5-22-2006

Essays on Fine Structure of Asset Returns, Jumps, and Stochastic Volatility

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ESSAYS ON FINE STRUCTURE OF ASSET RETURNS, JUMPS, AND STOCHASTIC VOLATILITY

A Dissertation

Submitted to the Graduate Faculty of the
University of New Orleans
in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy
in
Financial Economics

by

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May 2006

Dedication

I dedicate this dissertation to my loving family, especially my parents, parents-in-law, and wife, Sungyeon Kim. They all have been always supportive and considerate during my excellent and rigorous graduate studies in the U.S. to become a financial economist.

Acknowledgements

First of all, the greatest of thanks goes to my advisor, Elton Daal, for numerous discussions, strong support, insightful advice, and endless encouragements. I am also deeply grateful to my dissertation committee members, Atsuyuki Naka, Oscar Varela, Gerald A. Whitney, and Michael Morris at the University of New Orleans. I also thank M. Kabir Hassan for teaching me how to present and publish my research efficiently and for showing me how to be a productive financial researcher. They all motivated me throughout my doctoral studies and their constant commitment enabled me to complete this dissertation on time.

Tim Bollerslev, George Tauchen, and Bjorn Eraker at Duke University inspired me to take the first steps toward an exciting career in financial econometrics and empirical finance. Finally, I am also indebted to Yongjin Kim, Jaeho Jo, ChonPyo Lee, WunChan Jeong at Seoul National University. They all stimulated my intellectual curiosity to keep studying financial economics and I could learn the importance of hard work and the rewards of achieving excellence from so many mentors.

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Abstract

There has been an on-going debate about choices of the most suitable model amongst a variety of model specifications and parameterizations. The first dissertation essay investigates whether asymmetric leptokurtic return distributions such as Hansen's (1994) skewed *t*-distribution combined with GARCH specifications can outperform mixed GARCH-jump models such as Maheu and McCurdy's (2004) GARJI model incorporating the autoregressive conditional jump intensity parameterization in the discrete-time framework. I find that the more parsimonious GJR-HT model is superior to mixed GARCH-jump models. Likelihood-ratio (LR) tests, information criteria such as AIC, SC, and HQ and Value-at-Risk (VaR) analysis confirm that GJR-HT is one of the most suitable model specifications which gives us both better fit to the data and parsimony of parameterization. The benefits of estimating GARCH models using asymmetric leptokurtic distributions are more substantial for highly volatile series such as emerging stock markets, which have a higher degree of non-normality. Furthermore, Hansen's skewed *t*-distribution also provides us with an excellent risk management tool evidenced by VaR analysis.

The second dissertation essay provides a variety of empirical evidences to support redundancy of stochastic volatility for SP500 index returns when stochastic volatility is taken into account with infinite activity pure Lévy jumps models and the importance of stochastic volatility to reduce pricing errors for SP500 index options without regard to jumps specifications. This finding is important because recent studies have shown that stochastic volatility in a continuous-time framework provides an excellent fit for financial asset returns when combined with finite-activity Merton's type compound Poisson jump-diffusion models. The second essay

also shows that stochastic volatility with jumps (SVJ) and extended variance-gamma with stochastic volatility (EVGSV) models perform almost equally well for option pricing, which strongly imply that the type of Lévy jumps specifications is not important factors to enhance model performances once stochastic volatility is incorporated. In the second essay, I compute option prices via improved Fast Fourier Transform (FFT) algorithm using characteristic functions to match arbitrary log-strike grids with equal intervals with each moneyness and maturity of actual market option prices.

Introduction

The various models analyzed in this dissertation have been built upon pioneering work of Nobel Laureates Robert F. Engle (2003) and Robert C. Merton (1997) for methods of analyzing economic time series with time-varying volatility (ARCH) and a new method to determine the value of derivatives, respectively. This dissertation fills the gaps which another Nobel Laureate Harry M. Markowitz's (1990) mean-variance analysis fails to capture. Especially, this dissertation investigates dynamic processes of asset returns, volatility, and jumps which are time-varying and stochastic in discrete- and continuous-time settings. I demonstrate that these additional computational and modeling efforts provide us with significant benefits to better capture actual financial time-series data and to reduce option pricing errors.

If we only consider mean and variance (1^{st} and 2^{nd} moments) as in Markowitz, most likely we may not fully appreciate recent advances in risk managements, investments, and derivatives pricing since many researchers recognize the importance of economic and statistical roles of skewness and kurtosis (3^{rd} and 4^{th} moments). To better explain well-known skewness and excess kurtosis of financial time-series returns, I employ asymmetric fat-tailed distributions such as Hansen's skewed *t*-distribution in the first dissertation essay and Lévy jump models in the second dissertation essay.

I have been motivated by Torben G. Andersen, Tim Bollerslev, Francis X. Diebold (2002)'s paper titled “Parametric and Nonparametric Volatility Measurement” - Handbook of Financial Econometrics, Amsterdam : North Holland, Yacine Aït-Sahalia and Lars Peter Hansen (eds.) to investigate the financial modeling approaches analyzed in this dissertation.

"Since Engle's (1982) seminal paper on ARCH models, the econometrics literature has focused considerable attention on time-varying volatility and the development of new tools for volatility measurement, modeling and forecasting. These advances have in large part been motivated by the empirical observation that financial asset return volatility is time-varying in a persistent fashion, across assets, asset classes, time periods, and countries. Asset return volatility, moreover, is central to finance, whether in asset pricing, portfolio allocation, or risk management, and standard financial econometric methods and models take on a very different, conditional, flavor when volatility is properly recognized to be time-varying. The combination of powerful methodological advances and tremendous relevance in empirical finance produced explosive growth in the financial econometrics of volatility dynamics, with the econometrics and finance literatures cross-fertilizing each other furiously."

Chapter 1

A Comparison of mixed GARCH-Jump Models with Asymmetric Fat-tailed Asset Return Distributions

1. Introduction

A vast numbers of research endeavors have been made to explain well-documented stylized facts and empirical properties of asset returns and volatility – absence of autocorrelations, leptokurtosis (high peaks and heavy tails of the distribution), long memory and volatility persistence, mean-reverted volatility, aggregated Gaussianity, volatility clustering, slow decay of autocorrelation in absolute returns, leverage effect, and volume/volatility correlation.[1] However, most currently existing models including GARCH-families with normal error distributions and constant jump-diffusion models following Merton (1976), fail to reproduce all these statistical features simultaneously.

In order to improve overall goodness-of-fits for the models and especially to better describe unique features of conditional higher moments (extreme observations, outliers and non-zero skewness in returns) prevalent in most financial assets, financial researchers have proposed two distinct classes of econometric models. One emphasizes the economic and statistical roles of jumps and tries to capture infrequent/abnormal extreme events through Poisson-distributed jumps. The examples of this class are voluminous (Jorion (1988), Lin and Yeh (1999), Scott (1997), Carr et al. (2002), Wu (2003), Ait-Sahalia (2004), Eraker et al. (2003, 2004) and Johannes (2004)). Very recently, more sophisticated jump-diffusion models such as the mixed

[1] For an excellent summary of empirical properties and statistical issues of asset returns and volatility, see Ghysels et al. (1995), Cont (2001) and Engle and Patton (2001).

GARCH-Jump models incorporating time-varying autoregressive jump intensity have been developed by Chan and Maheu (2002) and extended by Maheu and McCurdy (2004).

The other continues to find promising theoretical probability distribution models that fit the empirical distributions for conditional returns better than traditional normal distributions. In this group, models of asset returns can be divided in two categories based on whether the parameters of return-generating process are time-dependent or not (Gillemot et al., 2000). For example, time-independent models of asset returns include student *t*-distribution, Levy-type distribution, general stable distribution, and mixture of normal distributions. Similarly, time-dependent models include GARCH-type models, stochastic volatility (SV) models and models based on chaos theory leading to complex dynamics. In any situation, to be considered a candidate for an empirical return distribution, a probability density function (PDF) with three essential parameters (location, scale, and shape) must be estimable and have sufficiently flexible shape parameters so as to explain the skewness and kurtosis that may be encountered in finance (Verhoeven and McAleer, 2003).

As such, more recently several capable alternative distributions that provide an effective means of modeling asymmetry and fat-tailedness of financial data have been applied to financial research in the past decade (e.g., Skewed *t*-distribution (Hansen, 1994), Non-central *t*-distribution (Harvey and Siddique, 1999), and Log generalized gamma distribution (Brannas and Nordman, 2003)). Most important distinctions of these new approaches are to incorporate asymmetry into the conditional density function explicitly since most attention has focused on the use of symmetric conditional density functions especially with GARCH-type models. Recently, Bond (2001) compares a review of some of asymmetric conditional density functions in GARCH-type models and put emphasis on the role of alternative return distributions to capture skewness in financial returns. This can be a particularly relevant issue in emerging stock markets as well as in matured stock markets such as G7 countries.

Therefore, there has been an on-going debate about choices of the most suitable model amongst a variety of model specifications and parameterizations. So far, numerous research has been performed to compare and evaluate model performances of different GARCH, SV, and/or jump-diffusion specifications to capture volatility clustering and leptokurtosis of asset returns. In the similar vein, there already exist several published papers that investigate a variety of models to seek desirable alternative return distributions to better explain conditional higher moments of asset returns.

However, very few papers examine superiority between mixed GARCH-jump models and alternative models assuming more generalized distributions compared to normal distribution. Furthermore, to the best of my knowledge, it is hard to find the papers which have compared and evaluated Hansen's (1994) skewed *t*-distribution and the GARJI model (a mixed GARCH-jump model incorporating the autoregressive conditional jump intensity parameterization) which was originally proposed by Chan and Maheu (2002) and then most recently extended by Maheu and McCurdy (2004). Therefore, the main contribution of the first chapter in this dissertation is to investigate whether alternative return distributions such as Hansen's skewed *t*-distribution combined with similar GARCH specifications can outperform mixed GARCH-jump models.

This question is very important because GARJI model has many more parameters to estimate and is relatively computationally-intensive compared to the Hansen's skewed *t*-distribution model. For example, in GARJI model, there are 11 parameters to estimate through maximum likelihood estimation (MLE). However, I only need to estimate 6 parameters for the Hansen's skewed *t*-distribution when combined with GJR-GARCH specification (henceforth, *GJR-HT*). In addition, it takes about 7-10 minutes to estimate GARJI model using highly-optimized algorithms of MATLAB V6.5 through Intel 2.8GHz Pentium 4 Processor with Hyper-Threading Technology. However, it only takes 1-2 seconds to obtain the stable estimation results for the GJR-HT.

Therefore, this study is the first to question whether complicated jumps specification can provide a better fit to the empirical distribution of the data and jump models have been parameterized in such a way that they can accommodate and explain the most common stylized facts observable in the data compared to the more parsimonious GJR-HT model can. To answer this empirical research question, I perform several goodness-of-fit diagnostic tests and Value-at-Risk (VaR) analysis and employ likelihood-ratio tests and information criteria to determine the most suitable model specifications which give me both better fit to the data and parsimony of parameterization. Additionally, I will examine the ability of each model to capture extreme outliers in Asian and Latin emerging stock markets through VaR analysis.

In the separate section of the first chapter in this dissertation, I try to answer the following fundamental question; why do the asymmetric fat-tail distributions such as Hansen's skewed *t*-distribution outperform the mixed-GARCH models? I find that Poisson-distributed jump-diffusion (JD) models fail to capture the excess fat tails and peakness around zero in equity returns, especially in the emerging stock markets, by using the boxplot methods (1997) and the probability analysis of Ait-Sahalia (2004). In my analysis, JD models can only capture a small portion of the extreme values prevalent in emerging stock markets. More importantly, the simulation results strongly support that the skewed *t*-distribution models have the flexibility of capturing both jump and non-jump extreme values depending on the degree of freedom parameter.

The first chapter is organized as follows. In Section 2 I setup five different model parameterizations and illustrate their likelihood functions. Section 3 describes the data covered in this study. Section 4 presents estimation methodology for model comparisons. Section 5 discusses the empirical results of the proposed models, along with goodness-of-fit diagnostic tests. In Section 6, I perform 95% and 99% Value-at-Risk (VaR) analysis to further evaluate each different model specification. Section 7 explains the rationales on why asymmetric fat-tailed distributions such as Hansen's skewed *t*-distribution outperform the mixed GARCH-jump models. Section 8 provides a summary and conclusion.

2. Model Specifications

I follow Harvey and Siddique's (1999) approach to develop different model specifications considered in this study because of nonlinearity of likelihood functions and choice of good starting values to obtain global maximum. Estimation of parameters in stages moving from simpler models to more complex specifications has also an advantage of goodness-of-fit diagnostic tests. That is, if more complex models nest simpler models, complex ones should have higher likelihood values than those of simpler ones to pass likelihood ratio (LR) test.

In addition, I will set up models in a discrete-time setting since GARCH processes can be seen as approximations to continuous-time diffusion processes with stochastic volatility. Nelson (1990) shows that most GARCH processes converge in distribution to diffusion processes. That is, the discrete-time GARCH (1, 1) model converges to a continuous-time diffusion model as the sampling interval gets arbitrarily small. Therefore, simple GARCH processes offer a discrete-time filter for stochastic volatility models and GARCH models with jumps can be similarly viewed as filters for continuous-time stochastic volatility models with jumps in returns and/or volatility. (For review of linkage between discrete-time and continuous-time framework, see an excellent Sundaresan (2000)'s survey paper and Duan et al. (2004)). This finding is important because it is much easier to perform MLE for GARCH processes with discretely recorded data.

2.1. GJR-N: GJR-GARCH with Normal Return Distributions

Although the simple GARCH (1, 1) model is usually a good starting point when modeling financial returns, there is substantial evidence that time-varying asymmetry is a major component of volatility dynamics (e.g., Nelson (1991), Engle and Ng (1993), Glosten et al. (1993)). Hence, in order to avoid misspecification of the conditional variance equation, I explicitly include a leverage term in the GARCH specification (Glosten et al., 1993) and I label *GJR-N* since its standardized error, $z_t = \frac{\varepsilon_t}{\sqrt{h_t}}$, is assumed to be a standard normal distribution, $NID(0,1)$ and the normal distribution is by far the most widely used distribution when estimating and forecasting GARCH-type models.

Suppose Ψ_{t-1} is the information set at time $t-1$, then I can define functional form of GJR-N as:

$$r_t = \ln(\frac{P_t}{P_{t-1}}) = \mu + \varepsilon_t \tag{1}$$

$$h_t = \alpha_0 + (\alpha_1 + \gamma_1 I_{t-1})\varepsilon_{t-1}^2 + \beta_1 h_{t-1} \tag{2}$$

$$\varepsilon_t \big| \Psi_{t-1} = \sqrt{h_t}\, z_t, \qquad z_t \sim NID(0,1) \tag{3}$$

where I_{t-1} is an indicator function which equals 1 when $\varepsilon_{t-1} < 0,$ and zero otherwise. In this model, good news (or positive shocks, $\varepsilon_{t-1} > 0$) have an impact of $\alpha_1 \varepsilon_{t-1}^2 \geq 0$ on volatility, while bad news (or negative shocks, $\varepsilon_{t-1} < 0$) have an impact of $(\alpha_1 + \gamma_1)\varepsilon_{t-1}^2 \geq 0$. Therefore, if $\gamma_1 \neq 0$, I can say that there exist asymmetric effects on conditional volatility. In equation (2) and (3), h_t is a time-varying, positive and measurable function of the information set at time $t-1$ and z_t is an independently and identically distributed process with $E\left(z_t\right) = 0$ and $Var\left(z_t\right) = 1$. By definition, ε_t is serially uncorrelated with mean zero, but its conditional variance equals h_t and, therefore, may change over time, contrary to what is assumed in OLS estimations (For overview of the most important theoretical developments in the parameterization and implementation of GARCH-type models, see Bollerslev et al. (1992)).

