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***Abstract*—It is well known that the Viterbi and Viterbi Monomial-Based Phase Estimator, which includes the M^{th} Power Estimator, performs poorly for cross QAM signals. However, it is shown here that by allowing the power of the monomial to be negative, much improved performance can be realized at medium to high signal-to-noise ratios (*SNR*). Monte Carlo simulations are used to demonstrate the efficacy of this novel simple extension, for 32- and 128-QAM systems. In principle, this extension can also be applied to other constellations, e.g., (4,12)-PSK.**

***Keywords*—Synchronization, blind phase estimation, quadrature amplitude modulation, blind carrier phase recovery.**

I. Introduction

THE need for blind phase recovery in quadrature amplitude modulation (QAM) systems is well established. In order to satisfy this need, many systems have been invented. These systems can be grouped into two areas – those that require established gain control and those that do not. The Fourth Power Phase Estimator [1]-[3], which is a special case of the Viterbi and Viterbi (V&V) monomial-based estimators [4], the Eighth-Order Estimator (EOE) [5], the Concentration Ellipse Orientation (CEO) [6], and more recently the iterative methods (DCA-a and DCA-b) of Alvarez-Diaz and Lopez-Valcarce [7] are systems in the latter category. Among the former category are the Reduced-Constellation Fourth Power Estimator [1], the two methods of Georghiades [1] which require finding the mode of the probability density of the phase, the rather complex Minimum Distance Estimator (MDE) [8], the Two-Stage Conjugate (2SC) algorithm which according to Rice et al. [8] is similar to the Two-Pass algorithm of [9, pg. 33], the optimal method, proposed by Wang and Serpedin [10], who along with Ciblat [11] have also introduced the APP Estimator, which approximately implements the optimal estimator, and more recently the Reduced Constellation Eighth-Order Estimator (RCEOE) for cross QAM signals [12].

The purpose of this paper is to propose another phase estimator that does not require established gain control and is no more complicated than the V&V monomial-based estimator, although with slightly increased computational expense due to the required reciprocal operation. In fact, it is identical to this estimator, except negative powers of the monomial are now allowed. Although this seems to be a trivial idea, it is not at all self-evident. Indeed, the authors could find no hint in the literature that negative powers would be of any interest. In fact, all previous authors have assumed non-negative powers.

This simple but effective extension is demonstrated for 32-QAM and 128-QAM. We show that for these systems operating at medium to high signal-to-noise ratios (*SNR*), the negative power monomial-based estimator can provide much improved performance over the conventional V&V monomial-based estimator. However, in principle, this new extension can also be used for other constellations, e.g. (4,12)-PSK which is useful in the non-linear satellite channel [13].

The organization of this paper is as follows: in Section II, a statement of the problem we are trying to solve is presented, followed in Section III by a review of the V&V monomial-based estimators that are known to solve this problem. In Section IV, we present our method that improves the performance of these estimators, and we demonstrate the effectiveness of our improvement in Section V. In Section VI, we consider some implementation issues for these new estimators. Finally, in Section VII, we provide some concluding remarks.

II. Statement of the Problem

The received signal is given by

$$Y(n) = e^{j\theta} X(n) + V(n), \qquad n = 0,\cdots,N-1 \qquad (1)$$

where $\{X(n) = X_r(n) + jX_i(n)\}$ is the sequence of zero-mean unit variance, i.e., $E\left(|X(n)|^2\right) = 1$, independently and identically distributed (i.i.d.) QAM complex transmitted symbols, $\{V(n) = V_r(n) + jV_i(n)\}$ is a zero-mean circular white Gaussian noise process, independent of $X(n)$ and with variance σ^2 in each component, and θ is the phase angle to be determined by observing the received signal $Y(n)$. Furthermore, the received signal-to-noise ratio is $SNR = 1/(2\sigma^2)$.

It will also be convenient to rewrite (1) in polar form to get

$$Y(n) = \rho(n) e^{j\phi(n)}, \qquad n = 0,\cdots,N-1. \qquad (2)$$

The blind estimation problem is to find an estimate for θ, denoted $\hat{\theta}$, without actually detecting the data X. Note that because X has quadrant symmetry, it is only possible to recover

θ within $\pi/2$ rad. Without loss of generality, we assume $-\pi/4 < \theta < \pi/4$ rad.

