ambiguity can be eliminated with proper coding). Without loss of generality, we assume $-\pi/P < \hat{\phi} \le \pi/P$.

III. Asymptotic Performance of the P^{TH} Power-Law Phase Estimator

According to Moeneclaey and de Jonghe [1], the linearized tracking error of the NDA algorithm (2), as $L \to \infty$, is given by

$$\phi - \hat{\phi} = \frac{1}{P\left|E\left[d^P\right]\right|^2}\,\text{Im}\left[E\left[d^{*P}\right]\frac{1}{L}\sum_{k=1}^{L} r_k^P e^{-jP\phi}\right]. \tag{3}$$

Using the identities $E\left[(\text{Im}\,x)^2\right] = \left(E\left[|x|^2\right] - \text{Re}\,E\left[x^2\right]\right)/2$ and $\text{Re}(x) = \left(x + x^*\right)/2$, the variance of this tracking error becomes

$$\text{var}_P\left(\phi - \hat{\phi}\right) = \frac{2\left|E\left[d^P\right]\right|^2 E\left[|r|^{2P}\right] - E^2\left[d^P\right]E\left[d^{*2P}\right] - E^2\left[d^{*P}\right]E\left[d^{2P}\right]}{4P^2\left|E\left[d^P\right]\right|^4 L}. \tag{4}$$

Eq. (4) is valid for general $2\pi/P$-rotationally symmetric constellations. On the other hand, Serpedin et al. [2] found the variance for QAM (P=4, as QAM has $2\pi/4$ symmetry), as $L \to \infty$, to be:

$$\text{var}_4(\phi - \hat{\phi}) = \frac{1}{L}\left(\frac{\mu_{Y,44} - E\left[d^8\right]}{32\left(E\left[d^4\right]\right)^2}\right), \tag{5}$$

where $\mu_{Y,44}$ can be written as $E|r|^8$, a fact which was not given in [2], but nonetheless is not only stated by us, but has also recently been independently recognized by Campisi et al. [11].

Applying the same techniques as used in [2], it can be shown that if $E\left[d^P\right]$ is a real number (as in the case of QAM, suitably defined PSK, Star 16-QAM, and others), (5) generalizes to

$$\text{var}_P(\phi - \hat{\phi}) = \frac{1}{L}\left(\frac{E|r|^{2P} - E\left[d^{2P}\right]}{2P^2\left(E\left[d^P\right]\right)^2}\right). \tag{6}$$

Note that (6) can also be derived from (4) by assuming that both $E\left[d^P\right]$ and $E\left[d^{2P}\right]$ are real numbers. Note also that requiring $E\left[d^P\right]$ to be a real number is not as restrictive as it sounds, as a simple rotation will force many constellations to obey this rule. For example, consider the general 3-PSK constellation which has points $\left\{e^{j\theta}, e^{j\left(\theta + \frac{2\pi}{3}\right)}, e^{j\left(\theta + \frac{4\pi}{3}\right)}\right\}$. Note that $E\left[d^3\right] = e^{j3\theta}$, which is a real number for $\theta = 0, \pm m\pi/3$, where m is an integer. Hence, rotating the original 3-PSK constellation by $-\theta$ can ensure the desired result.

Furthermore, it is not difficult to show that the following is valid:

$$\begin{aligned} E|r|^{2P} &= E\left[\left|d + ne^{-j\phi}\right|^{2P}\right] \\ &= E\left[\left(d + n'\right)^P\left(d^* + n'^*\right)^P\right] \\ &= E\left[\sum_{l=0}^{P}\left(\frac{P!}{l!(P-l)!}\right)^2 |d|^{2P-2l}|n'|^{2l}\right] \\ &= \sum_{l=0}^{P}\left(\frac{P!}{l!(P-l)!}\right)^2 E\left[|d|^{2P-2l}\right]E^l\left[|d|^2\right]\frac{E\left[|n'|^{2l}\right]}{E^l\left[|d|^2\right]}. \end{aligned} \tag{7}$$

Note that the expectations in (7) are with respect to the constellation points d and the modified noise $n' = ne^{-j\phi}$, which are assumed to be independent.