The log-likelihood function of the sample under a normality assumption for the disturbances is:

$$L(\Theta; r_1, \cdots r_T) = -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\left(\ln h_t + \frac{\varepsilon_t^2}{h_t}\right) \tag{4}$$

where *T* is the number of observations. To ensure that the GJR-N model is stationary, $L(\Theta)$ is maximized under the restriction $\alpha_1 + \beta_1 + \frac{1}{2}\gamma_1 < 0$. When $\gamma_1 = 0$, the second moment regularity condition of GJR-N reduces to the well-known second moment condition for the GARCH (1, 1) model. It follows that unconditional variance implied by GJR-N model is

$$E\left[h_t\right] = \frac{\alpha_0}{1 - \left(\frac{\gamma_1}{2} + \alpha_1 + \beta_1\right)}.$$

2.2. JD-GJR: Jump-Diffusion Model with GJR-GARCH process

A way to obtain more realistic distributions is to assume that the stock price follows a jump-diffusion process. For example, in pricing and hedging with financial derivatives, jump-diffusion models are particularly important, since ignoring jumps in financial prices will cause pricing and hedging risks. Merton (1976) develops a model in which the arrival of normal information is modeled as a diffusion process, while the arrival of abnormal information is modeled as a Poisson process. As an extension, Jorion (1988) combines jump processes with ARCH models in a discrete-time framework. Therefore, in this study, I consider a jump-diffusion process, which is a mixture of normal error distribution and a compound Poisson jump process for the US and emerging stock markets. Then, this constant jump-diffusion model is combined with GJR-GARCH specification to reflect time-varying volatility and asymmetric effects (henceforth, *JD-GJR*). Therefore, JD-GJR is one of the simplest forms among the mixed GARCH-Jump models considered in this paper.

However, note that there is an important difference between my JD-GJR specification and existing Jorion-type mixed jump-diffusion models because I explicitly assume that jumps are incorporated in both returns and volatility. Much recent research has provided strong evidence of the presence of positive jumps in volatility (See Bates (2000), Duffie et al. (2000), Pan (2002), and Eraker (2003, 2004)). All of them argue that the specification of the volatility process should include jumps, possibly correlated with the jumps in returns.

Eraker et al. (2003, pp. 1269-1970) explain that "Jumps in returns can generate large movements such as the crash of 1987, but the impact of jump is transient: A jump in returns today has no impact on the future distribution of returns. On the other hand, diffusive volatility is highly persistent, but its dynamics are driven by a Brownian motion. For this reason, diffusive stochastic volatility can only increase gradually via a sequence of small normally distributed increments. Jumps in volatility fill the gap between jumps in returns and diffusive volatility by providing a rapidly moving but persistent factor that drives the conditional volatility of returns."

Therefore, for model comparison purpose and to follow the direction of modern financial research, I incorporate jumps in returns and volatility across all mixed GARCH-Jump models examined in this paper. The JD-GJR model is formally described as follows:

$$r_t = \ln(\frac{P_t}{P_{t-1}}) = \mu + \varepsilon_t, \quad \varepsilon_t = \varepsilon_{1,t} + \varepsilon_{2,t} \tag{5}$$

where

$$\varepsilon_{1,t} \left| \Psi_{t-1} = \sqrt{h_t} z_t, \qquad z_t \sim NID(0,1) \right. \tag{6}$$

$$\varepsilon_{2,t} \left| \Psi_{t-1} = \sum_{i=1}^{N} J_{i,t} - \theta\lambda, \qquad J_{i,t} \sim NID(\theta, \delta^2) \right. \tag{7}$$

The returns process consists of three components, a linear drift (μ), 'normal' price vibrations ($\varepsilon_{1,t}$), and jump innovation representing a compound Poisson process that accounts for 'abnormal' change in prices (jumps) due to arrival of news ($\varepsilon_{2,t}$). The GJR-GARCH specification is the same as that of GJR-N except squared innovations term (ε_{t-1}^2) of JD-GJR model contain previous shocks from both normal price vibrations and abrupt jump innovations such as crashes, devaluations, corrections or defaults. In equation (7), the jump innovation at time *t* is the sum of the N_t jumps which arrived over the time interval ($t-1$, *t*), where jump magnitude $J_{i,t}$ is determined by sampling from an *iid* normal distribution with mean jump size (θ) and jump size variability (δ^2). This special case makes estimation and hypothesis testing tractable and has become one of the most important representations of the mixed GARCH-jump models.

2.3. ARJI-GJR: Autoregressive Jump Intensity with GJR-GARCH

Johannes and Kumar (1999) mention that standard state-independent models of returns such as Merton (1976)'s jump-diffusion model, Jorion (1988)'s mixed GARCH-jump model and JD-GJR model cannot capture the persistence in the jump processes. In reality, in contrast to the *iid* specification of jump size distribution, actual financial data exhibit jump-times clustering and temporally-dependent jump sizes. As such, the structure of jumps plays an important role in most financial applications including derivatives pricing, hedging, risk and portfolio management such as Value-at-Risk (VaR) which identify the maximum allowed movement in asset prices for a given threshold probability.

To overcome weaknesses of state-independent models such as the JD-GJR model, Chan and Maheu (2002) propose jump dynamics in stock market returns using the ARJI (Autoregressive conditional jump intensity) model coupled with a GARCH specification. For comparison with other models discussed in this paper, I consider a model which permits time variation in the jump intensity combined with GJR-GARCH specification (henceforth, *ARJI-GJR*). Return process and volatility equation have identical parameterizations as those of the JD-GJR model, which were already shown in equation (2) and (5).

The time variation in the jump intensity and the jump intensity residual to capture jump clustering are modeled as:

$$\lambda_t = \lambda_0 + \rho\lambda_{t-1} + \varphi\xi_{t-1} \tag{8}$$

where

$$\xi_{t-1} = \sum_{j=0}^{\infty} jP(n_{t-1} = j | \Psi_{t-1}) - \lambda_{t-1} \tag{9}$$

2.4. GARJI: The mixed GARCH-Jump model with Autoregressive Jump Intensity

Most recently, Maheu and McCurdy (2004) propose *GARJI* specification, which is the extended mixed GARCH-jump model with autoregressive jump intensity of Chan and Maheu (2002). The only difference between ARJI-GJR model and GARJI model is that the variance equation of GARJI model includes the effect of both past normal innovations and past jump innovations to returns. In other words, GARJI model estimates the number of jumps that occurred during period $t-1$ and allow it to directly affect the feedback that ε_{t-1} has on the GARCH variance process.

Formally, return process is the same as those of previous models such as JD-GJR model and ARJI-GJR model. However, variance equation of GARJI model has somewhat different characterization compared to those of previous GARCH-families. Specifically, the GARCH specification of GARJI model is defined as:

$$h_t = \alpha_0 + g(\Lambda, \Psi_{t-1})\varepsilon_{t-1}^2 + \beta h_{t-1}, \tag{10}$$

where

$$g(\Lambda, \Psi_{t-1}) = \exp\left(\alpha + \alpha_j E\left[n_{t-1} | \Psi_{t-1}\right] + I\left(\varepsilon_{t-1}\right)\left(\alpha_a + \alpha_{a,j} E\left[n_{t-1} | \Psi_{t-1}\right]\right)\right) \tag{11}$$

Time-varying jump intensity structure of GARJI model is assumed to be identical as that of ARJI-GJR in the equation (8) and (9).

For estimation, the log-likelihood function of GARJI model (including JD-GJR model and ARJI-GJR model) is defined as[2]:

$$L(\Theta; r_1, \cdots r_T) = \sum_{t=2}^{T} \ln\left[f(r_t \middle| \Psi_{t-1}) \right] \tag{12}$$

where

$$f(r_t | \Psi_{t-1}) = \sum_{j=0}^{\infty} P(N_t = j | \Psi_{t-1}) f(r_t | N_t = j, \Psi_{t-1}) \tag{13}$$

$$= \sum_{j=0}^{\infty} \frac{\exp(-\lambda_t)\lambda_t^{\,j}}{j!} \cdot \frac{1}{\sqrt{2\pi(h_t + j\delta^2)}} \exp\left(-\frac{(r_t - \mu + \lambda_t\theta - j\theta)^2}{2(h_t + j\delta^2)} \right) \tag{14}$$

For comparison with other estimated conditional density functions and computations of corresponding empirical quantiles to evaluate VaR, the standardized version of conditional density function for GARJI model can be derived as:

$$f(z_t | \Psi_{t-1}) = \sum_{j=0}^{\infty} f(z_t | N_t = j, \Psi_{t-1}) \cdot P(N_t = j | \Psi_{t-1}) \tag{15}$$

where

$$z_t = \frac{r_t - \mu}{\sqrt{h_t + (\theta^2 + \delta^2)\lambda_t}} \tag{16}$$

and

[2] To see more detailed economic intuitions of each specification in equation (8)-(14), refer to Chan and Maheu (2002) and Maheu and McCurdy (2004).

$$f(z_t \mid N_t = j, \Psi_{t-1}) = \frac{1}{\sqrt{2\pi(h_t + j\delta^2)}} \exp\left(-\frac{(z_t \cdot \sqrt{h_t + (\theta^2 + \delta^2)\lambda_t} + \theta\lambda_t - \theta j)^2}{2(h_t + j\delta^2)} \right) \times \sqrt{h_t + (\theta^2 + \delta^2)\lambda_t} \tag{17}$$

In addition, I can also show that the standardized version of conditional density function for GARJI model is qualified as a distribution since integration of Equation (15) is:

$$\int_{-\infty}^{\infty} f(z_t \mid \Psi_{t-1}) dz_t = 1 \tag{18}$$

Therefore, standardizing r_t as in Equation (16) results in a zero-mean and unit-variance random variable with the distribution in Equation (15).

2.5. GJR-HT: GJR-GARCH with Hansen's Skewed t-Distribution

It is well-documented that even asymmetric GARCH models fail to fully account for sample skewness and leptokurtosis of high frequency financial time series when they are assumed to follow normal or symmetric Student's *t* distributions with ν degree of freedom[3] in order to allow for excess kurtosis in the conditional distribution. This has naturally led to the use of asymmetric non-normal distributions to better model conditional higher moments. To make progress, Hansen (1994) has assumed that the distribution of the error term z_t can be skewed.

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[3] While financial time series can be skewed, the unconditional skewness is still zero since the distribution of the error process z_t is symmetric around zero. Another disadvantage of using *t*-distribution as one of candidates in return distributions is that it is not high-peaked around zero returns, which is one of important features of leptokurtosis (high peak + heavy tails) of financial asset returns.

The same specification for the GJR-HT is applied to define mean and variance equations as those of GJR-N in the equation (1) and (2). However, underlying return distribution is now assumed to follow Hansen's skewed *t*-distribution.

$$\varepsilon_t \left| \Psi_{t-1} = \sqrt{h_t} z_t, \quad z_t \sim HT(z \left| \eta, \phi \right.) \right. \tag{19}$$

where

$$HT(z|\eta,\phi) = \begin{cases} bc\left(1+\dfrac{1}{\eta-2}\left(\dfrac{bz+a}{1-\phi}\right)^2\right)^{-\frac{\eta+1}{2}} & \text{if } z < -a/b \\ bc\left(1+\dfrac{1}{\eta-2}\left(\dfrac{bz+a}{1+\phi}\right)^2\right)^{-\frac{\eta+1}{2}} & \text{if } z \geq -a/b \end{cases} \tag{20}$$

The values a, b, and c in the distribution are defined as:

$$a \equiv 4\phi c\frac{\eta-2}{\eta-1}, \quad b \equiv 1+3\phi^2-a^2, \quad \mathrm{c} \equiv \frac{\Gamma\left(\dfrac{\eta+1}{2}\right)}{\sqrt{\pi(\eta-2)}\Gamma\left(\dfrac{\eta}{2}\right)} \tag{21}$$

The conditional distribution of the standard residuals z_t is characterized by two parameters: kurtosis parameter (η) and asymmetry parameter (ϕ). These are restricted to $4 < \eta < 30$ and $-1 < \phi < 1$. This distribution nests the symmetric Student *t*-distribution when the asymmetry parameter (ϕ) equals 0 and also nests the standard normal distribution when kurtosis parameter (η) goes ∞.

As an alternative parameterization of Hansen's model, Harvey and Siddique (1999) employ variants of Hansen's (1994) autoregressive conditional density model (ARCD) with a skewed version of the *t*-distribution specified for the error term. The model extends the standard

GARCH-M model by allowing the conditional skewness and degrees of freedom of the skewed *t*-distribution to depend linearly on functions of lagged error terms. They also investigate the existence of coskewness using a bivariate GARCH on daily and monthly stock indexes. The log-likelihood function of the GJR-HT is similarly defined as:

$$L(\Theta;\Psi_{t-1}) = \sum_{t=2}^{T} \ln\left[HT(z \,|\, \eta, \phi; \Psi_{t-1}) \right] \tag{22}$$

3. Data and Descriptive Statistics

I use the daily returns r_t from the index series collected in each trading day at closing time from 07/05/1995 to 08/07/2002 for the US and four emerging stock markets (2 Asian stock markets: Korea, Indonesia and 2 Latin American stock markets: Mexico, Brazil). The data are obtained from CRSP (Center for Research in Security Prices) for the US and EMDB (Emerging Markets Database) for emerging stock markets. Zero returns associated with non-trading days have been excluded from the dataset.

The r_t values are the continuously compounded returns calculated as the natural logarithm of two consecutive index values, $r_t = \ln\left(P_t / P_{t-1} \right)$ to obtain approximately stationary series. The rationales I consider daily index returns for this study are that 1) a research has shown that jump-diffusion models are not supported for monthly data, due to the fact that less frequently observed data is not appropriate for detecting the jump component in asset returns process (see Lin and Yeh (1999)) and 2) VaR analysis has more important practical roles in daily basis.

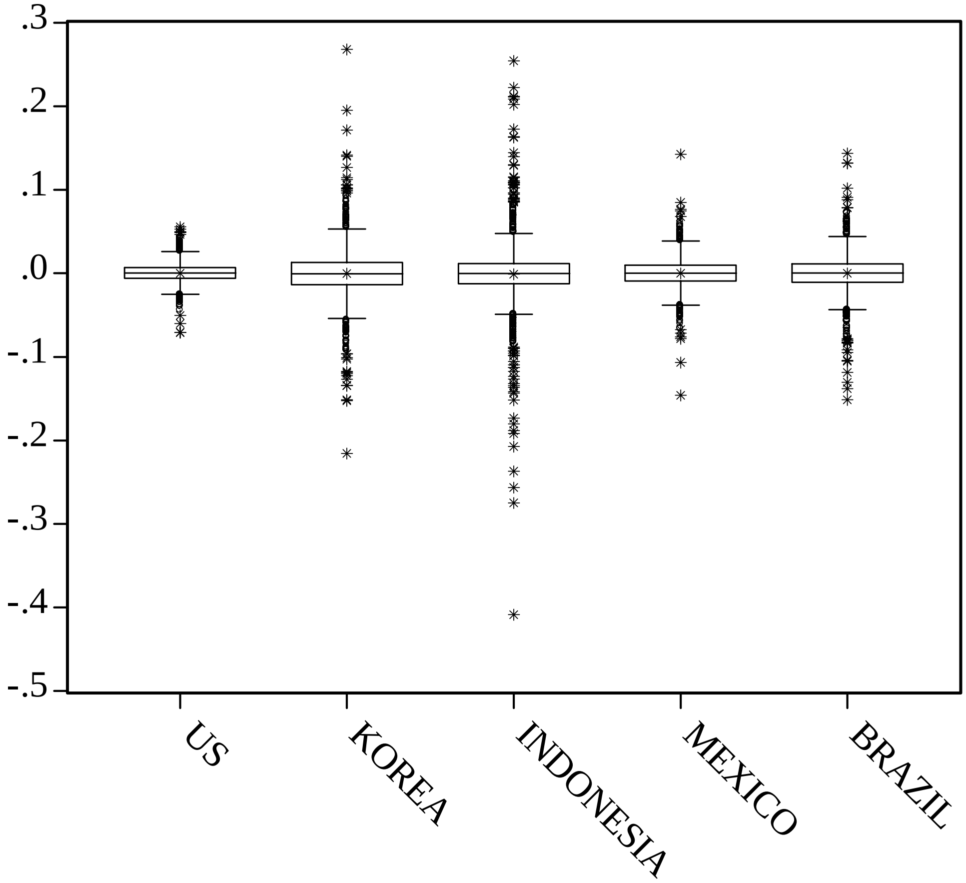


Figure 1. Boxplots of Daily Returns for the US and Four Emerging (Asian and Latin American) Stock Market Indices. This figure summarizes the distribution of a set of data by displaying the centering and spread of the data using a few primary elements. The box portion of a boxplot represents the first and third quartiles (middle 50 percent of the data). Data points outside the inner fence are considered as outliers.

Figure 1 displays boxplots of daily returns for my samples. This figure summarizes the distribution of a set of data by displaying the centering and spread of the data using a few primary elements. The box portion of a boxplot represents the first and third quartiles (middle 50 percent of the data). Data points outside the inner fence are considered as outliers. The US has relatively few outliers in the sample. However, as expected, Korea and Indonesia include a lot of extreme outliers since these countries were severely affected by Asian currency crisis in 1997. Mexico and Brazil have also several outliers in between the US and Asian stock markets.