III. Review of the V&V Phase Estimator

In [4], Viterbi and Viterbi introduced phase estimators suitable for M-ary Phase-Shift Keying (M-PSK). Specifically, the V&V phase estimate is given by

$$\hat{\theta} = \frac{1}{M}\,\text{angle}\left[\sum_{n=0}^{N-1} F(\rho(n))e^{jM\phi(n)}\right], \qquad (3)$$

where $F(\cdot)$ is a real-valued arbitrary nonlinear function.

The monomial V&V estimators result from the special case of $F(\rho(n)) = \rho^k(n),\ k = 0,1,2,\cdots,M$. Note that if $k = M$, (3) reduces to the M^{th} Power Estimator [1], whose phase estimate is usually stated as

$$\hat{\theta} = \frac{1}{M}\,\text{angle}\left[E\left[X^{*M}\right]\sum_{n=0}^{N-1} Y^M(n)\right]. \qquad (4)$$

For QAM signals —all of which have quadrant symmetry— $M = 4$ and $E\left[X^{*M}\right]$ is negative. Hence, Wang and Serpedin [10] and Wang et al. [11] investigated estimators for QAM that were given specifically by

$$\hat{\theta} = \frac{1}{4}\,\text{angle}\left[-\sum_{n=0}^{N-1} F(\rho(n))e^{j4\phi(n)}\right]. \qquad (5)$$

Wang and Serpedin [10] were able to find the optimum non-linear function $F(\cdot)$ that minimizes the variance of the estimator (5). (Much earlier, Paden [14] had done the same for QPSK). However, this function is a complicated function of the constellation and *SNR*. (Please see [10] and [11] for details). Therefore, these authors also considered the monomial estimators, i.e.,

$$\hat{\theta} = \frac{1}{4}\,\text{angle}\left[-\sum_{n=0}^{N-1} \rho^k(n)e^{j4\phi(n)}\right], \qquad k = 0,\cdots,4. \qquad (6)$$

However, as shown in Fig. 3 and Fig. 11 of [11], these monomial-based estimators perform poorly for cross QAM constellations, a fact that is also well-known for the Fourth Power Estimator [1]-[3].

IV. Suggested Improvement to the V&V Monomial-Based Estimator

The reason for the poor performance of the V&V monomial-based estimator is that the symbols of the constellation with the highest energy are not on the diagonal lines $X_r = X_i$ or $X_r = -X_i$. Indeed, elimination of all the received points that are not on these lines is the basis for the APP Estimator [11], (which approximates the optimal estimator, i.e., (5) with the optimum nonlinearity) and the separation of 16-QAM into two classes for the 2SC estimator [8]. However, in order to do this, established gain control is necessary.

This gain control requirement can be removed by taking the reciprocal of the received symbol. In the absence of noise, this action would produce a received constellation whose symbols of highest energy will lie on the required lines, as demonstrated in Figs. 1(a) and 1(b) —note the four symbols of the inner square of the 32-QAM constellation of Fig. 1(a), shown emphasized, have been mapped to the points of the outer square of Fig. 1(b). Hence, improved performance can be expected at medium to large *SNR* when this transformed constellation is applied to the V&V monomial-based estimator of (6). However, the same effect can be achieved with the original received constellation in (6) but now allowing k to be negative. As simple as this sounds, no one has apparently suggested this before. The next section presents simulations that demonstrate the efficacy of such an approach. Note that for $k = -4$, (4) can be used with $M = -4$ instead of (6). Indeed, more generally, for $k = -M$, (4) can be used in place of (3).

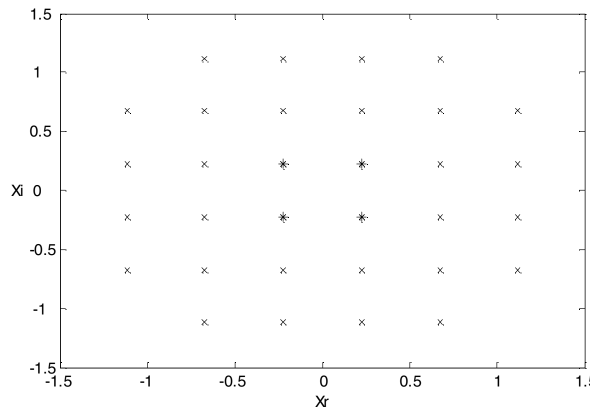

Fig. 1(a). Constellation of 32-QAM.