Recall that n' is a zero-mean complex Gaussian random variable with independent real and imaginary parts, each having variance of σ^2. Therefore, from (1.1.108) and (1.1.133) of [12], $E\left[|n'|^{2l}\right] = l!\left(2\sigma^2\right)^l$. Also, remember that $SNR = E\left(|d|^2\right)/(2\sigma^2)$. Using these results, (7) becomes

$$E|r|^{2P} = \sum_{l=0}^{P}\left(\frac{P!}{l!(P-l)!}\right)^2 l!\,E\left[|d|^{2P-2l}\right]E^l\left[|d|^2\right]\frac{1}{(SNR)^l}. \tag{8}$$

Substituting (8) into (4) gives the following expression for the variance:

$$\text{var}_P(\phi - \hat{\phi}) = \frac{1}{L}\left(k_{P0} + \sum_{l=1}^{P}\frac{k_{Pl}}{(SNR)^l}\right) \tag{9}$$

where

$$k_{P0} = \frac{2\left|E\left[d^P\right]\right|^2 E\left[|d|^{2P}\right] - E^2\left[d^P\right]E\left[d^{*2P}\right] - E^2\left[d^{*P}\right]E\left[d^{2P}\right]}{4P^2\left|E\left[d^P\right]\right|^4},$$

and

$$k_{Pl} = \frac{1}{2P^2 l!}\left(\frac{P!}{(P-l)!}\right)^2\frac{E^l\left[|d|^2\right]E\left[|d|^{2P-2l}\right]}{\left|E\left[d^P\right]\right|^2}.$$

If $E\left[d^P\right]$ is a real number, $k_{P0} = \dfrac{E|d|^{2P} - E\left[d^{2P}\right]}{2P^2\left(E\left[d^P\right]\right)^2}$.

As SNR goes to zero, the $l = P$ term in (9) dominates, k_{Pl} becomes k_{PP}, and (9) becomes

$\mathrm{var}_P\left(\phi - \hat{\phi}\right) \approx \dfrac{P!}{2LP^2} \dfrac{E^P\left[|d|^2\right]}{\left|E\left[d^P\right]\right|^2 SNR^P}$. This is the same result that is given by (9) of Steendam and Moeneclaey [9] for the low *SNR* CRB, if we remember that $E\left[|d|^2\right] = 1$ in that paper. This confirms that the P^{th} power phase estimator attains the CRB for small *SNR* because the asymptotic expression (9) reaches the CRB at sufficiently low *SNR*. Also, utilizing just the first two terms (k_{P0} and k_{P1}) in (9) produces the linear approximation of Moeneclaey and de Jonghe [1].

For *M*-PSK, $k_{P0} = 0, \dfrac{E^l\left[|d|^2\right]E\left[|d|^{2P-2l}\right]}{\left|E\left[d^P\right]\right|^2} = 1$, and *P*=*M*, as *M*-PSK has 2π/*M*-rotational symmetry. Hence, (9) becomes

$$\mathrm{var}_{M-PSK}(\phi - \hat{\phi}) = \frac{1}{2LM^2}\left(\sum_{l=1}^{M}\left(\frac{M!}{(M-l)!}\right)^2 \frac{1}{l!}\frac{1}{(SNR)^l}\right). \qquad (10)$$

It is reassuring that (10) is consistent with Wang et al. [4]. When the phase, frequency offset, and Doppler rate for *M*-PSK are simultaneously estimated, the variance of $M\hat{\phi}$ is given by Theorem 3 of [4], with $k = M$ in (22) and (23) of [4]; hence,

$$\mathrm{var}_{M-PSK}[M(\phi - \hat{\phi})] = \frac{9}{2L}\left\{E\left(|r|^{2M}\right) - 1\right\}, \qquad (11)$$

where, from (23) of [4],

$$E\left(|r|^{2M}\right) = \sum_{l=0}^{M}\left(\frac{M!}{(M-l)!}\right)^2 \frac{1}{l!}\frac{1}{(SNR)^l}. \qquad (12)$$

Substituting (12) into (11) shows that (11) is $9/M^2$ times as big as (10). The M^2 term is not in the denominator of (11) because this equation is finding the variance of $M\hat{\phi}$ and not $\hat{\phi}$, as in (10). Furthermore, the factor of nine is in (11) because phase, frequency offset, and Doppler rate are being simultaneously estimated, whereas in (10), only phase is being estimated.