Table I

Descriptive Statistics

This table presents summary statistics for returns on the US and four emerging (Asian and Latin American) stock indices for the period from 07/05/1995 to 08/07/2002. JB is the Jarque-Bera statistic for testing normality. This is asymptotically distributed as χ^2 (2) with 2 degree of freedom. KS is the Kolmogorov-Smirnov statistic for testing the null hypothesis of normality. ARCH is the Engle's hypothesis test to examine for the presence of ARCH/GARCH effects. $Q^2(30)$ represents the Ljung-Box test statistics for serial autocorrelation in the squared returns at a lag of 30 trading days for each stock index. The *p*-values are reported in square brackets.

	US	Korea	Indonesia	Mexico	Brazil
Observations	1788	1784	1815	1832	1816
Mean	0.0003	-0.0002	-0.0008	0.0003	0.0000
Median	0.0004	-0.0006	-0.0004	0.0002	0.0006
Maximum	0.0557	0.2679	0.2543	0.1426	0.1436
Minimum	-0.0711	-0.2156	-0.4085	-0.1460	-0.1514
Std. Dev.	0.0120	0.0307	0.0368	0.0185	0.0229
Skewness	-0.1745	0.3049	-0.8631	-0.0430	-0.3191
	[0.0013]	[0.0000]	[0.0000]	[0.2263]	[0.0000]
Kurtosis	6.1418	11.6586	22.1726	8.9594	9.1843
	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[0.0000]
JB	744.48	5600.47	28024.25	2711.47	2924.75
	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[0.0000]
KS	0.4810	0.4553	0.4500	0.4723	0.4675
	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[0.0000]
ARCH	60.9868	70.2904	82.4824	14.5321	87.9282
	[0.0000]	[0.0000]	[0.0000]	0.0001	[0.0000]
$Q^2(30)$	408.34	2652.59	1511.63	357.85	813.81
	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[0.0000]

Table I presents summary statistics on my samples. The Jarque-Bera (JB) tests and Kolmogorov-Smirnov statistics (KS) show non-normality of all of five indexes distributions and in all cases this is mainly due to the presence of skewness and excess kurtosis while the coefficient of skewness for the Mexico is not significant at the conventional 5% level. Engle's LM test indicates the presence of ARCH/GARCH effects in the conditional variance and the Ljung-Box tests, $Q^2(30)$, show strong autocorrelations in the squared returns at a lag of 30 trading days for each stock index.

I also perform non-parametric runs test to check normality of returns distributions for my samples because severe non-randomness and higher predictability of stock returns strongly imply non-normality in return distributions. Table II shows the estimation results of a non-parametric runs test used for detecting the frequency of the changes in the direction of a time series. The runs test determines whether the total number of runs in the sample is consistent with the hypothesis that the changes are independent. The standard normal *Z*-test statistic used to conduct a runs test is given by:

$$Z = -\left(\frac{|R - E(R)| - 0.5}{\sqrt{Var(R)}} \right) \tag{23}$$

where *R* is the observed number of runs and

$$E(R) = \frac{2n_1 n_2}{n_1 + n_2} \tag{24}$$

where n_1 and n_2 are the number of observations below respectively above the mean[4], and

$$Var(R) = \frac{2n_1 n_2 (2n_1 n_2 - n_1 - n_2)}{(n_1 + n_2)^2 (n_1 + n_2 - 1)} \tag{25}$$

[4] Other versions of the test use the median or mode instead of the mean.

The hypothesis of randomness is then rejected at the 100-α significance level if $2 \cdot P\{Z > |z|\} < \alpha$ where $Z \sim NID(0,1)$. A runs test carried out on the daily data for indices of the US and four (Asian and Latin American) emerging stock markets gives *Z*-values of -0.3247 (US), -5.6424 (Korea), -5.8863 (Indonesia), -4.3439 (Mexico), and -3.7380 (Brazil) which have *p*-values that are basically equal to zero except the US, thus giving huge evidence to reject the hypothesis of randomness[5] or normality of asset returns for four emerging stock markets.

Table II

Nonparametric Runs Test for Randomness

This table shows the estimation results of a nonparametric runs test used for detecting the frequency of the changes in the direction of a time series. A runs test is conducted for each of the US and four emerging (Asian and Latin American) stock markets indices.

Runs Test	US	Korea	Indonesia	Mexico	Brazil
Observations	1788	1784	1815	1832	1816
Below Cutoff	862	922	920	908	871
Above Cutoff	926	862	895	924	945
Number of Runs	887	773	783	824	828
Cutoff	0	0	0	0	0
E[*R*]	893.8546	891.9910	908.3278	916.9301	907.4923
Std Dev.	21.1094	21.0889	21.2915	21.3935	21.2660
Z-Value	-0.3247	-5.6424	-5.8863	-4.3439	-3.7380
P-Value	[0.7454]	[0.0000]	[0.0000]	[0.0000]	[0.0002]

[5] Positive *Z*-values indicate the sample contains greater-than-expected numbers of runs and negative Z-values indicate the sample contains fewer-than-expected numbers of runs.

4. Estimation Methodology

For the five different model specifications, parameters are estimated using the maximum likelihood principle (MLE), i.e., I maximize the conditional log-likelihood functions of each model for a sample of *T* observations $r_1, \cdots, r_T$ with respect to parameter vector Θ. Especially, estimation of parameter values for the GJR-HT through MLE is quite easy to implement and its estimation does not face serious problems of convergence[6]. The optimization algorithm used is the Broyden, Fletcher, Goldfarb and Shanno (BFGS) Quasi-Newton updating scheme. To avoid the unexpected explosions of parameter values during the iterations of numerical optimizations, I use a constrained optimization algorithm (*fmincon*) built in the MATLAB. Furthermore, I also ensure global maximum by trying a few qualified starting values to see whether I obtain same (at least almost identical) parameter values and likelihood values for each different model specification.

For comparison among nested models such as GJR-N, JD-GJR and ARJI-GJR, I employ the likelihood ratio (LR) test since within the MLE framework, it is the standard method for testing goodness of different models. The idea of this test is that the distributions f_1 and f_2 with dimension of parameter spaces d_1 and d_2 ($d_1 < d_2$) determine the likelihood ratio:

$$LR = 2\left(\ln\left(L_{f_2}\left(r\middle|\Theta_{d_2}\right)\right) - \ln\left(L_{f_1}\left(r\middle|\Theta_{d_1}\right)\right)\right) \sim \chi^2_{d_2-d_1} \tag{26}$$

where L_{f_1} and L_{f_2} are the likelihood functions; d_1 and d_2 are the respective parameter spaces, and r is the data. LR follows asymptotically χ^2 distribution with $(d_2 - d_1)$ degrees of freedom.

6 I am grateful to Dr. Andrew Patton who provides me MATLAB source codes to estimate Hansen (1994)'s skewed *t*-distribution. Although I considerably modified original m-files for this study, I appreciate his generosity to allow programs to be available to other researchers.

However, to compare ARJI-GJR, GARJI and GJR-HT model, I need alternative measures because these three models are non-nested, which means that the parameter space of the PDF with smaller number of independent parameters is not derived from the parameter space of the PDF with higher dimensionality.

Therefore, the goodness-of-fit for the models in this paper is also measured using the following three information criteria since past research has not provided common single conventional test criteria[7]: Akaike information criteria (AIC), Schwarz criteria (SC), and Hannan-Quinn criteria (HQ). All of them are applicable when the parameter spaces of the different model being compared are non-nested.

$$\begin{aligned} AIC &= -2(l/T) + 2(k/T) \\ SC &= -2(l/T) + k\log(T)/T \\ HQ &= -2(l/T) + 2k\log(\log(T))/T \end{aligned} \tag{27}$$

Note that all the above three information criteria include an adjustment for the degrees of freedom that depends on both the number of parameters (k) and the sample size (T). Therefore, the model that minimizes the above information criteria values provides the best fit, since it is the model that has the highest likelihood value while controlling for the number of parameters in the model and the sample size.

5. Empirical Results and Goodness-of-fit Diagnostic Tests

Tables III presents estimation results and parameter values for each model specification for the US and four emerging markets (see Panel A to E). LR test statistics and several model selection measures strongly suggest that GJR-HT outperforms GARJI model, let alone all other

[7] McAleer (1995) surveys more than one hundred empirical papers in which models have been tested against one or more non-nested alternatives. Granger et al. (1995) argue that it is often better to use model selection criteria rather than formal hypothesis testing when evaluating alternative models.

model specifications for every country considered in this study. Although the likelihood value of GARJI (5608.96) is slightly higher than that of GJR-HT (5608.60) for the US (See Panel A), GJR-HT model has the smallest values of three information criteria. Of course, in this case, information criteria should be used to select better model specification since GARJI and GJR-HT model are non-nested. For emerging stock markets, superiority of GJR-HT over GARJI model is even more astounding. GJR-HT model has not only the highest likelihood values, but also the smallest values of information criteria, which imply that GJR-HT model provide not only the best but also parsimonious model specification for my sample.

Table III

Parameter Estimates for the Maheu and McCurdy's (2004) GARJI Model and GJR-GARCH with Skewed *t* Error Distribution (GJR-HT)

The tables (Panel A~E) present parameter values and test statistics through maximum likelihood estimates (MLE) for the GARJI model and GJR-GARCH with skewed *t* error distribution. For comparison, estimation results of Merton's constant jump model combined with GJR-GARCH are also provided. Ln *L* is log-likelihood values with the *k* parameters estimated using *T* observations for each model specification and data series. For model comparison purpose, three information criteria are also computed by AIC (Akaike Information Criterion), SC (Schwarz Criterion), and HQ (Hannan-Quinn Criterion). The *p*-values are reported in square brackets.

Mean Equation : $r_t = \ln(P_t / P_{t-1}) = \mu + \varepsilon_t$

$$\text{GJR-HT}: \ \varepsilon_t \,|\, \Psi_{t-1} = \sqrt{h_t} z_t, \qquad z_t \sim HT(z_t \,|\, \eta, \phi)$$

$$\text{GARJI}: \varepsilon_t = \varepsilon_{1,t} + \varepsilon_{2,t}, \ \ \varepsilon_{1,t} \,|\, \Psi_{t-1} = \sqrt{h_t} z_t, \ \ \varepsilon_{2,t} \,|\, \Psi_{t-1} = \sum_{i=1}^{N} J_{i,t} - \theta\lambda_t, \ \ z_t \sim NID(0,1), \ \ J_{i,t} \sim N(\theta, \delta^2)$$

Variance Equation

$$\text{GJR-HT}: h_t = \alpha_0 + (\alpha_1 + \gamma_1 I_{t-1})\varepsilon_{t-1}^2 + \beta h_{t-1}, \ \ I_{t-1} = 1 \text{ if } \varepsilon_{t-1} < 0, \text{ otherwise } 0$$

$$\text{GARJI}: h_t = \alpha_0 + g(\Lambda, \Psi_{t-1})\varepsilon_{t-1}^2 + \beta h_{t-1},$$

$$g(\Lambda, \Psi_{t-1}) = \exp\big(\alpha + \alpha_j E[n_{t-1} \,|\, \Psi_{t-1}] + I(\varepsilon_{t-1})\big(\alpha_a + \alpha_{a,j} E[n_{t-1} \,|\, \Psi_{t-1}]\big)\big)$$

Autoregressive Jumps Intensity

$$\lambda_t = \lambda_0 + \rho\lambda_{t-1} + \varphi\xi_{t-1}, \ \ \xi_{t-1} = \sum_{j=0}^{\infty} jP(n_{t-1} = j \,|\, \Psi_{t-1}) - \lambda_{t-1}$$

The Information Criteria

$$\text{AIC} = -2(\ln L / T) + 2(k / T) \qquad \text{SC} = -2(\ln L / T) + k\log(T) / T \qquad \text{HQ} = -2(\ln L / T) + 2k\log(\log(T)) / T$$

Panel A: US

	GJR-N	JD-GJR	ARJI-GJR	GARJI	GJR-HT
α_0	0.0000	0.0012	0.0011	0.0000	0.0000
	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[0.2616]
β	0.9025	0.8961	0.9178	0.9041	0.8999
	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[0.0000]
α_1	0.0188	0.0051	0.0002		0.0000
	[0.0653]	[0.0947]	[0.1928]		[0.4993]
γ_1	0.1970	0.1513	0.1329		0.1712
	[0.0000]	[0.0000]	[0.0000]		[0.0000]
α				-6.0885	
				[0.0000]	
α_j				-3.5436	
				[0.0001]	
α_a				4.1337	
				[0.0000]	
$\alpha_{a,j}$				3.2791	
				[0.0035]	
η					10.3639
					[0.0000]
ϕ					-0.1208
					[0.0001]
λ_0			0.0566	0.0310	
			[0.0000]	[0.0000]	
ρ			0.3891	0.5462	
			[0.0083]	[0.0000]	
φ			0.4037	0.3008	
			[0.0050]	[0.0011]	
λ		0.1175			
		[0.0006]			
θ		-0.0058	0.0097	-0.0160	
		[0.0000]	[0.0000]	[0.0000]	
δ		0.0100	0.0088	0.0068	
		[0.0000]	[0.0000]	[0.0196]	
$\ln L$	5583.32	5602.39	5605.26	5608.96	5608.60
AIC	-6.2409	-6.2588	-6.2598	-6.2617	-6.2669
SC	-6.2286	-6.2373	-6.2322	-6.2279	-6.2485
HQ	-6.2363	-6.2509	-6.2496	-6.2492	-6.2601

Panel B: Korea

	GJR-N	JD-GJR	ARJI-GJR	GARJI	GJR-HT
α_0	0.0000	0.0019	0.0021	0.0000	0.0000
	[0.0371]	[0.0000]	[0.0000]	[0.0003]	[0.1846]
β	0.9333	0.9063	0.9234	0.9225	0.9217
	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[0.0000]
α_1	0.0457	0.0482	0.0322		0.0502
	[0.0000]	[0.0004]	[0.0036]		[0.0000]
γ_1	0.0433	0.0547	0.0562		0.0542
	[0.0000]	[0.0001]	[0.0000]		[0.0009]
α				-3.3262	
				[0.0000]	
α_j				-0.0513	
				[0.4661]	
α_a				0.8989	
				[0.0001]	
$\alpha_{a,j}$				0.2854	
				[0.3738]	
η					6.2460
					[0.0000]
ϕ					0.0156
					[0.2916]
λ_0			0.0152	0.0184	
			[0.0214]	[0.0035]	
ρ			0.7217	0.6634	
			[0.0000]	[0.0000]	
φ			0.4338	0.3125	
			[0.0274]	[0.0039]	
λ		0.0897			
		[0.0000]			
θ		0.0040	0.0018	0.0018	
		[0.0010]	[0.3832]	[0.3707]	
δ		0.0331	0.0466	0.0461	
		[0.0000]	[0.0000]	[0.0000]	
$\ln L$	4079.91	4097.38	4106.08	4102.15	4112.53
AIC	-4.5694	-4.5856	-4.5931	-4.5865	-4.6037
SC	-4.5571	-4.5641	-4.5655	-4.5527	-4.5853
HQ	-4.5649	-4.5777	-4.5829	-4.5739	-4.5969

Panel C: Indonesia

	GJR-N	JD-GJR	ARJI-GJR	GARJI	GJR-HT
α_0	0.0000	0.0001	0.0025	0.0000	0.0000
	[0.0000]	[0.3175]	[0.0000]	[0.0001]	[0.0253]
β	0.9318	0.8816	0.8350	0.7984	0.8627
	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[0.0000]
α_1	0.0467	0.0706	0.0643		0.0880
	[0.0000]	[0.0000]	[0.0002]		[0.0000]
γ_1	0.0483	0.0456	0.1350		0.0986
	[0.0000]	[0.0002]	[0.0000]		[0.0001]
α				-2.5460	
				[0.0000]	
α_j				-1.4485	
				[0.1959]	
α_a				0.6850	
				[0.0232]	
$\alpha_{a,j}$				1.4272	
				[0.2084]	
η					4.3000
					[0.0000]
ϕ					-0.0386
					[0.0762]
λ_0			0.0670	0.0005	
			[0.0000]	[0.0007]	
ρ			0.1475	0.9072	
			[0.1772]	[0.0000]	
φ			0.1843	0.0450	
			[0.0390]	[0.0017]	
λ		0.1524			
		[0.0000]			
θ		-0.0019	-0.0108	-0.0043	
		[0.0095]	[0.0120]	[0.1354]	
δ		0.0329	0.0560	0.0548	
		[0.0000]	[0.0000]	[0.0000]	
$\ln L$	4172.33	4213.00	4233.38	4256.43	4277.06
AIC	-4.5932	-4.6347	-4.6550	-4.6782	-4.7064
SC	-4.5811	-4.6135	-4.6277	-4.6448	-4.6882
HQ	-4.5887	-4.6269	-4.6449	-4.6659	-4.6997