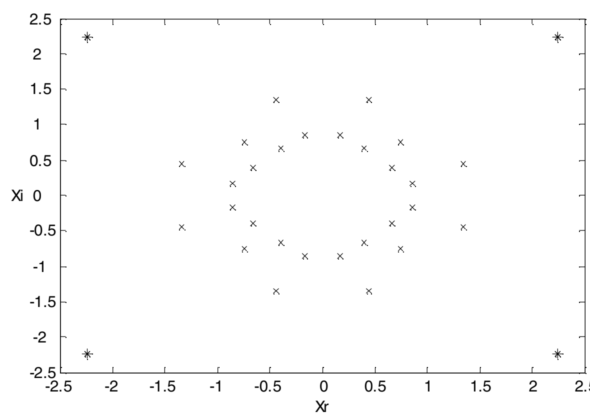

Fig. 1(b). Constellation of the transformed received 32-QAM, in the absence of noise. Each point is the reciprocal of the corresponding point in the original constellation, above.

V. Performance Verification

In order to determine the mean-square-error (MSE) of the phase estimate of the V&V monomial estimator with negative k values, Monte Carlo (MC) simulations were performed for 32-QAM and 128-QAM. In each case, unless otherwise noted, 1,000 MC trials were utilized, and θ was assumed to be 0.2 radians, although simulations show the performance does not depend upon this value.

The results of these simulations with 1,000 MC trials for 32-QAM with $N = 500$ are shown in Fig. 2, and for 128-QAM with $N = 7500$ are shown in Fig. 3. From these figures, it is clear that it is possible to optimize the performance over a given SNR interval by the appropriate choice of k. For example, for 32-QAM and $20 \le SNR \le 25$ dB, $k = -2$ provides the best performance. This is also the case for 128-QAM and $27 \le SNR \le 31$ dB. As will be shown later in this section, these SNR ranges correspond to the operating ranges for probability of symbol error for practical systems.

It is also interesting to note that the performance with $k = 0$ is better than the performance with $k = 4$ for 32-QAM for SNR above about 20 dB. This is opposite to the case for 128-QAM where the performance for $k = 4$ is always better.

Also, by observation of Fig. 2 and Fig. 3, it is clear that the maximum mean-square error is 0.2 rad^2. This is because the phase error becomes uniformly distributed between $-\pi/4$ and $\pi/4$, as explained by Tavares et al. [15]. Hence, the variance is $\pi^2/48 = 0.2$ rad^2.

Note that in all the figures above, experimental results are given for the Fourth Power Estimator ($k = 4$); however, theoretical results could have been found with (13) of [2], or more explicitly, with (8) of [16].

Additionally, in Fig. 2 and Fig. 3, simulation results are given for the EOE. This estimator's performance was chosen to represent the performance of the other estimators which do not require established gain control, as [6] and [7] show that the performance of EOE, CEO, DCA-a and DCA-b are similar for 32-QAM with N=500 and 128-QAM with N=7500. From Fig. 2, it is clear that our new extension provides improved performance over EOE for 32-QAM with $k = -2, -3$ or -4 and $SNR \ge\approx 23$ dB. Similarly, Fig. 3 shows that for 128-QAM, our method provides improved performance over EOE with $k = -3$ or $k = -4$ and $SNR \ge\approx 32$ dB. For $k = -2$, however, SNR must be greater than 33 dB.