IV. Monte Carlo Verification of (9)

To show experimentally that (9) is valid, Monte Carlo simulations were performed for the Star 16-QAM, V.29, QPSK, and 256-QAM constellations, assuming the true phase ϕ is zero.

A. Star16-QAM

Monte Carlo (MC) simulations were first performed using Star 16-QAM with outer to inner ring ratio RR=1.8 [6]; hence, $d \in \left\{2.172\left(\pm 1, \pm j1, \pm e^{\pm j(\pi/4)}, \pm 1.8, \pm j1.8, \pm 1.8e^{\pm j(\pi/4)}\right)\right\}$, with average symbol energy of 10. The points of this constellation lie on two concentric circles (rings), with eight equally spaced points on each circle.

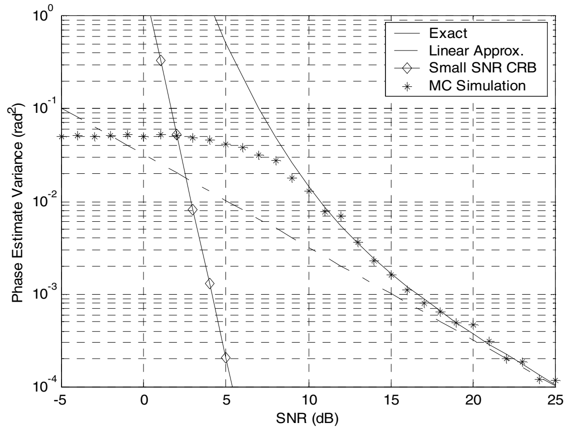

Fig. 1(a). Phase estimate variance for Star 16-QAM (RR=1.8), 500 Monte Carlo trials, and L=20.

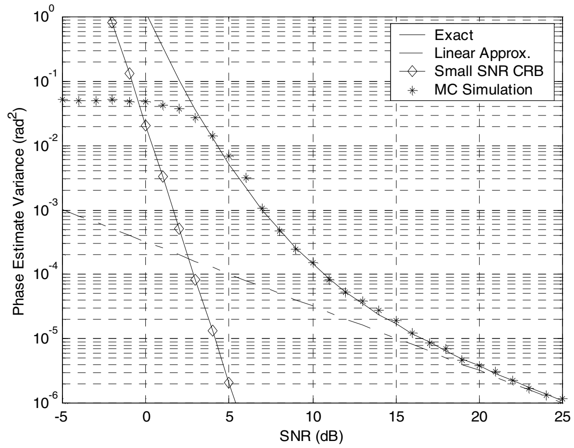

Fig. 1(b). Phase estimate variance for Star 16-QAM (RR=1.8), 500 Monte Carlo trials, and L=200.

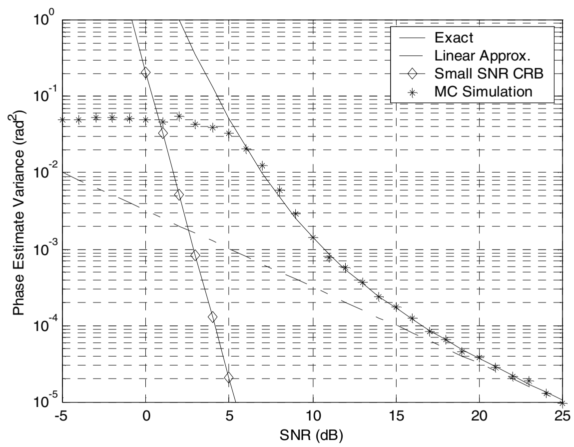

Fig. 1(c). Phase estimate variance for Star 16-QAM (RR=1.8), 500 Monte Carlo trials, and L=2000.