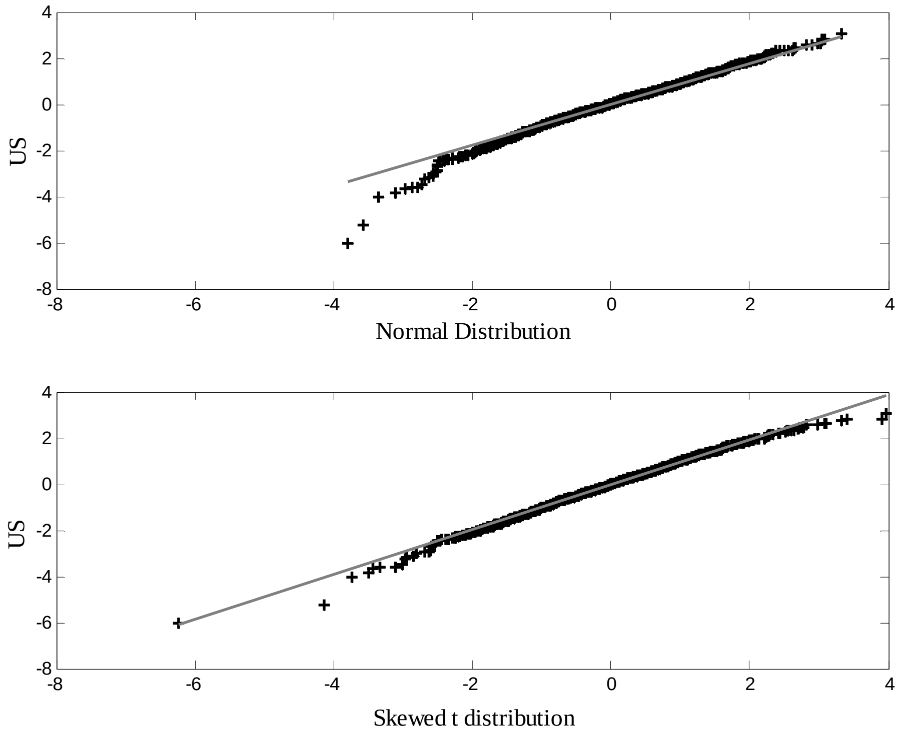
Panel D: Mexico

	GJR-N	JD-GJR	ARJI-GJR	GARJI	GJR-HT
α_0	0.0000	0.0026	0.0021	0.0000	0.0000
	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[0.0153]
β	0.8830	0.8428	0.9212	0.8817	0.8774
	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[0.0000]
α_1	0.0247	0.0533	0.0164		0.0289
	[0.0080]	[0.0000]	[0.0461]		[0.0012]
γ_1	0.1373	0.1346	0.0680		0.1358
	[0.0000]	[0.0000]	[0.0000]		[0.0000]
α				-3.6319	
				[0.0000]	
α_j				-0.2742	
				[0.4137]	
α_a				1.8286	
				[0.0000]	
$\alpha_{a,j}$				3.4104	
				[0.0033]	
η					6.9359
					[0.0000]
ϕ					0.0269
					[0.1873]
λ_0			0.0205	0.0372	
			[0.0946]	[0.0000]	
ρ			0.7021	0.2576	
			[0.0000]	[0.0435]	
φ			0.4044	0.2567	
			[0.0377]	[0.0292]	
λ		0.1220			
		[0.0000]			
θ		0.0041	-0.0042	-0.0025	
		[0.0000]	[0.3156]	[0.2853]	
δ		0.0235	0.0263	0.0312	
		[0.0000]	[0.0000]	[0.0000]	
$\ln L$	4905.81	4928.25	4932.44	4935.40	4945.55
AIC	-5.3513	-5.3725	-5.3749	-5.3760	-5.3925
SC	-5.3393	-5.3515	-5.3479	-5.3429	-5.3745
HQ	-5.3469	-5.3648	-5.3649	-5.3638	-5.3859

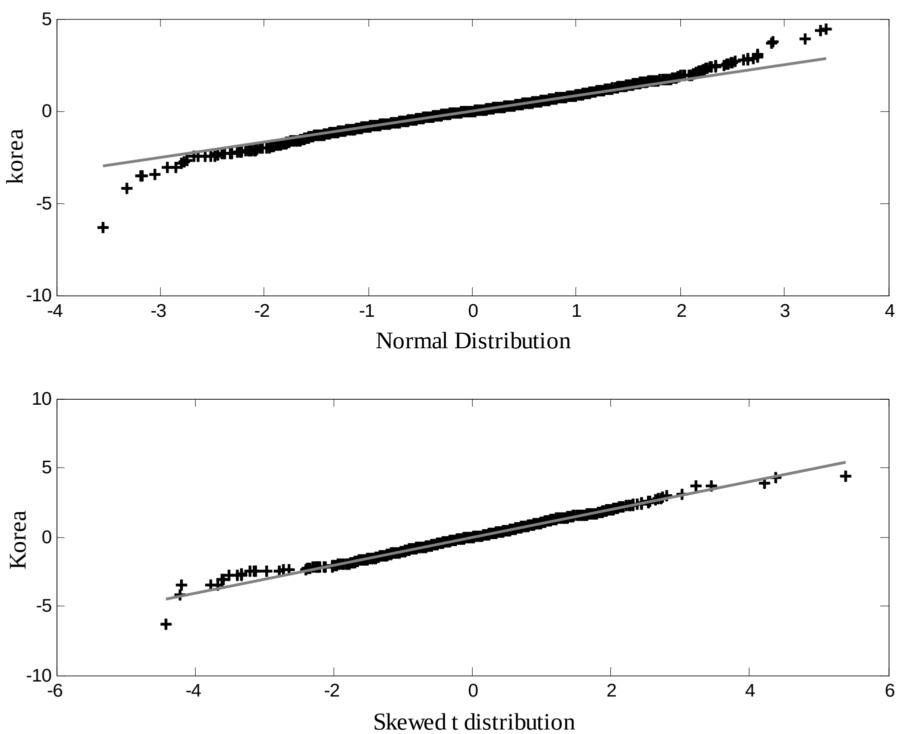
Panel E: Brazil

	GJR-N	JD-GJR	ARJI-GJR	GARJI	GJR-HT
α_0	0.0000	0.0029	0.0022	0.0000	0.0000
	[0.0000]	[0.0000]	[0.0000]	[0.1471]	[0.0158]
β	0.1733	0.8439	0.8895	0.8944	0.8586
	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[0.0000]
α_1	0.0626	0.0509	0.0364		0.0600
	[0.0000]	[0.0000]	[0.0001]		[0.0000]
γ_1	0.8175	0.1226	0.0862		0.1276
	[0.0000]	[0.0000]	[0.0000]		[0.0000]
α				-1.8997	
				[0.0000]	
α_j				-20.4076	
				[0.0034]	
α_a				-0.1010	
				[0.4073]	
$\alpha_{a,j}$				19.8977	
				[0.0041]	
η					6.9236
					[0.0000]
ϕ					-0.0918
					[0.0013]
λ_0			0.0685	0.0242	
			[0.0000]	[0.0619]	
ρ			0.4922	0.7695	
			[0.0002]	[0.0000]	
φ			0.3228	0.2941	
			[0.0139]	[0.0027]	
λ		0.0809			
		[0.0000]			
θ		-0.0236	-0.0137	-0.0165	
		[0.0000]	[0.0007]	[0.0001]	
δ		0.0166	0.0240	0.0253	
		[0.0000]	[0.0000]	[0.0000]	
$\ln L$	4605.93	4629.19	4637.39	4643.02	4645.49
AIC	-5.0682	-5.0905	-5.0973	-5.1013	-5.1096
SC	-5.0561	-5.0693	-5.0701	-5.0680	-5.0914
HQ	-5.0637	-5.0827	-5.0873	-5.0890	-5.1029

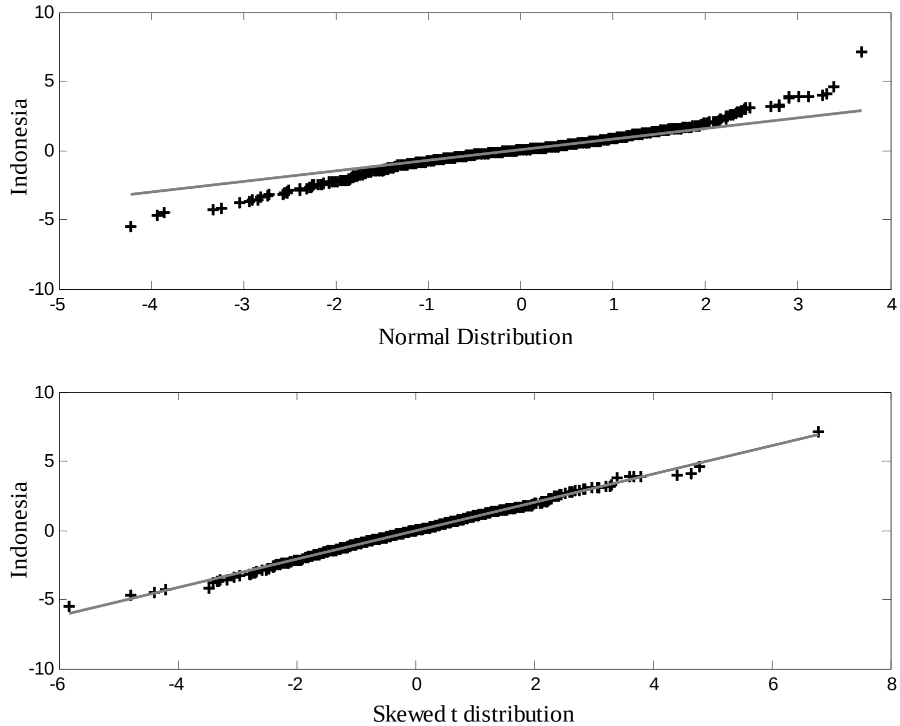
Panel A: US



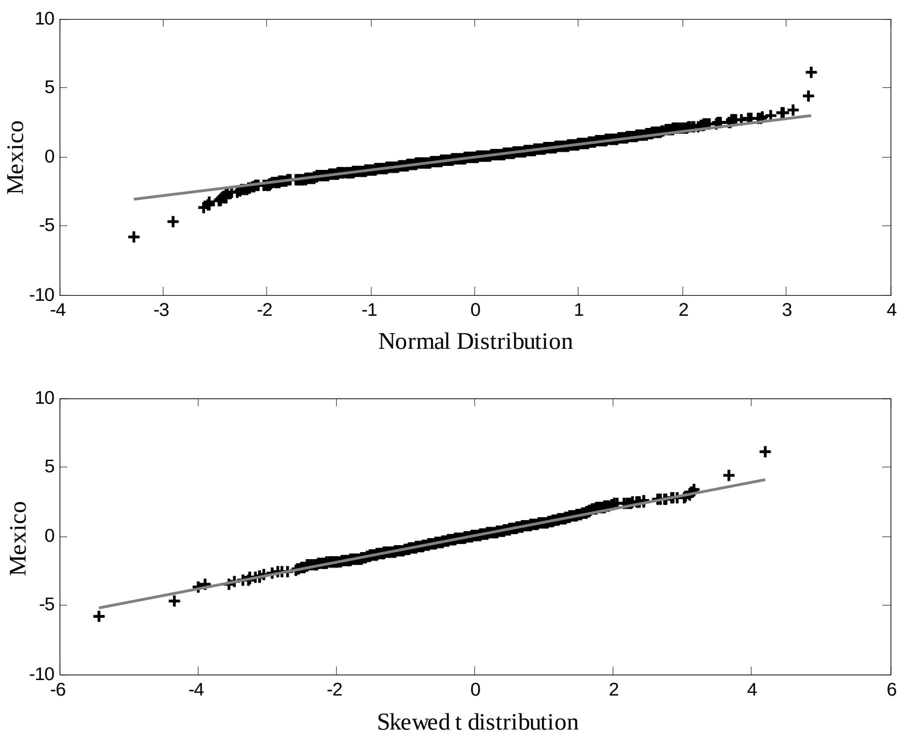
Panel B: Korea



Panel C: Indonesia



Panel D: Mexico



Panel E: Brazil

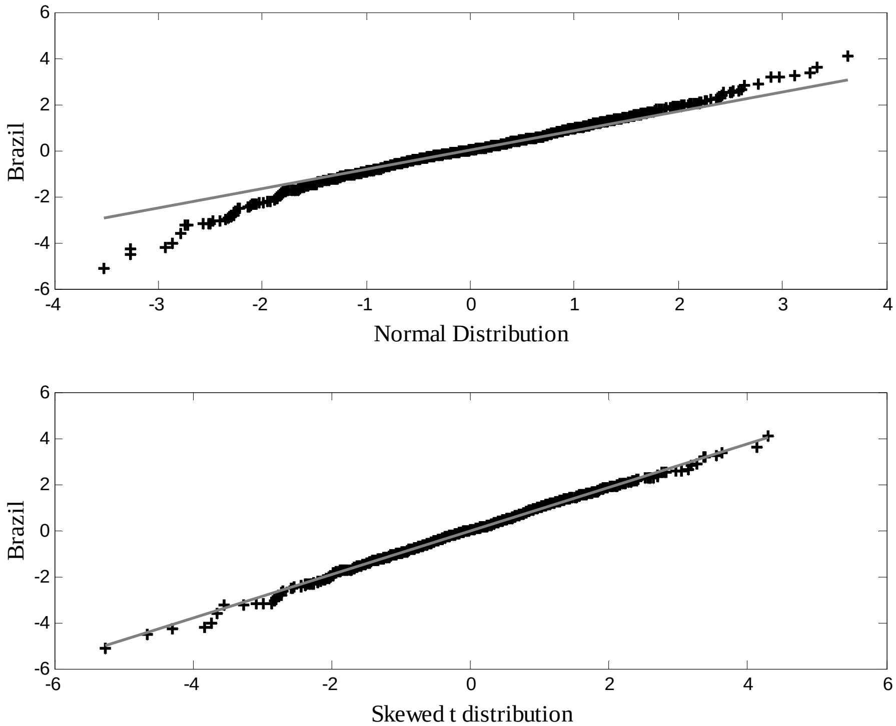


Figure 2. QQ-plots of Standardized Innovations for GJR-N and GJR-HT. The figures illustrate the QQ-plots of the standardized innovations for the GJR-GARCH model against normally distributed and Hansen's skewed *t* distributed random values using random number generator.

To see if GJR-HT model can describe equity returns well for the US and emerging markets, I also draw QQ-plots of standardized innovations for GJR-N and GJR-HT using random number generator in Figure 2. QQ-plots are useful in comparing distributions of observed sample values with values from a known distribution. If the sample data were generated from a random sample of the reference distribution, the plot should look roughly linear. If the reference distributions have heavier tails than the sample distribution, the plot curves down at the left end and up at the right end. Therefore, the QQ-plot is particularly helpful for analysis of tail behavior as shown in Hilgers (2004).

To generate random numbers from the Hansen's skewed *t*-distribution at η and ϕ, I transform draws from the uniform [0,1] distribution using the inverse of the following cumulative distribution function of skewed *t*-distribution:

$$\int_{-\infty}^{r} HT(z)dz = \begin{cases} (1-\phi)\int_{-\infty}^{s} t(\nu)d\nu & \text{if } s<0 \\ \\ (1-\phi)\int_{-\infty}^{0} t(\nu)d\nu + (1+\phi)\int_{0}^{s} t(\nu)d\nu & \text{if } s\geq 0 \end{cases} \tag{28}$$

where

$$s = \frac{br+a}{(1-\phi)}\sqrt{\frac{\eta}{\eta-2}} \tag{29}$$

In Equation (28), $t(\nu)$ is a *t*-distribution with degree of freedom, ν, and the values a and b in Equation (29) were already defined in Equation (21).

Therefore, denoting $\int_{-\infty}^{r} HT(z)dz$ and $\int_{-\infty}^{s} t(\nu)d\nu$ by $HT(r)$ and $T(s)$ respectively, the random number r from Hansen's skewed *t*-distribution is obtained by Equation (30) (For more detailed mathematical derivation, CDF, generating random samples from the Hansen's skewed *t*-distribution, refer to Hashmi and Tay (2001)).

$$r = HT^{-1}(u) = \begin{cases} \frac{1}{b}\left[(1-\phi)\sqrt{\frac{\eta-2}{\eta}}\cdot T^{-1}\left(\frac{u}{1-\phi}\right)-a\right] & \text{if } u<\frac{1-\phi}{2} \\ \\ \frac{1}{b}\left[(1+\phi)\sqrt{\frac{\eta-2}{\eta}}\cdot T^{-1}\left(\frac{u+\phi}{1-\phi}\right)-a\right] & \text{if } u\geq\frac{1-\phi}{2} \end{cases} \tag{30}$$

Figure 2 presents QQ-plots of the sample data against normal values. Under the assumption of normal distribution, for all countries, the plot is only approximately linear in the center of the distribution, while it curves up at the right end and down at the left end. This strongly indicates that return distributions of the US and emerging stock markets have much heavier tails compared to a normal distribution.