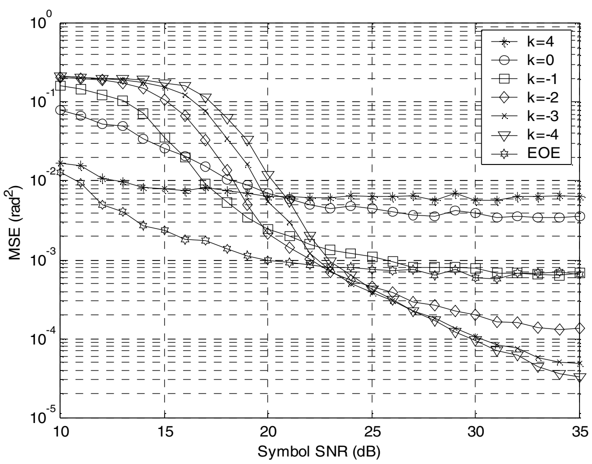

Fig. 2. Mean Square Error of Phase Estimates for 32-QAM. N = 500.

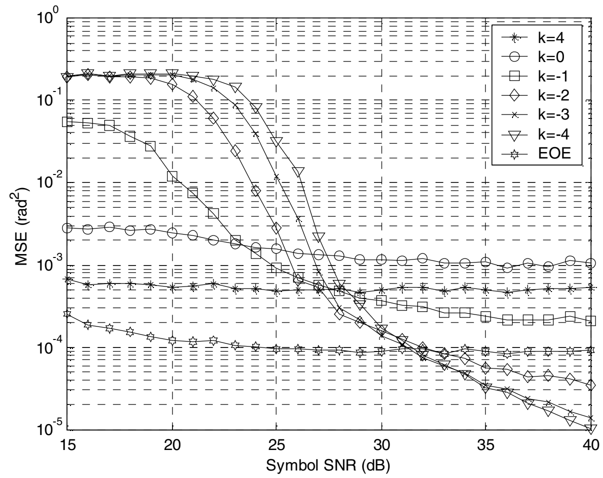

Fig. 3. Mean Square Error of Phase Estimates for 128-QAM. N = 7500.

It is also of particular interest to find the effects of the phase estimate on the probability of symbol error, P_{es}. This was done and the results for 32-QAM are shown in Fig. 4, where it is clearly seen that $k = -2$ provides the best performance for the SNR values normally of interest, thereby confirming the results in Fig. 2. However, for $SNR \ge 25$ dB, $k = -4$ provides the best results. In addition, our new extension gives improved performance over EOE for $SNR \ge 22$ dB, again confirming the results in Fig. 2.

Note that the probability of symbol error was computed using MC simulations to find the expected value of the following:

$$P_{es} \approx 0.5E\Big[\mathrm{erfc}\Big(\Big(X_r - \hat{X}_r' + 1/\sqrt{20}\Big) * \sqrt{SNR_a}\Big) + \mathrm{erfc}\Big(\Big(-X_r + \hat{X}_r' + 1/\sqrt{20}\Big) * \sqrt{SNR_a}\Big) + \mathrm{erfc}\Big(\Big(X_i - \hat{X}_i' + 1/\sqrt{20}\Big) * \sqrt{SNR_a}\Big) + \mathrm{erfc}\Big(\Big(-X_i + \hat{X}_i' + 1/\sqrt{20}\Big) * \sqrt{SNR_a}\Big)\Big] \quad (7)$$

where

$$\hat{X}_r' = X_r \cos(\theta - \hat{\theta}) - X_i \sin(\theta - \hat{\theta})$$

$$\hat{X}_i' = X_r \sin(\theta - \hat{\theta}) + X_i \cos(\theta - \hat{\theta}),$$

and SNR_a is the SNR in absolute units, i.e. not in dB.

(This method of simulating P_{es} is called quasi-analytical estimation [17]. Strictly speaking, (7) is a very tight upper bound).

Note also that 10,000 phase estimates were used at each SNR to generate the curves in Fig. 4. As each phase estimate requires $N = 500$ symbols, this means that 5×10^6 symbols were utilized to estimate P_{es} at each SNR. Furthermore, the Gaussian noise only curve in Fig. 4 was generated using $P_{es} \approx 2\mathrm{erfc}\left(\sqrt{SNR/20}\right)$.