The results of these MC simulations using $L = 20$, 200, and 2,000 are shown in Figs. 1(a), 1(b), and 1(c), respectively, which also show the theoretical variance as obtained by (9) with $P = 8$, the linear approximation, and the small *SNR* CRB as determined by (9) with just the k_{88} term.

Note that each MC point in Fig. 1 used 500 phase estimates. Also, note that the term "Exact" in Fig.1 and all the figures below refer to the theoretical values as determined by (9).

As can be seen, there is excellent agreement between the theoretical and simulated values, for a much larger range of *SNR* values than just those values where the linear approximation applies. However, as *SNR* becomes small, the phase estimate actually becomes uniformly distributed between $-\pi / P$ and π / P, as pointed out by Tavares et al. [13]. Hence, the variance becomes $\pi^2 /(3P^2) \cong 0.05$ as seen in the MC simulations of Fig. 1. Because of this, the low *SNR* CRB is not valid if its value exceeds $\pi^2 /(3P^2)$, a fact that was not noted in [9]. Hence, as seen in Fig. 1, the *SNR* range of *practical usefulness* of that bound is very limited (approximately 0 to 2.5 dB for *L*=2000, and even smaller for smaller *L*; the linear approximation provides a better bound for $SNR \geq 2.5$ dB). A reasonable rule of thumb (at least, for the constellations simulated in this paper) appears to be: use the larger of the linear approximation or the low *SNR* bound, provided the latter does not exceed $\pi^2 /(3P^2)$, as discussed above.

Note that as *L* increases, the range of *SNR* for which (9) is valid also increases, because the lower limit of *SNR* validity, SNR_L, decreases. Indeed, as $L \to \infty$, $SNR_L \to 0$ (or $-\infty$ dB). Also, from Fig. 1, the *SNR* which causes the variance to be a half of the maximum value of $\pi^2 /(3P^2)$ seems to be a reasonable estimate of SNR_L .

Actually, practical systems need the variance to be substantially less than 0.05. For example, from Fig. 6-15 of [14], a 2.2° rms (or variance of 0.00147 rad^2) phase jitter gives a 1 dB loss in performance at a bit error rate of 10^{-6}, for 8-PSK, which is similar in performance to Star 16-QAM. Hence, *L* should be chosen to ensure that the variance of the phase error is sufficiently small, and in so doing, (9) will automatically be valid for such practical systems.

B. V. 29 Constellation

Simulations were also performed with the V.29 constellation, which is used in fax modems (see, for example, [15]). The constellation points are given by

$$d \in \left\{ \begin{matrix} \pm(1+j1), \pm(3+j3), \pm(-1+j1), \\ \pm(-3+j3), \pm3, \pm5, \pm j3, \pm j5 \end{matrix} \right\},$$ with average symbol energy of 13.5. These simulations with *L*=50, *L*=200, and *L*=2000 are shown in Figs. 2(a), 2(b), and 2(c), respectively.

For all of these, the theoretical variance is obtained by (9) with $P = 4$ (as V.29 has quadrant, i.e., $2\pi/4$, symmetry), and the small *SNR* CRB is determined by (9) with just the k_{44} term. Notice that the small *SNR* performance converges to $\pi^2 / 48 \approx 0.2$, as it should.

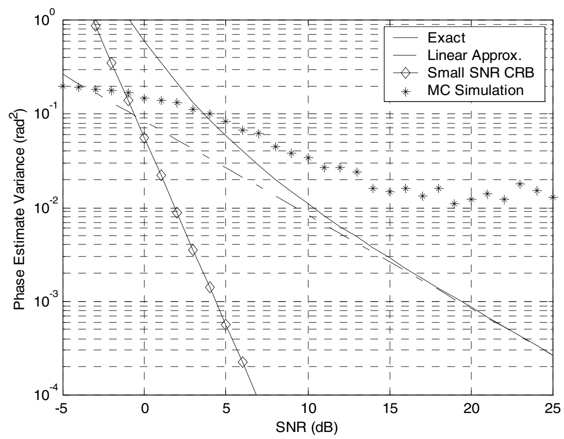

Fig. 2(a). Phase estimate variance for the V.29 constellation, 1000 Monte Carlo trials, and L=50.