QQ-plots of the stock market data against the simulated skewed *t*-distributed random numbers are also displayed in Figure 2. All of these plots show an approximately linear dependence of the quantiles over the largest part of the distributions. This suggests that the market time series are much better described by skewed *t*-distributed than by normal random numbers. Therefore, the skewed *t*-distribution gives an excellent description for most of the sample values, and in particular for the large negative values. Moreover, the QQ-plots imply that fat tails are not symmetric.

Table IV reports summary statistics of the standardized residuals from the estimated five different model specifications. The Ljung-Box statistics for the first order autocorrelation in the squared residuals are all insignificant. This indicates that all of models I consider effectively remove the relationship among the squared returns since I have used same GJR-GARCH specifications for the variance equation except GARJI model. Note that all the raw squared returns displayed significant first order autocorrelation in Table I. Therefore, it might be more interesting to examine model implied higher moments to compare performances on different models.

Table IV

Goodness-of-Fit Diagnostic Tests

The tables present the Ljung-Box Portmanteau statistics (Panel A) and Engle's ARCH tests (Panel B) of squared standardized residuals for each of the six different model specifications. The figure in square brackets is the *p*-value of the Ljung-Box $Q^2(30)$ test and ARCH test against the null hypothesis of no serial correlation.

Panel A: Ljung-Box Q-statistic Lack-of-fit Hypothesis Tests

	GJR-N	JD-GJR	ARJI-GJR	GARJI	GJR-HT
US	21.8938	22.4375	19.6887	19.9094	20.5907
	[0.8578]	[0.8377]	[0.9244]	[0.9189]	[0.9002]
Korea	22.7161	26.5992	25.1750	24.5985	23.1463
	[0.8267]	[0.6442]	[0.7165]	[0.7443]	[0.8092]
Indonesia	35.7587	34.5111	31.8396	18.4733	36.4404
	[0.2161]	[0.2609]	[0.3750]	[0.9504]	[0.1940]
Mexico	28.5057	29.0669	27.2744	15.5045	28.9697
	[0.5437]	[0.5141]	[0.6088]	[0.9866]	[0.5192]
Brazil	26.1925	29.2633	32.9575	29.0389	28.6627
	[0.6657]	[0.5038]	[0.3244]	[0.5156]	[0.5354]

Panel B: Engle's hypothesis test for the presence of ARCH/GARCH effects

	GJR-N	JD-GJR	ARJI-GJR	GARJI	GJR-HT
US	2.5637	2.0571	0.9462	1.4185	1.7269
	[0.1093]	[0.1515]	[0.3307]	[0.2337]	[0.1888]
Korea	2.1771	1.7838	3.3546	2.9128	2.4768
	[0.1401]	[0.1817]	[0.0670]	[0.0879]	[0.1155]
Indonesia	0.5045	0.3607	2.4919	0.0024	0.0604
	[0.4775]	[0.5481]	[0.1144]	[0.9611]	[0.8058]
Mexico	0.4635	0.1698	1.5910	0.5365	0.3643
	[0.4960]	[0.6802]	[0.2072]	[0.4639]	[0.5461]
Brazil	1.3275	4.2520	0.0104	0.2738	0.3710
	[0.2493]	[0.0392]	[0.9185]	[0.6008]	[0.5425]

For the JD-GJR, ARJI-GJR, and GARJI models, conditional higher moments can be computed using the following formula derived by Das and Sundaram (1999).

$$Sk(r_t \mid \Psi_{t-1}) = \frac{\lambda_t(\theta^3 + 3\theta\delta^2)}{(h_t + \lambda_t\delta^2 + \lambda_t\theta^2)^{3/2}} \tag{31}$$

$$Ku(r_t \mid \Psi_{t-1}) = 3 + \frac{\lambda_t(\theta^4 + 6\theta^2\delta^2 + 3\delta^4)}{(h_t + \lambda_t\delta^2 + \lambda_t\theta^2)^2} \tag{32}$$

Therefore, the distributions for the mixed GARCH-jump models are leptokurtic when $\lambda_t > 0$ and skewed for $\theta \neq 0$.

Conditional higher-order moments (model implied skewness and kurtosis) of GJR-HT are also obtained by the following equations (For mathematical proof, see Jondeau and Rockinger, 2000).

$$\begin{aligned} SK[z] &= E[z^3] = [m_3 - 3am_2 + 2a^3]/b^3 \\ Ku[z] &= E[z^4] = [m_4 - 4am_3 + 6a^2m_2 - 3a^4]/b^4 \end{aligned} \tag{33}$$

where

$$\begin{aligned} m_2 &= 1 + 3\phi^2 \\ m_3 &= 16c\phi(1+\phi^2)\frac{(\eta-2)^2}{(\eta-1)(\eta-3)} && \text{if } \eta > 3 \\ m_4 &= 3\frac{(\eta-2)}{(\eta-4)}(1 + 10\phi^2 + 5\phi^4) && \text{if } \eta > 4 \end{aligned} \tag{34}$$

Hence, Hansen’s skewed *t*-distribution is also fat-tailed, and is skewed to the left (right) when ϕ is less (greater) than 0.

Table V shows that assuming from fat-tailed distributions is somewhat consistent with the mixed GARCH-jump models over the short term in some degree. However, unlike GJR-HT, even GARJI model could not capture extreme outliers for the Indonesia since jump times and jump sizes are very difficult to estimate precisely. In addition, jump models are hard to explain high peak around zero returns since the role of jumps is to capture infrequent extreme events, which is another important feature of leptokurtosis of empirical return distribution.

Figure 3 present graphical illustrations of the conditional density functions of Hansen's skewed *t* distribution with estimated kurtosis (η) and skewness (ϕ) parameters for standardized innovations of the GJR-HT model. For comparison, each figure additionally provides the standard normal density function and standardized mixed conditional density of JD-GJR, sharing with identical distribution of Maheu and McCurdy's (2004) GARJI model. All three densities are scaled to have zero mean and unit variance for comparison purpose. Standardized version of conditional density function of GARJI model was shown in Equation (15).

The skewed *t*-distribution assigns greater probability mass close to zero (for small returns of either sign) than does the normal and mixture of GARCH and jump distribution. In addition, from the plots, the skewed *t*-model appears to capture skewness when it is present. Furthermore, GJR-HT can match the high conditional kurtosis of returns (tail thickness) documented in the literature for many classes of financial assets for emerging stock markets.

Table V

Model Implied Higher Moments

These tables present the model implied skewness (Panel A) and kurtosis (Panel B) across each different model specification. For the mixed jump-GARCH models (JD-GJR, ARJI-GJR, and GARJI), I report the average values of conditional skewness and conditional kurtosis of the index returns implied by their corresponding models. For comparison, I also provide sample skewness and sample kurtosis in the first columns. The model implied higher moments of each model are computed from:

<u>JD-GJR, ARJI-GJR, and GARJI (Das and Sundaram, 1999)</u>

$$Sk(r_t \mid \Psi_{t-1}) = \frac{\lambda_t(\theta^3 + 3\theta\delta^2)}{(h_t + \lambda_t\delta^2 + \lambda_t\theta^2)^{3/2}} \qquad Ku(r_t \mid \Psi_{t-1}) = 3 + \frac{\lambda_t(\theta^4 + 6\theta^2\delta^2 + 3\delta^4)}{(h_t + \lambda_t\delta^2 + \lambda_t\theta^2)^2}$$

GJR-HT (Jondeau and Rockinger, 2000)

$$SK[z]=E[z^3]=[m_3-3am_2+2a^3]/b^3 \qquad Ku[z]=E[z^4]=[m_4-4am_3+6a^2m_2-3a^4]/b^4$$

where $$a\equiv 4\phi c\frac{\eta-2}{\eta-1}, \qquad b\equiv 1+3\phi^2-a^2, \qquad \text{c}\equiv\frac{\Gamma\left(\frac{\eta+1}{2}\right)}{\sqrt{\pi(\eta-2)}\Gamma\left(\frac{\eta}{2}\right)}$$

$$m_2=1+3\phi^2$$

$$m_3=16c\phi(1+\phi^2)\frac{(\eta-2)^2}{(\eta-1)(\eta-3)} \qquad \text{if } \eta>3$$

$$m_4=3\frac{(\eta-2)}{(\eta-4)}(1+10\phi^2+5\phi^4) \qquad \text{if } \eta>4$$

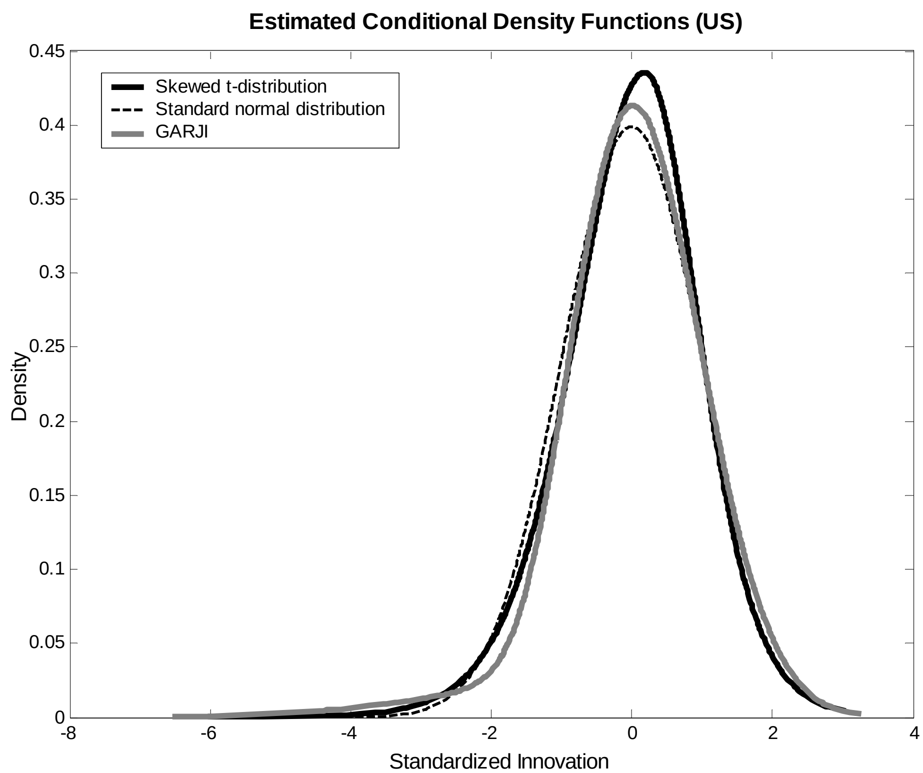
Panel A: Model implied Skewness

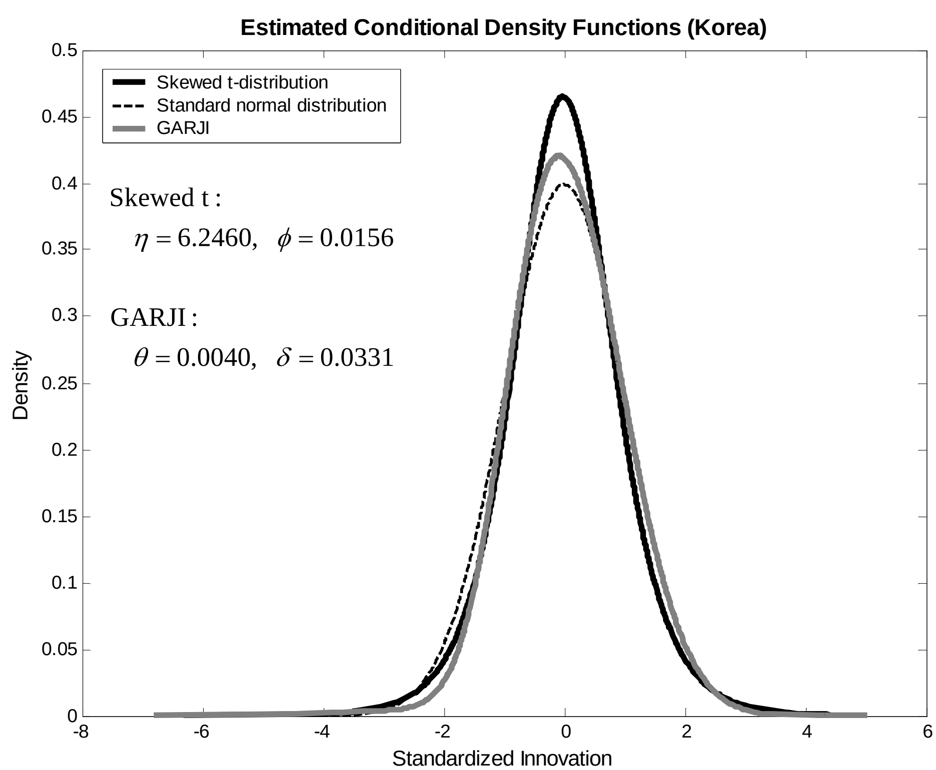
Country	Sample (Unconditional)	JD-GJR	ARJI-GJR	GARJI	GJR-HT
US	-0.1711	-0.2874	-0.3171	-0.3814	-0.8188
Korea	0.3049	0.1169	0.0454	0.0509	0.1077
Indonesia	-0.8631	-0.1131	-0.6081	-0.2321	-0.3120
Mexico	-0.0430	0.0363	-0.1401	-0.0896	0.1853
Brazil	-0.3191	-0.2750	-0.5659	-0.5831	-0.6328

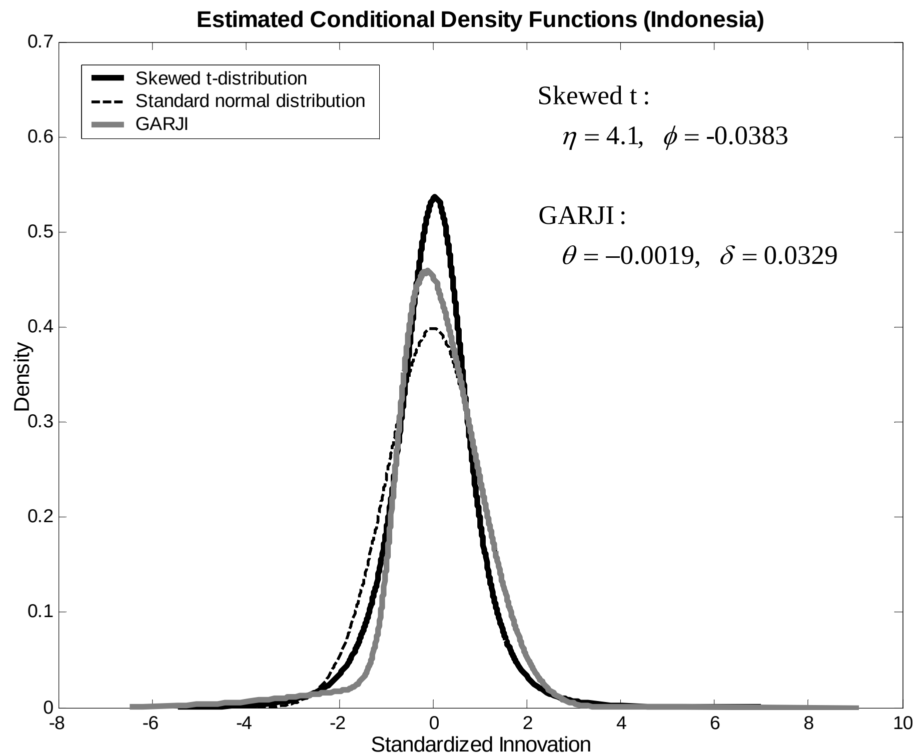
* The sample skewness of Mexico is statistically insignificant at the 5% significance level. (*p*-value: 0.2263)