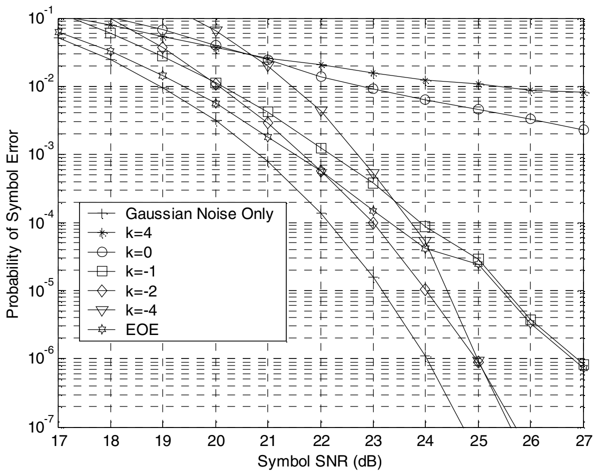

Fig. 4. Probability of Symbol Error for 32-QAM. N=500.

For 128-QAM and $N = 7500$, P_{es} is given in Fig. 5, which again shows that for the monomial V&V estimators, $k = -2$ provides the best performance for the *SNR* values normally of interest. Clearly, however, for $SNR \geq 31$ dB, $k = -4$ provides the best results, even providing improved performance over EOE for $SNR \geq 32$ dB, Nevertheless, the performance for $k = -2$ is close to optimum for this range, as well (at least for $SNR \leq 33$ dB). Thankfully, these results are consistent with those in Fig. 3.

Note that (7) was again used; however, with the 20 replaced by 82, and with 1,000 phase estimates for each *SNR* to generate the curves in Fig. 5. As each phase estimate requires $N = 7500$ symbols, this means that 7.5×10^6 symbols were utilized to estimate P_{es} at each *SNR*. Additionally, the curve labeled 'Gaussian Noise Only' in Fig. 5 was generated using $P_{es} \approx 2\mathrm{erfc}\left(\sqrt{SNR/82}\right)$.

By inspection of Fig. 4 and Fig. 5, it is clear that substantially more samples were used for 128-QAM than for

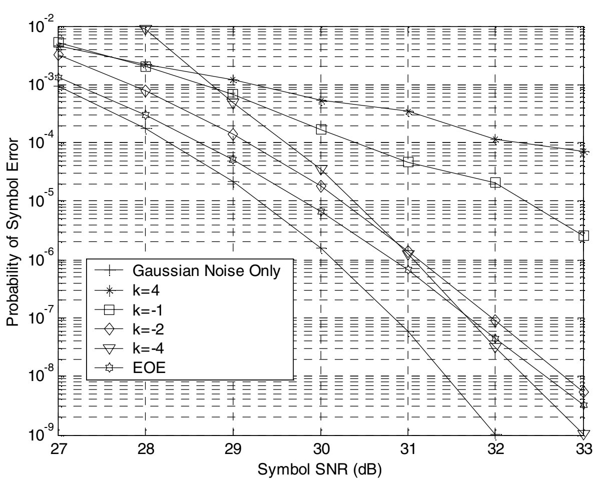

Fig. 5. Probability of Symbol Error for 128-QAM, N=7500.

32-QAM; this is required in order to obtain similar probability of symbol error. This is also the case for more complex estimators which are gain independent, such as EOE.

It is also of interest to determine how these estimators behave with respect to the number of samples, N. Simulations were conducted for 32-QAM for low *SNR* and high *SNR* and are shown in Fig. 6(a) and Fig. 6(b), respectively.

As can be seen, in general, the performance is inversely proportional to the number of samples. This, however, is not the case for $k = -3$ and $k = -4$: for these, the performance is independent of the number of samples for low *SNR*. This effect is also seen for 128-QAM in Figs. 7(a) and 7(b). Note that 10,000 Monte Carlo trials were used for each point in Fig. 7(a).

The question —important if coding is used— then presents itself: for what range of SNR, is the performance independent

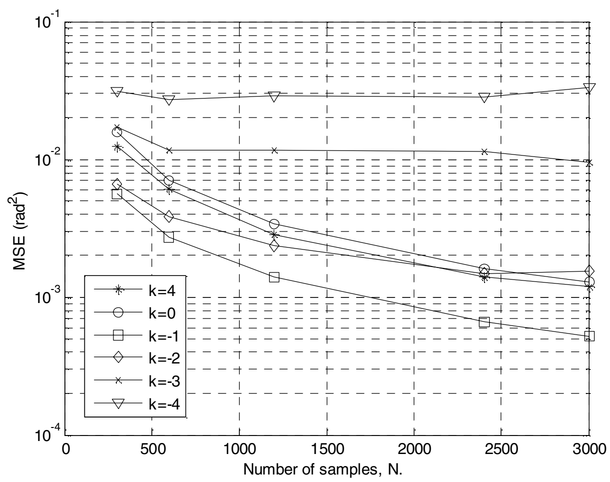

Fig. 6(a). Mean Suare Error of Phase Estimates as a function of N for 32-QAM . *SNR*=19 dB.