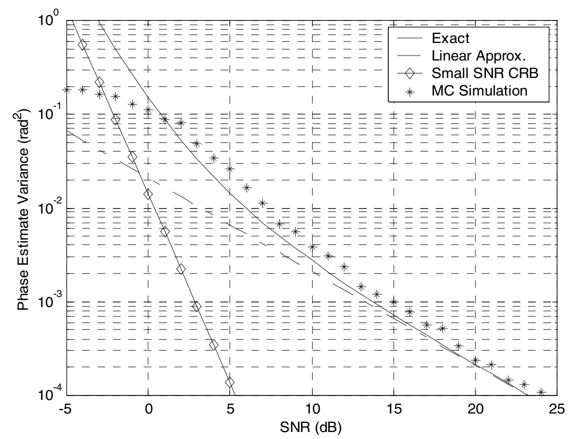

Fig. 2(b). Phase estimate variance for the V.29 constellation, 1000 Monte Carlo trials, and L=200.

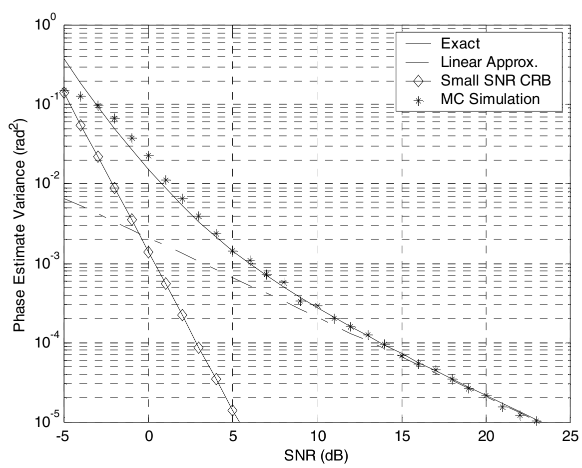

Fig. 2(c). Phase estimate variance for the V.29 constellation, 500 Monte Carlo trials, and L=2000.

Notice also that the predicted performance for L=50 is disappointing, behaving more like a lower bound. However, one must keep in mind that (9) was derived from (4), which assumed $L \to \infty$. Therefore, each constellation will require a minimum number of samples before (9) gives predicted values that are sufficiently accurate. Indeed, using L=2000 for V.29 gives excellent accuracy, as shown in Fig. 2(c).

C. QPSK Constellation

In Fig. 3, we present simulation results for QPSK: these show that the linear approximation is inadequate, especially if coding is employed or the system is operating at relatively high probability of error. For example, the linear approximation appears to have reasonable accuracy for $SNR \geq\approx 15$ dB. However, one should keep in mind that for QPSK, $SNR = 8.22$ dB produces a probability of symbol error of 10^{-2}. Hence, (9) clearly has better accuracy when *SNR* is not large. Note that, as above, the theoretical variance is obtained by (9) with $P = 4$, and the small *SNR* CRB is determined by (9) with just the k_{44} term.

D. 256-QAM Constellation

Simulations with 256-QAM were also performed. These are shown in Fig. 4. We wish to compare our results with Fig. 8 of Serpedin et al. [2]. Hence, we used the same number of samples in each trial, and the same number of trials, as they did.

As can be seen, the linear approximation is quite accurate for $SNR \geq\approx 10$ dB. However, Serpedin et al. [2] claimed that $SNR \geq\approx 20$ dB is needed. Unfortunately, the linear approximation in Fig. 8 of [2] is plotted incorrectly, which gives rise to this erroneous statement. Indeed, Fig. 4 is also consistent with Fig. 1 of [1].

Notice that the variance does not decrease with increasing *SNR* for high *SNR*. This is due to k_{P0} being non-zero for this constellation, as noted in [1].