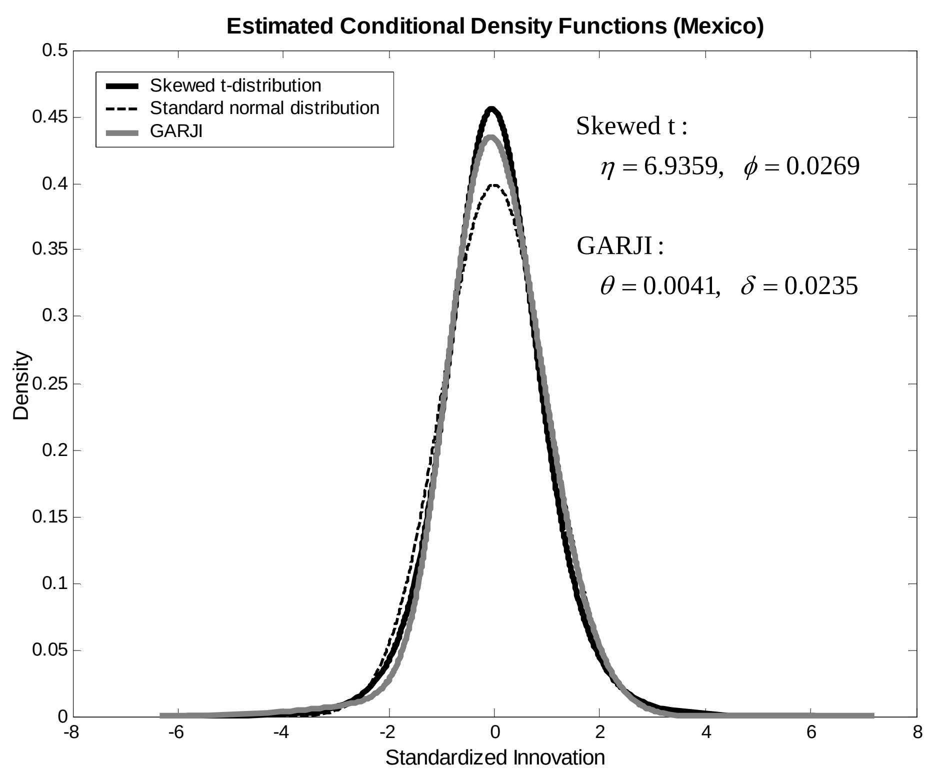
Panel B: Model implied Kurtosis

Country	Sample (Unconditional)	JD-GJR	ARJI-GJR	GARJI	GJR-HT
US	6.1418	3.9561	3.8573	4.0567	4.5321
Korea	11.6586	5.0286	5.9019	6.0683	5.6869
Indonesia	22.1726	6.6469	11.9501	11.7916	23.3423
Mexico	8.9594	4.7625	4.6813	5.4919	5.0790
Brazil	9.1843	4.2302	5.2710	5.4411	5.4844


Estimated Conditional Density Functions (US)
Skewed t-distribution
Standard normal distribution
GARJI
Density
Standardized Innovation
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-8
-6
-4
-2
0
2
4


Estimated Conditional Density Functions (Korea)
Skewed t-distribution
Standard normal distribution
GARJI
Skewed t :
$\eta = 6.2460, \quad \phi = 0.0156$
GARJI :
$\theta = 0.0040, \quad \delta = 0.0331$
Density
Standardized Innovation
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-8
-6
-4
-2
0
2
4
6


Estimated Conditional Density Functions (Indonesia)
Skewed t-distribution
Standard normal distribution
GARJI
Skewed t :
η = 4.1, φ = -0.0383
GARJI :
θ = –0.0019, δ = 0.0329
Density
Standardized Innovation


Estimated Conditional Density Functions (Mexico)
Skewed t-distribution
Standard normal distribution
GARJI
Skewed t :
η = 6.9359, φ = 0.0269
GARJI :
θ = 0.0041, δ = 0.0235
Density
Standardized Innovation

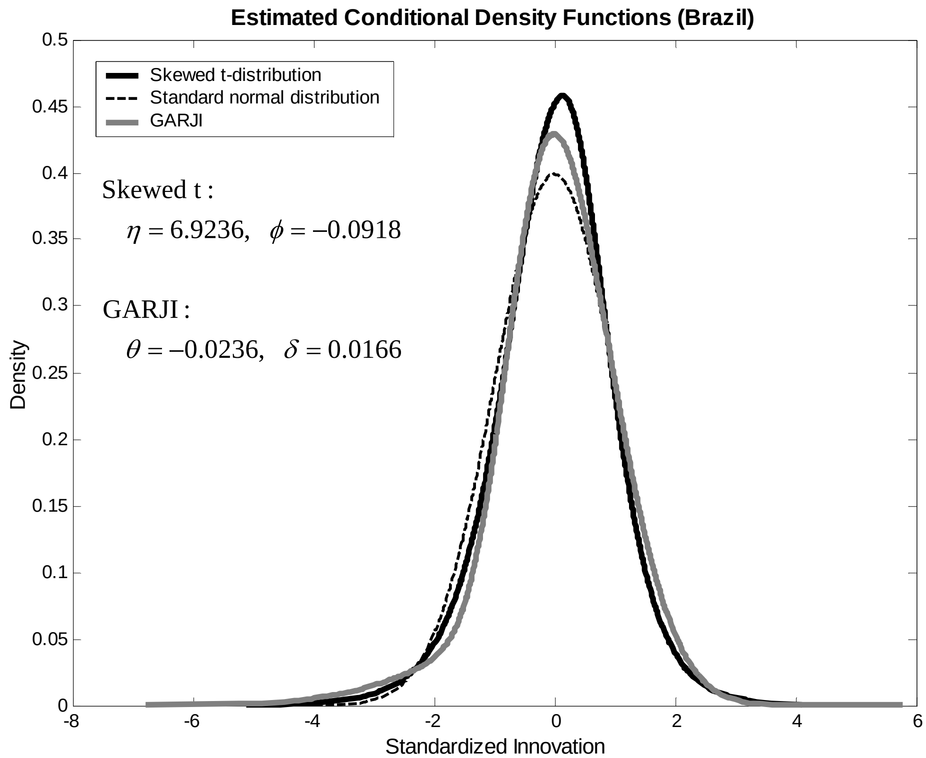


Figure 3. Estimated Conditional Density Functions. The figures present graphical illustrations of the conditional density functions of Hansen's skewed *t* distribution with estimated kurtosis(η) and skewness(ϕ) parameters for standardized innovations of the GJR-HT model. For comparison, each figure additionally provides the standard normal density function and standardized mixed conditional density of JD-GJR, sharing with identical distribution of Maheu and McCurdy's (2004) GARJI model. All three densities are scaled to have zero mean and unit variance.

6. Value-at-Risk Analysis

VaR model has been widely used over the past decades as a primary tool since senior executives could have easily interpreted the financial risk exposures of their corporations using the VaR computations. In addition, VaR is useful since 1) it concentrates on downside risk and subsequently on the behavior of a lower tails for the assumed distribution and 2) accurate estimation of VaR requires precise modeling of the unconditional kurtosis of returns distribution

(For recent evaluations of VaR forecasts at commercial banks, see Berkowitz and O'Brien (2002)).

J.P. Morgan (1996)'s EWMA model (also called RiskMetrics) puts more weight on the more recent observations, and thus takes some account of the dynamic ordering in returns. The formal description can be written as

$$\begin{aligned} \varepsilon_t &= \sqrt{h_t} z_t \ , \qquad\qquad z_t \sim NID(0,1) \\ h_t &= (1-\psi)\sum_{j=1}^{t-1} \lambda^{j-1} r_{t-j}^2 = \psi h_{t-1}^2 + (1-\psi) r_{t-1}^2 \end{aligned} \tag{35}$$

where $\psi = 0.94$ is smoothing constants and governs the degree of dependence of h_t on the past history of r_t^2. Therefore, the larger the value ψ the more weight is placed on past observations and so the smoother the series becomes. The first term, ψh_{t-1}^2, determines the persistence in volatility. The second term, $(1-\psi) r_{t-1}^2$, determines the intensity of reaction of volatility to market events. From Equation (35), I can see that the variance at time *t* is a weighted average of the variance at time *t*-1 and the magnitude of the return at time *t*-1. To compare and evaluate model performances on five different model specifications along with EWMA model, I choose the following criteria. That is, I consider how much each model underestimates the losses since this criterion is reasonable in VaR perspective.

The total numbers of violations (N) and percentages of violations ($Perc$) are computed from:

$$I_{t+1} = \begin{cases} 1, & \text{if } r_{t+1} < VaR_{t+1|t} \\ 0, & \text{if } r_{t+1} \geq VaR_{t+1|t} \end{cases} \tag{36}$$

$$N = \sum_{t=1}^{T} I_{t+1}, \qquad Perc = \frac{N}{T} \times 100\% \tag{37}$$

where

$$VaR_{t+1|t} = F(\alpha) \cdot \sqrt{\hat{h}_{t+1|t}} \tag{38}$$

and $F(\alpha)$ is the corresponding empirical quantile (95th or 99th) of the assumed distribution and $\sqrt{\hat{h}_{t+1|t}}$ is the forecast of the conditional standard deviation at time *t*+1 given the information at time *t*.

Therefore, the VaR forecast is the quantity, $VaR_{t+1|t} = F(\alpha) \cdot \sqrt{\hat{h}_{t+1|t}}$, such that $\Pr\left(r_{t+1} < VaR_{t+1|t}\right) = \alpha$ over the next trading day. Here $\alpha = 0.05$ ($\alpha = 0.01$), so that the model predicts a lower bound on losses not to be exceeded with 95% (99%) confidence (see Berkowitz and O'Brien (2002)).

Therefore, for the GJR-N model, the values of $F(\alpha)$ are always -1.96 (95%) and -2.58 (99%). However, for JD-GJR, ARJI-GJR and GARJI models, $F(\alpha)$ should be computed numerically by integrating their standardized PDF in Equation (15) because cumulative distribution functions (CDF) are unfortunately unknown analytically. For GJR-HT model, $F(\alpha)$ can be obtained by inverting CDF at the probability α computed from Equation (28). Therefore, except GJR-N model, the values of $F(\alpha)$ are always changed depending on data and sample periods since the shapes of distribution functions will be determined by estimated parameter values. This implies that the mixed jump-GARCH models and skewed *t*-distribution allow for asymmetric VaR forecasts and fully takes into account for the fact that the density distribution of asset returns can be substantially skewed.

Table VI

Evaluations of Performances on VaR

The tables compare the total numbers and percentages of violations of 95% and 99% Value-at-Risk (VaR) across the whole sample periods for each different model specification. For comparison, EWMA model created by J. P. Morgan, which is a standard benchmark for risk management models, is also considered. The total numbers of violations (N) and percentages of violations ($Perc$) are computed from:

$$I_{t+1} = \begin{cases} 1, & \text{if } r_{t+1} < VaR_{t+1|t} \\ 0, & \text{if } r_{t+1} \geq VaR_{t+1|t} \end{cases}, \qquad N = \sum_{t=1}^{T} I_{t+1}, \qquad Perc = \frac{N}{T} \times 100\%$$

where $VaR_{t+1|t} = F(\alpha) \cdot \sqrt{\hat{h}_{t+1|t}}$

$F(\alpha)$: the corresponding quantile (95th or 99th) of the assumed distribution

$\sqrt{\hat{h}_{t+1|t}}$: the forecast of the conditional standard deviation at time *t*+1 given the information at time *t*

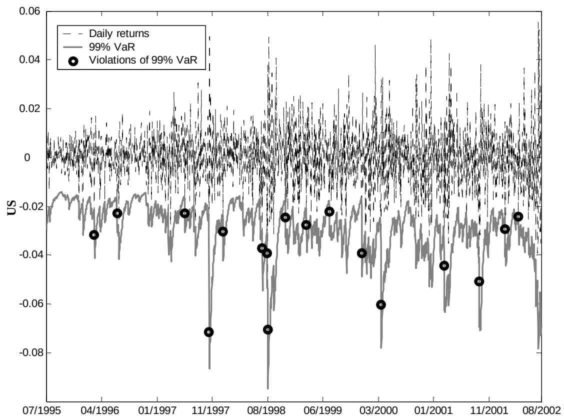
Panel A: 95% Value-at-Risk (VaR)

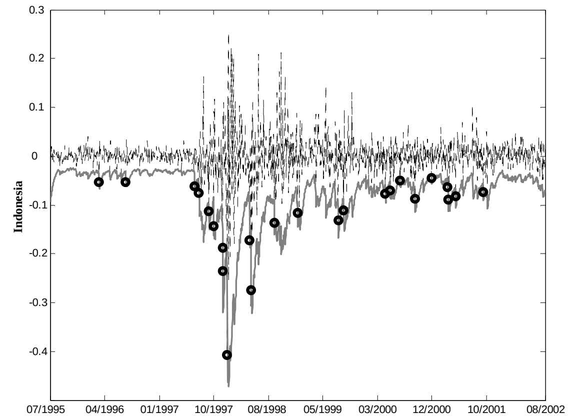
Country	EWMA	GJR-N	JD-GJR	ARJI-GJR	GARJI	GJR-HT
US	67 (3.75%)	55 (3.08%)	73 (4.08%)	62 (3.46%)	61 (3.41%)	45 (2.52%)
Korea	65 (3.65%)	50 (2.80%)	76 (4.26%)	76 (4.26%)	73 (4.09%)	44 (2.47%)
Indonesia	66 (3.64%)	55 (3.03%)	92 (5.07%)	83 (4.57%)	102 (5.61%)	51 (2.81%)
Mexico	64 (3.50%)	43 (2.35%)	75 (4.09%)	66 (3.60%)	59 (3.22%)	43 (2.35%)
Brazil	66 (3.64%)	58 (3.19%)	96 (5.29%)	66 (3.63%)	69 (3.80%)	48 (2.64%)

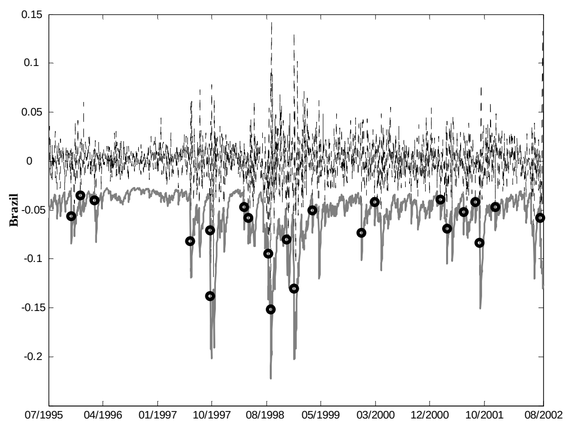
Panel B: 99% Value-at-Risk (VaR)

Country	EWMA	GJR-N	JD-GJR	ARJI-GJR	GARJI	GJR-HT
US	25 (1.40%)	17 (0.95%)	21 (1.17%)	17 (0.95%)	15 (0.84%)	11 (0.62%)
Korea	21 (1.18%)	11 (0.62%)	16 (0.90%)	14 (0.78%)	14 (0.79%)	7 (0.39%)
Indonesia	32 (1.76%)	24 (1.32%)	58 (3.20%)	15 (0.83%)	18 (0.99%)	10 (0.55%)
Mexico	24 (1.31%)	18 (0.98%)	21 (1.15%)	18 (0.98%)	17 (0.92%)	13 (0.71%)
Brazil	34 (1.87%)	22 (1.21%)	40 (2.20%)	19 (1.05%)	18 (0.99%)	12 (0.66%)

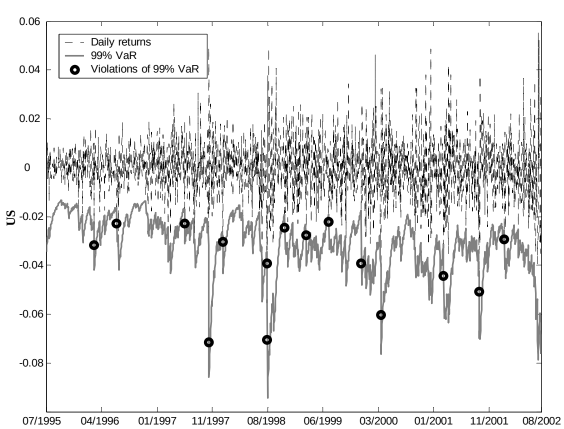
Panel A: Violations of 99% VaR for the GJR-N Model