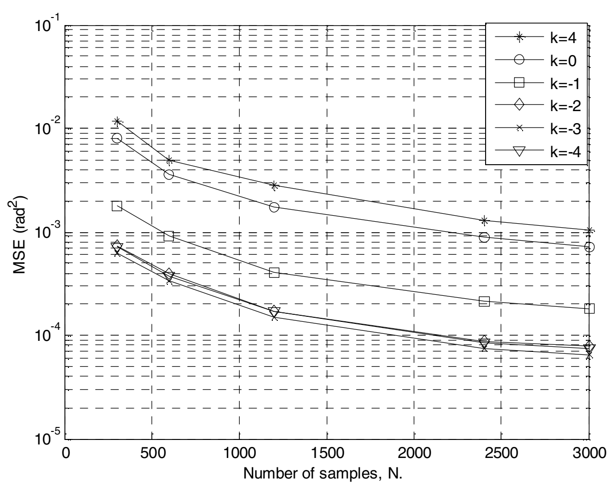

Fig. 6(b). Mean Square Error of Phase Estimates as a function of N for 32-QAM. *SNR*=25 dB.

of the number of samples? Fig. 8 shows that for 32-QAM and $k = -4$, increasing the number of samples does not improve the performance if $SNR \leq 20$ dB. Similar curves can be determined for other powers.

II. Some Implementation Considerations

Having established that the monomial-based V&V estimators with negative powers are of benefit, especially for 32-QAM or 128-QAM at high *SNR*, the question now becomes how complex is it to implement them. The answer to this question depends upon what assumptions are made concerning the available data. For example, it is straightforward to create $e^{j\phi(n)}$ in hardware, simply by using a bandpass limiter on the modulated signal and then translating to baseband. Indeed, by adding a bandpass multiplier (as used in FM transmitters) before translating to baseband, it is also easy to create $e^{j4\phi(n)}$. Furthermore, by amplitude demodulation of the modulated signal, $\rho(n)$ can be made readily available. If we were to assume these latter two signals are available, then each phase estimate would require about 3N real multiplications and N real reciprocal operations for $k = -2$. (We are ignoring the final angle determination, as this occurs once every N samples).

However, we will assume a worst case scenario, i.e., only the real part and the imaginary part of (1) are available, as is typical in QAM systems.

The $k = -2$ estimator can be implemented in two ways as $\hat{\theta} = \frac{1}{4}\text{angle}\left[-\sum_{n=0}^{N-1} Z_i(n)\right], i = 1,2,$ where $Z_1(n) = \frac{Y^4(n)}{\rho^6(n)}$ or $Z_2(n) = \left(\frac{Y(n)}{\rho^2(n)}\right)^4 \rho^2(n)$. The advantage to the latter is that the order of each calculated term is smaller than in the former. This may be important to reduce overflow problems in fixed-point implementations.

For the $k = -4$ estimator, we can use

$$\hat{\theta} = \frac{1}{4}\text{angle}\left[-\sum_{n=0}^{N-1}\left(\frac{Y(n)}{\rho^2(n)}\right)^4\right].$$

The number of calculations needed for each of these is shown in Table I, where the computational burden of other gain-independent estimators as determined by [7] is also shown. Note that DCA-a and DCA-b are iterative estimators which also require some method of phase initialization which further adds to their computational burden. Please see [7] for details.

As can be seen from Table I, in order for the negative power monomial-based V&V estimator to remain competitive in terms of computational burden, there has to be a fast method to accomplish reciprocation. Fortunately, such methods exist: see, for example, [18]-[19].