V. A CLARIFICATION OF THE MC SIMULATIONS OF (9)

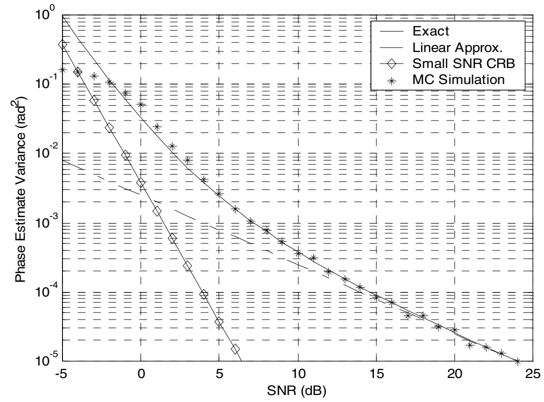

Fig. 3. Phase estimate variance for the QPSK constellation, 500 Monte Carlo trials, and L=200.

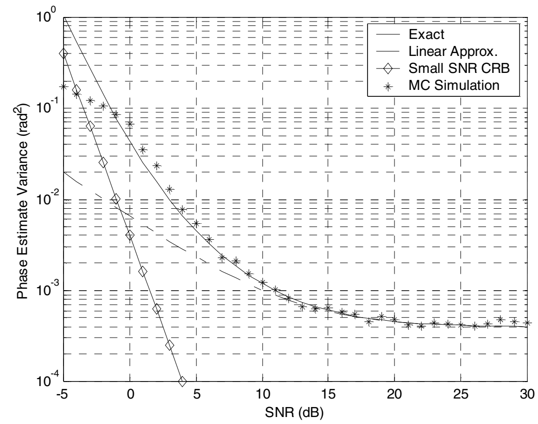

Fig. 4. Phase estimate variance for the 256-QAM constellation, 300 Monte Carlo trials, and L=512.

The simulations in the above section were done with the assumption that the true phase ϕ is zero. This was done to avoid equivocation complications [16]-[17]. Indeed, if the true phase is *near* an odd multiple of $\pm\pi / P$, then the estimated phase jumps between values near π / P and its negative, i.e., it equivocates, from one trial to the other. For example, if the true phase for QPSK is 44°, the estimated phase might be 44.5° on one trial and then -44° on the other. (We use degrees solely for illustration purposes). Hence, the apparent phase variance will be huge. However, the phase estimate of -44° is equivalent to 46°. Hence, using the 44.5° and 46° estimates would produce the smaller variance, which might be a fairer assessment of the performance of the system, as in TDMA systems for which the phase estimate of one block is not needed for another block. Note that for systems where equivocation is a problem, phase unwrapping can be used to eliminate the problem, as discussed in [17].

Rather than assuming $\phi = 0$, $-\pi / P < \phi \leq \pi / P$ can be used, provided the experimental variance, var_e, is estimated fairly.

This can be done by computing $\mathrm{var}_e = \frac{1}{MC}\sum_{i=1}^{MC}\varepsilon_i^2$, where ε_i^2 is the minimum of $(\phi-\hat{\phi}_i)^2$, $(\phi-\hat{\phi}_i-2\pi/P)^2$, or $(\phi-\hat{\phi}_i+2\pi/P)^2$, and MC is the number of Monte Carlo trials. If the variance is calculated this way, simulations verify (9) is indeed valid for $-\pi/P<\phi\le\pi/P$.

VI. Conclusions and Further Work

An expression for the variance of the P^{th} power-law phase estimator has been given which is valid for a wide range of *SNR*. This expression is an extension to the linear approximation of Moeneclaey and de Jonghe [1] and is valid for general $2\pi/P$-rotationally symmetric constellations. This expression also generalizes the known expressions for QAM and PSK. Additionally, we showed that the expression reduces to the CRB given by Steendam and Moeneclaey [9], as *SNR* goes to zero, and we also pointed out a limitation to this bound, not stated in [9]. Also, using (9), an error in [2] was corrected. Finally, Monte Carlo simulations provided experimental verification of the theoretical expression for the Star 16-QAM, V.29, QPSK, and 256-QAM constellations.

The work presented here may be extended to derive the Barankin bound which provides a better bound than the CRB for low *SNR* and limited number of samples.

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