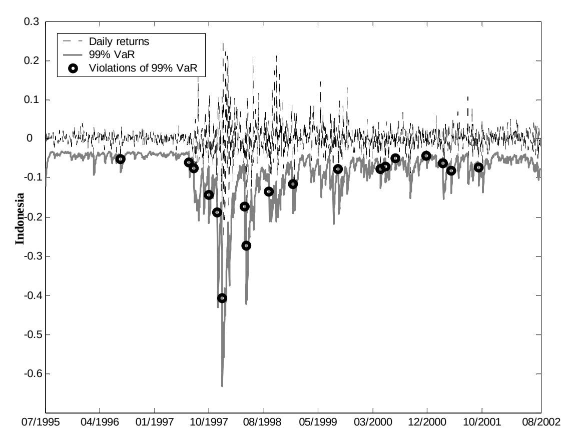


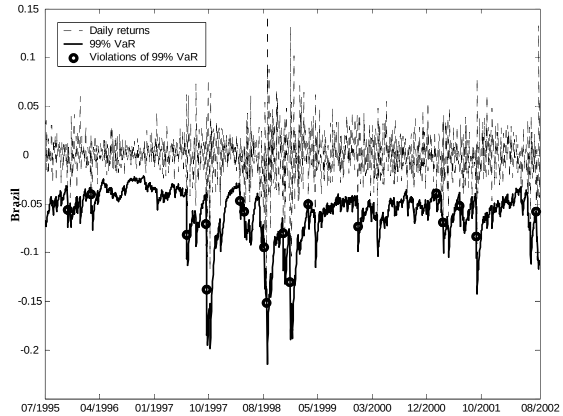




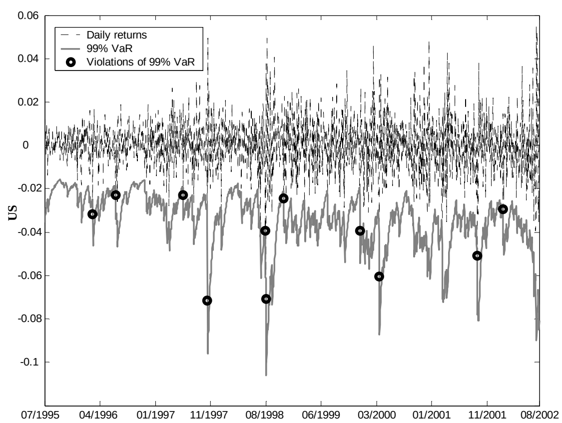
Panel B: Violations of 99% VaR for the GARJI Model

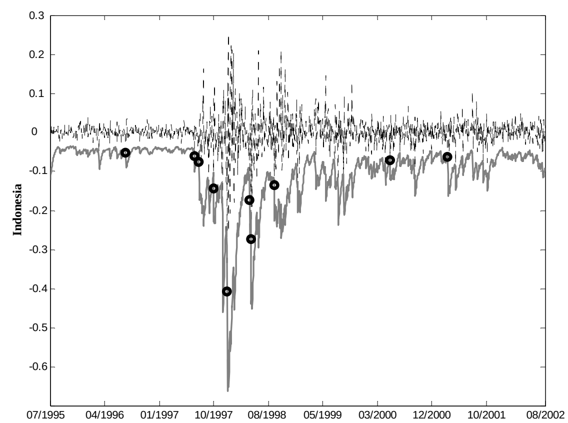






Panel C: Violations of 99% VaR for the GJR-HT Model





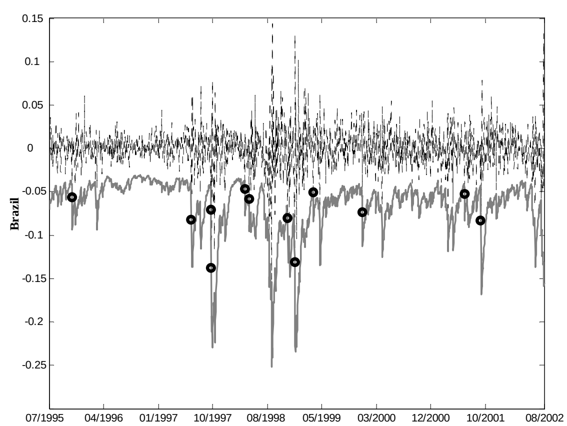


Figure 4. Violations of 99% VaR for GJR-N, GARJI and GJR-HT Model. The figures (Panel A ~ C) present graphical illustrations of continuously compounded daily returns, 99% VaR, and its violations for the GJR-N, GARJI and GJR-HT model.

Figure 4 illustrates the time series of daily returns for the US, Indonesia, and Brazil and corresponding one-day-ahead 99% VaR and its violations for the GJR-N, GARJI, and GJR-HT models. The total number of violations, $N = \sum_{t=1}^{T} I_{t+1}$, may be regarded as a random variable that has a binomial distribution. If the model is correctly specified, $Perc = \frac{N}{T} \times 100\%$ should be less than 5% (95% VaR) and 1% (99% VaR) since model has the tendency to underestimate losses as $Perc$ becomes larger. Consequently, if $Perc$ is greater than thresholds, it strongly implies that VaR does not work well. As I can see from the Table VI and Figure 4, GJR-HT model outperforms all other model specifications and the time-series of VaRs achieve the smallest violation rates of 95% (99%) VaRs compared to other model parameterizations.

7. Why do asymmetric fat-tail distributions outperform the GARCH-Jump models?

To make our analysis manageable, I assume constant volatility and zero skewness to exclude GARCH effects and asymmetric effects from our analysis. In this section, I show that the jump-diffusion model (JD) only capture a small portion of the extreme values and Poisson-type jumps cannot capture the component that is related to extreme (non-jump) realization. For this analysis, I will make use of the boxplot and the probability analysis of Ait-Sahalia (2004). To further support the empirical findings in this chapter, I will perform simulation analysis to show that the skewed *t* model has the flexibility of capturing both jump and non-jump extreme values.

7.1. Boxplot Methods and Probability Analysis

I employ the boxplot (Figure 1) as an informal nonparametric way to detect jumps[8]. Ait-Sahalia (2004) explains the difficulty of inferring jumps from large realized returns in discretely sampled data. Ait-Sahalia discusses the fact that an extreme realization can be a jump or a large realization of the Brownian noise. More importantly, he derives the probability that an observed large value of the return is a jump or a large realization of a diffusion noise. Therefore, I exploit the probability analysis of Ait-Sahalia to determine a threshold for the jumps.

Ait-Sahalia explains that as far into the tail as 3.5 times standard deviation, it is still more likely that a large observed log-return was produced by Brownian noise only. He emphasizes the fact that is difficult to rely on large observed returns as a means of identifying jumps. In addition, he also shows, among others, the probability of a jump as it relates to the size of the observed returns, whereby the size of the returns is expressed in terms of the number of standard deviations. From Ait-Sahalia' analysis, I can observe a large value 5 or 6 times the standard deviation that it is a jump almost surely. Therefore, the values greater than the threshold value of 5 times standard deviation definitely can be considered as jumps. For example, Ait-Sahalia shows that when one observes a value of 10% for the daily absolute returns, there is a 60% probability that this value was generated by a jump.

To examine whether Poisson-based Jump-diffusion can capture all the jumps that lead to fat tails, I use 5 times standard deviation as the threshold to identify the jumps in daily Indonesia returns. Using 5 times standard deviation allows us to capture only few extremely large jumps.

[8] There are many formal nonparametric ways to capture jumps including the bi-power variation method, the wavelet method, the median approach, and the variance swap strategy. However, the informal and adhoc boxplot way of capturing extreme values as in Figure 1 also serves very well for our purpose to show why it fails to capture the excess kurtosis in emerging stock markets. However, the choice of the threshold for extreme values is somewhat arbitrary, although it is derived from the probability analysis of Ait-Sahalia (2004), and that it will be only used for illustration purposes and not as any formal tests.

Since this threshold is based on the probabilities derived from the Poisson based jump-diffusion model with constant volatility in Ait-Sahalia, this result clearly underscores the fact the Poisson based jump-diffusion model only captures the extreme large jumps.

To double check this empirical finding, I also count how many observations should be considered as jumps based on 5 times standard deviation threshold value. For Indonesia, there are exactly 13 jumps based on 5 times standard deviation threshold during our sample period from 07/05/1995 to 08/07/2002. Next, I determine how many jumps are recorded by the JD model when applied to Indonesia. According to the estimation results of JD model, the annualized jump intensity parameter has a value of 0.3658 for Indonesia. Therefore, this value is equivalent to one large jump per about 2.73 years. Finally, I compare the boxplot number of jumps based on 5 times standard deviation threshold value with the number of jumps based on JD model. Indonesia should have no more than 3 jumps during our sample periods (7.2 years) based on JD model. Therefore, it appears that JD model underestimates the number of jumps compared to boxplot method.

7.2. Simulations

To examine whether Hansen's skewed *t*-distribution has the flexibility to exhibit jumps depending on the degree of freedom parameter, I use simulated data derived from skewed *t*-distribution based on the following three categories. First, I consider very low levels of degree of freedom (df = 2.5 and df = 4) since extreme leptokurtosis can be frequently observed in the emerging stock markets such as Indonesia where severely affected by Asian currency crisis. Second, I consider a medium value of degree of freedom (df = 7) to capture moderate leptokurtosis which can be observed in most of emerging markets and developed countries.

Finally, I employ very high levels of degree of freedom (df = 15 and df = 30) which resemble normal innovations to show difficulty in distinguishing small jumps from large normal innovations based on jump-diffusion model.

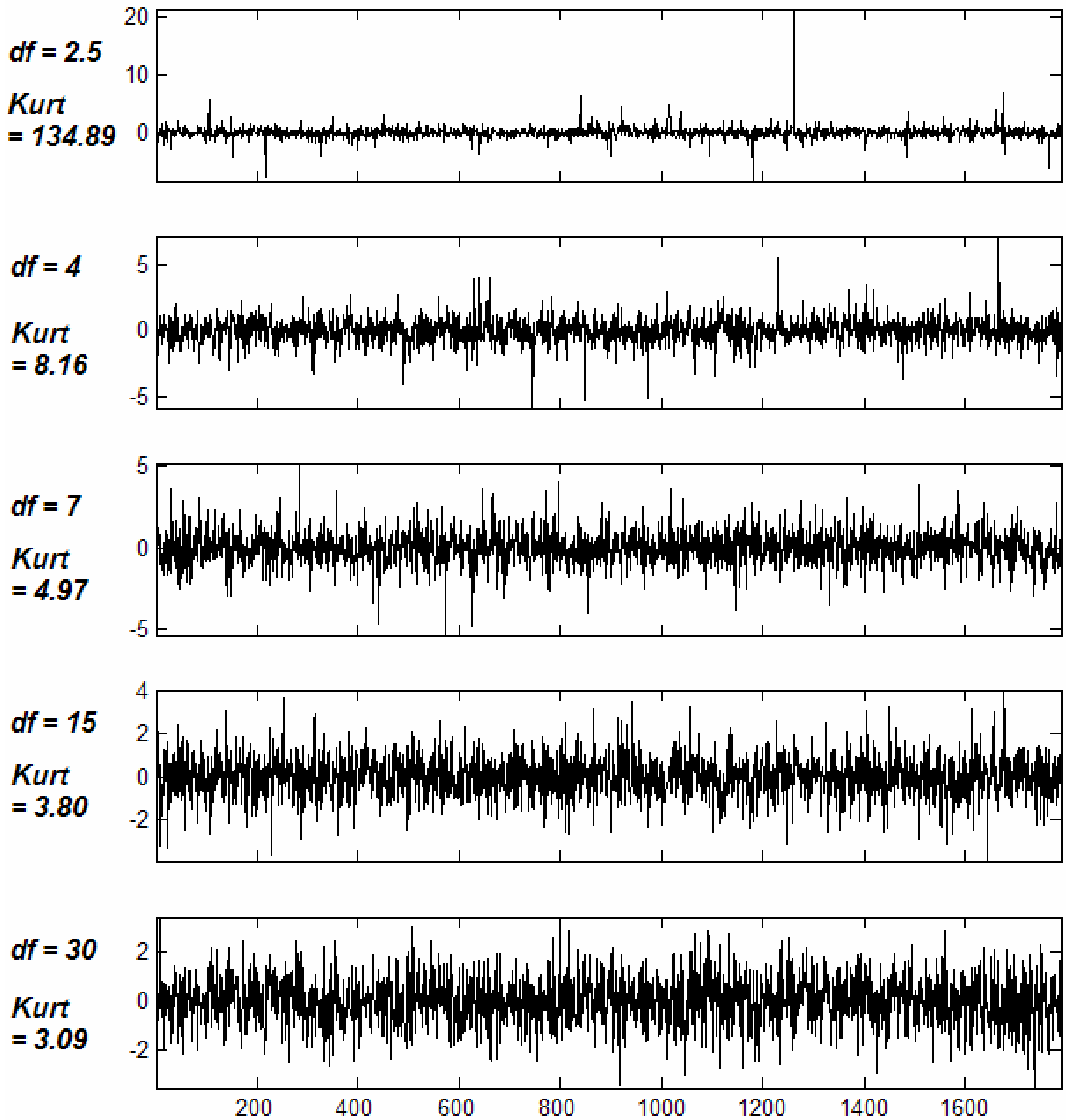


Figure 5. Simulations of Hansen's skewed *t*-distribution. The figures present simulated time-series plots derived from Hansen's skewed *t*-distribution depending on the different degrees of freedom parameters. All of the plots are scaled to have zero skewness and unit standard deviation to exclude asymmetric effects and GARCH effects.

Table VII

Parameter Estimates for Jump-Diffusion Model based on Simulated Time-Series Data derived from Hansen's skewed *t*-distribution

The table compares the parameter estimates through maximum likelihood estimates (MLE) for jump-diffusion model based on simulated time-series data derived from Hansen's skewed *t*-distribution depending on the different degrees of freedom parameters. The *p*-values are reported in square brackets.

	Extreme Leptokurtosis (Emerging Markets)		Moderate Leptokurtosis (Developed Countries)	Normal Innovations	
	df = 2.5	df = 4	df = 7	df = 15	df = 30
μ	0.0057	-0.0001	0.0000	-0.0001	0.0006
	[0.0000]	[0.4678]	[0.3685]	[0.2372]	[0.0094]
σ	0.0035	0.0064	0.0067	0.0076	0.0080
	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[0.0000]
λ	0.6547	0.4396	0.5947	0.5394	0.6862
	[0.0000]	[0.0000]	[0.0000]	[0.0014]	[0.0000]
θ	0.0085	0.0001	0.0001	-0.0004	-0.0003
	[0.0000]	[0.4820]	[0.4704]	[0.2006]	[0.2735]
δ	0.0190	0.0101	0.0100	0.0083	0.0070
	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[0.0000]

If we assume df = 2.5 or df = 4, skewed *t*-distributions are capable of capturing extreme realizations up to excess kurtosis 134.89 and 8.16, respectively[9]. It appears that skewed *t*-simulated data mimic daily stock returns data very well, showing several jumps-looking extreme realizations, which can be frequently observed in emerging financial markets. It is almost impossible for us to distinguish these plots from actual daily returns or simulated time-series from Poisson-type JD model. Especially, in the extreme case, if I assume df = 2.5, simulated

[9] Note that in this analysis, the shapes of plots and the values of excess kurtosis will be changed per each simulation. However, the important statistical feature simulated from skewed *t*-distribution will remain the same.

time-series data from skewed *t*-distribution are capable of representing 21.35 times standard deviation.

Since most countries will have larger values of degree of freedom, this simulation result strongly implies that we can capture leptokurtosis of daily returns almost surely using the flexibility of degree of freedom of skewed *t*-distribution. More realistically, simulated time-series based on df = 4 can capture infrequent, extreme realizations up to 7.23 times standard deviation. Now I perform similar analysis to compare the above simulation results with others based on df = 7, df = 15, and df = 30, then I compare the estimation results of JD model using the above simulated data. The estimation results are reported in Table VII.

8. Summary and Concluding Remarks

In this study, I have shown that for the US and emerging stock markets simple GJR-N model and even more sophisticated model incorporating time-varying jumps such as Maheu and McCurdy's (2004) GARJI model failed to capture these extremely fat tails of conditional return distributions, especially for the Indonesia. However, unlike other model specifications, GJR-HT model was capable of capturing leptokurtosis represented by both extreme outlying observations and high peaks of asset returns distribution, which cannot be adequately captured by a time-varying conditional variance and mixed GARCH-jump models including recently proposed GARJI models.

Using boxplot methods and simulations, I have also shown that skewed *t*-distribution is capable of capturing extreme realizations by allowing the flexibility of degree of freedom (df) parameter. As evidenced by the estimation results of jump-diffusion model based on simulated data from skewed *t*-distribution using different values of degree of freedom, skewed *t*-

distribution can mimic jump-looking abnormal and large events quite well. In addition, as the values of degree of freedom parameter increase, jump variability parameters in the estimation results of jump-diffusion model decrease, although jump intensity parameters are still non-zero significant, implying that jump-diffusion model cannot disentangle small jumps from large normal innovations. Therefore, jump-diffusion estimation tends to misguide us by providing us with spurious values of jump parameters.