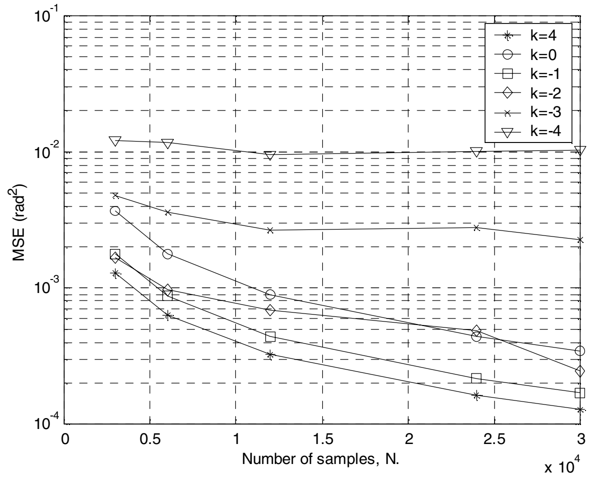

Fig. 7(a). Mean Square Error of Phase Estimates as a function of N for 128-QAM. *SNR*=26 dB.

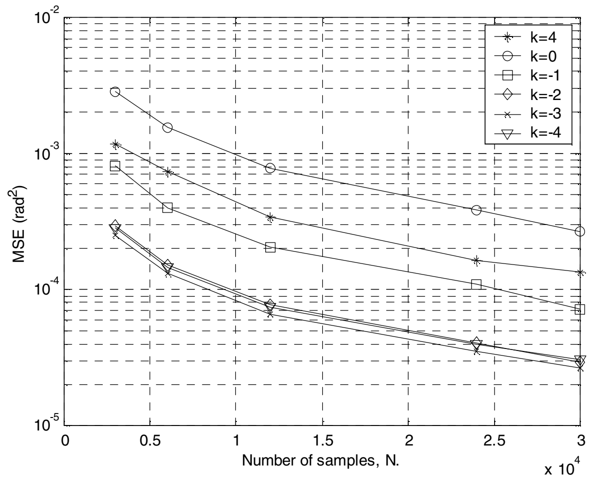

Fig. 7(b). Mean Square Error of Phase Estimates as a function of N for 128-QAM. *SNR*=31 dB.

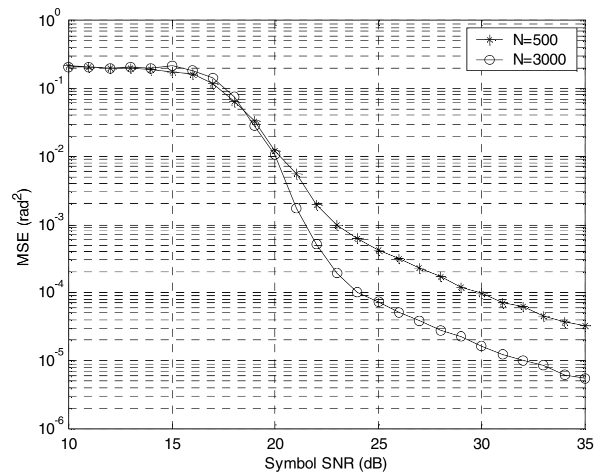

Fig. 8. Mean Square Error of Phase Estimates for 32-QAM and k = -4.

TABLE I. COMPUTATIONAL BURDEN OF THE VARIOUS GAIN-INDEPENDENT PHASE ESTIMATORS

Estimator	No. of Real Multiplications	No. of Real Reciprocations	No. of Real Additions
EOE	11N	0	8N
DCA-a	2N/iteration	0	3N/iteration
DCA-b	4N/iteration	0	4N/iteration
Fourth Power $(k=4)$	5N	0	4N
$k=-2, Z_1$	11N	N	5N
$k=-2, Z_2$	12N	N	6N
$k=-4$	9N	N	6N

III. CONCLUSION

It has been demonstrated using Monte Carlo simulations that blind recovery of the phase for cross QAM signals can be greatly improved at medium and high *SNR* by allowing negative powers in the V&V monomial phase estimator. Even though this is a simple idea, it appears to be novel, as previous authors have assumed non-negative powers, up until now. It has further been established that $k=-2$ provides the best (or close to) performance for 32-QAM and 128-QAM for the *SNR* values normally of interest.

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