Therefore, asymmetric non-normal distributions, especially Hansen's skewed *t*-distribution, provide better estimation results than the Gaussian and mixed jump-diffusion distribution do. As expected, the benefits of estimating GARCH models using asymmetric leptokurtic distributions are more substantial for highly volatile series such as emerging stock markets, which have a higher degree of non-normality. In addition, skewed *t*-distribution also provides me an excellent risk management tool evidenced by VaR analysis.

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Chapter 2

An Examination on the Roles of Diffusions and Stochastic Volatility in the Exponential Lévy Jumps Models

1. Introduction

Academic research on financial modeling based on Lévy processes has been extensive[10] even though independence of increments is not a property observed in historical time series of returns such as volatility clustering. They allow flexible modeling of tail behavior at various time scales by generalizing the Black-Scholes model by introducing jumps but conserving independence of log-returns. It is very well known that the traditional Black-Scholes option pricing model assumes that returns follow Brownian motion, but actual return processes differ from this benchmark in several ways. For example, Carr and Wu (2004) summarize three important deviations of actual returns distributions from traditional Black-Scholes' assumptions – jumps which lead to non-normal return innovations, time-varying return volatilities, and negative correlation between equity returns and their stochastic volatilities.

To address these issues simultaneously, at least two research endeavors have been made to incorporate a stochastically changing volatility effect. First, the constant volatility parameter of Black-Scholes model is made to be stochastic in a proper manner (Hull and White (1987); Heston (1993)) and their volatility process is driven by a Brownian motion. This idea can be easily extended to the case of more general infinite activity pure jump Lévy processes[11] such as

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[10] For comprehensive textbook reviews of Lévy processes on historical financial time-series and pricing financial derivatives, refer to Schoutens (2003) and Cont and Tankov (2004).

[11] In contrast to a standard Poisson or compound Poisson process, pure infinite activity jump process has an infinite number of jumps over any time interval, allowing it to capture the extreme activity traditionally handled by diffusion

symmetric variance-gamma (VG) (Madan and Seneta (1990)), and its asymmetric extension (Madan and Milne (1991); Madan et al. (1998)), normal inverse Gaussian (NIG) (Barndorff-Nielsen (1998)), and its generalization to the generalized hyperbolic class (Eberlein et al. (1998)) and generalization of the VG model (CGMY model; Carr et al. (2002)). The main weakness of these homogeneous Lévy processes is that volatility is assumed to be constant and does not change stochastically over time.

Second, to capture the evidence on stochastic volatility, Carr et al. (2003) and Carr and Wu (2004) proposed a stochastic time change to the Lévy processes. This amounts to increasing or decreasing the level of uncertainty by speeding up or slowing down the rate at which time passes. According to the Brownian motion scaling property[12], random changes in volatility can alternatively be captured by changes in time after I employ a mean-reverting positive process as a measure of the local rate of time change such as the Ornstein-Uhlenbeck (OU) process or the classical Cox-Ingersoll-Ross (CIR) process.

Empirical work has supported the need for both stochastic volatility to calibrate the longer maturities and jumps to reflect shorter maturity option prices (See Carr et al. (2003); Carr and Wu (2004); and Huang and Wu (2004)). More specifically, while jumps are necessary to explain the variation in strike at shorter terms, stochastic volatility appears to explain the variation in strike of option prices at longer terms. In addition, unlike approximately uncorrelated stock returns, their volatility exhibits strong serial dependence. The time-varying

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processes. Most of jumps are very small and may be regarded as approximating the transition from one decimalized price to another one nearby (Carr and Wu (2003)).

[12] There is a well-known set of transformations of Brownian motion which produce another Brownian motion. One of these is the scaling property which says that if $W = \{W_t,\ t \geq 0\}$ is a Brownian motion, then, for every $c \neq 0$, $\tilde{W} = \{\tilde{W}_t = cW_{t/c^2},\ t \geq 0\}$ is also a Brownian motion. (For an overview of the classical properties of Brownian motion, refer to Schoutens (2003).)

term structure of the implied volatility also suggests that stochastic volatility still remains under the risk-neutral measure.

However, most recent papers have mainly focused on time-changed volatility combined with infinite activity Lévy jump model to tackle this difficulty. Therefore, in this paper, I apply Heston (1993) stochastic volatility process to variance-gamma (VG) process proposed by Madan and Milne (1991) and Madan et al. (1998) to see if this specification improves fits of financial time-series data and reduces option pricing errors. Since the characteristic function is known analytically, option prices can be readily computed using efficient fast Fourier transform (FFT).

Although some of theoretical/empirical approaches to combine Heston (1993) stochastic volatility with Lévy processes have been proposed for the Finite Moment Log Stable (FMLS) process (Carr and Wu (2003)) and the Meixner process (Schoutens (2003)), none of the authors report empirical performance on both historical time-series underlying asset returns and options pricing and Schoutens (2003) only focuses on calibration of option prices to estimate parameters to compare pricing errors among competitive model specifications.

However, I should distinguish statistical models (under a probability measure $\mathbb{P}$) for financial time series returns behavior of underlying assets from option pricing models (under a risk-neutral measure $\mathbb{Q}$) for risk-neutral dynamics. For example, although the processes defined by measures $\mathbb{P}$ and $\mathbb{Q}$ share the identical paths, they can have quite different analytical and statistical properties (Cont and Tankov (2004)). Formally, if $\mathbb{P}$ defines a Lévy process X, the process Y defined by $\mathbb{Q}$ is not necessarily a Lévy process. That is, it may have increments which are neither independent nor stationary. Recall that a Lévy process is defined as a continuous in probability, càdlàg (right continuous left limit) stochastic process $x(t)$, $t > 0$ with independent and stationary increments and $x(0) = 0$.

Therefore, in this second chapter, I will assume structure-preserving changes of measure so that X and Y are both Lévy processes and the equivalence of their measures gives one-to-one correspondence between their parameters for risk-neutral pricing. If so, I can price any financial instruments in an arbitrage-setting under an appropriate risk-neutral measure since the price of a derivative security can be expressed as an expectation of discounted payoffs, which become martingales. In addition, it allows me to estimate parameters and analyze their properties using both historical financial time-series underlying asset returns and option prices.

Using option prices for calibration to estimate parameters gives me flexible tools to examine different model specifications by comparing measures of pricing errors such as mean squared errors (MSE), mean absolute errors (MAE), mean absolute relative errors (MARE), and root mean squared errors (RMSE). In addition, their estimated parameters using option prices are more informative since the parameters coming out of the calibration procedure by minimizing sum of squared errors between the market and model prices resemble the current view on the market. This is the main rationale why many authors do not explicitly take into account any historical data of underlying assets.

Much cares, however, have to be taken when I use option prices to compare model performances. One of pitfalls of this procedure is that pricing errors, in general, will be reduced as the number of parameters increases. This implies that more complicated model specifications incorporated with jumps and stochastic volatility will frequently outperform more parsimonious ones. It makes me challenged to empirically investigate the statistical and economic roles of diffusions, jumps, and stochastic volatility parameters. Carr et al. (2002) show that a diffusion parameter is statistically insignificant for many financial assets if infinite activity Lévy jumps are included using both time series on 13 individual stock prices and 5 closing option prices on 5 underlying assets, AMZN, IBM, INTC, MSFT, and SPX. Eraker (2004) finds that complex jump specification with state-dependent arrival intensity fits better in options and returns data simultaneously, but it adds little explanatory power in fitting only options data. Therefore, analyzing both time-series underlying asset returns and option prices is necessary and required to draw a full picture for my purpose.

In this chapter, I will answer to the following empirical questions; Does stochastic volatility perform differently between returns and options? Unlike a diffusion term such as CGMYe model (Carr et al. (2002)), does stochastic volatility still remain significant in EVGSV and SVJ models? Does stochastic jump volatility have the same impact as stochastic diffusion volatility? Do my empirical results coincide with the hypothesis by Carr and Wu (2003)? Carr and Wu (2003) provide a comprehensive analysis of when infinite activity jump and diffusions should matter. The main contribution of the infinite activity process should be primarily on the short term OTM options. So, I expect diffusions with constant volatility to give a substantial contribution for ATM options and diffusions with stochastic volatility to improve the pricing of both ATM and longer term maturity OTM.

This chapter contributes to the literature in three ways by providing comprehensive analysis of empirical performance on different model specifications using both returns and options together. First, I extend Carr et al. (2002)'s work, which showed unimportance of a diffusion parameter when it is incorporated with infinite activity of Lévy jumps process such as CGMY model, to see if similar conclusion can be made for the stochastic volatility. To achieve my goal, I examine the role of stochastic volatility when it is taken into account with infinite/finite activity pure jump Lévy processes. It is well-known that stochastic volatility significantly improves fits of data when it is incorporated with compound Poisson jump process of Merton (1976) which has finite activity of jumps (Bates (1996, 2000); Bakshi et al. (1997); Duffie et al. (2000)). On the contrary, I confirm that stochastic volatility does not play a key role when incorporated with infinite-activity Lévy type pure jump models such as variance-gamma and normal inverse Gaussian processes to model high and low frequency historical time-series SP500 index returns.

Second, I show that unlike SP500 index returns, stochastic diffusion volatility with infinite-activity Lévy jumps processes considerably reduces SP500 index call option in-sample and out-of-sample pricing errors of long-term ATM and OTM options, which contributed to a substantial improvement of pricing performances of SVJ and EVGSV models, compared to constant volatility Lévy-type pure jumps models and/or stochastic volatility models without jumps. Therefore, I also figure out why stochastic volatility performs differently depending on infinite/finite activity of jumps specifications.

Finally, I find that whether sources of stochastic volatility are diffusions or jumps are not relevant to improve the overall empirical fits of SP500 index returns. Interestingly, I also reveal that unlike asset returns, whether pure Lévy jumps specifications are finite or infinite activity is not an important factor to enhance option pricing model performances once stochastic volatility is incorporated.

This chapter is organized as follows. In Section 2, I set up the models and characteristic functions along with important econometric properties considered in this paper. Then, I describe returns and options data briefly in Section 3 and illustrate FFT estimation methodologies proposed by Carr and Madan (1999) in Section 4 to estimate parameters and plot density fits of underlying asset returns using characteristic functions of each model specification. Section 5 analyzes estimation results and compares model performances based on likelihood ratio tests along with information criteria for spot index returns and a variety of measures of in-sample and out-of-sample pricing errors for SP 500 index options. Section 6 concludes and suggests subjects for future research.

2. Lévy Model Specifications and their Characteristic Functions

2.1.   Pure Diffusions and Stochastic Volatility Models

2.1.1.   Black-Scholes-Merton Model

The characteristic function for an exponential Brownian motion with volatility σ, which is the only continuous Lévy process generated by a normal distribution, is given by

$$\phi_X(u) = E\left[e^{iuX_t}\right] = \exp\left\{-\frac{1}{2}u(u+i)\sigma^2 t\right\} \tag{39}$$

This result can be obtained by performing the integration explicitly or directly from the Lévy-Khintchine representation, which justifies that any Lévy process can be expressed as the sum of a linear drift term, a Brownian motion and a jump process. In generalized multivariate setting on Lévy-Khintchine theorem, the characteristic function of X, has the form

$$\phi_X(\theta) = E\left[e^{i\theta^T X}\right] = e^{-t\Psi_x(\theta)}, \qquad t \geq 0 \tag{40}$$

where the characteristic exponent $\Psi_x(\theta)$, $\theta \in \mathbb{R}^d$, is given by

$$\Psi_x(\theta) = -iw^{\mathrm{T}}\theta + \frac{1}{2}\theta^{\mathrm{T}}\Sigma\theta + \int_{\mathbb{R}_0^d}\left(1 - e^{i\theta^{\mathrm{T}}x} + i\theta x 1_{|x|<1}\right)\Pi(dx) \tag{41}$$

The Lévy process X is specified by the vector $\mathbb{R}^d$, the positive semi-definite matrix Σ on $\mathbb{R}^{d\times d}$, and the Lévy measure Π defined on $\mathbb{R}_0^d$ ($\mathbb{R}_0^d \overset{d}{=} \mathbb{R}^d \setminus \{0\}$), satisfying

$$\Pi(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}_0^d} (|x|^2 \wedge 1) \Pi(dx) < \infty \tag{42}$$

The triplet (μ, Σ, Π) is referred to as the Lévy characteristics of X. Since a Lévy process is uniquely characterized by its triplet of Lévy characteristics (μ, Σ, Π), the Lévy process is determined by individual specification of the components of this triplet. The first component μ is the constant drift term. This component is often determined by no-arbitrage or equilibrium pricing relations and thus depends on the specification of the other two elements of the triplet. To get the drift parameter μ, I restrict $\phi_T(-i) = E\left[e^{X_t}\right] = 1$ by imposing that the risk-neutral expectation of the stock price be the forward price with the assumption of zero interest rates and dividends. The second component Σ denotes the constant covariance matrix of the continuous components of the Lévy process. Finally, the third component is the Lévy measure $\Pi(dx)$, which describes the jump structure and controls the arrival rate of jumps of every possible size (x) for each component X. By definition, it is obvious that this third jump component, Π, should be orthogonal to the second diffusion component, Σ (Carr and Wu (2004)).

2.1.2. Stochastic Volatility (SV) Model

The SV model allows the instantaneous variance $v_t = \sigma^2$ to be stochastic. This process is not a Lévy process since its increments are path-dependent. As in Heston (1993), I assume that the instantaneous variance follows a mean reverting square root process

$$\begin{aligned} \frac{dS}{S} &= \mu dt + \sqrt{V_t} dW_t \\ dV_t &= k(\theta - V_t) dt + \sigma_V \sqrt{V_t} dZ_t \end{aligned} \tag{43}$$

where Z_t is another Brownian motion, possibly correlated with the Brownian motion W_t in the return process: $E\left[dW_t dZ_t\right] = \rho dt$. Parameter κ measures the speed of mean reversion, θ is the average level of volatility and σ_ν is the volatility of volatility. Then, the characteristic function of $r(t) = \ln\left[\frac{S(t)}{S(0)}\right]$ is

$$\phi(r(V,t),t;u) = \exp\left[C(u,t) + D(u,t)V(0)\right] \tag{44}$$

where

$$\begin{aligned} C(u,t) &= (iu)\mu t + \frac{\theta\kappa}{\sigma_\nu^2}\left(\left(\kappa - iu\rho\sigma_\nu - \alpha\right)t - 2\ln\left(\frac{1-\beta e^{-\alpha t}}{1-\beta}\right)\right) \\ D(u,t) &= \frac{\left(\kappa - iu\rho\sigma_\nu - \alpha\right)\left(1-e^{-\alpha t}\right)}{\sigma_\nu^2\left(1-\beta e^{-\alpha t}\right)} \end{aligned} \tag{45}$$

with

$$\alpha = \sqrt{\left(iu\rho\sigma_\nu - \kappa\right)^2 - \sigma_\nu^2\left[-iu - u^2\right]}$$

$$\beta = \frac{\kappa - iu\rho\sigma_\nu - \alpha}{\kappa - iu\rho\sigma_\nu + \alpha}$$

The Heston characteristic function can be derived as follows. By definition, the characteristic function is given by $\phi_X(u) = E\left[e^{iuX_t} \left| X_0 = 0\right.\right]$. The probability of the final log-stock price X_T being greater than the strike price is given by

$$
\begin{aligned}
\mathbf{Pr}\left(x_t > x\right) &= P_0\left(x, V, t\right) \\
&= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty dk \,\mathrm{Re}\left\{\frac{\exp\left\{C\left(k,t\right) + D(k,t)V(0) + ikx\right\}}{ik}\right\}
\end{aligned} \tag{46}
$$

where $x = \ln\left(S_t / K\right)$. Let the log-strike y be defined by $y = \ln\left(K / S_t\right) = -x$. Then, the probability density function $p(y)$ must be given by

$$
\begin{aligned}
p(y) &= -\frac{\partial P_0}{\partial y} \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty dk \exp\left\{C\left(k,t\right) + D(k,t)V(0) + iky\right\}
\end{aligned} \tag{47}
$$

Then,

$$
\begin{aligned}
\phi(r(V,t),t;u) &= \int_{-\infty}^\infty dy p(y) e^{iuy} \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty dk \exp\left\{C\left(k,t\right) + D(k,t)V(0)\right\}\int_{-\infty}^\infty dy e^{i(u-k)y} \\
&= \exp\left\{C\left(u,t\right) + D(u,t)V(0)\right\}
\end{aligned} \tag{48}
$$

2.2. Finite Activity Jumps with/without Diffusions and Stochastic Volatility Models

2.2.1. Jump-diffusion Model (JD)

Under the JD model (Merton (1976)), the log return has both a diffusion component and a compound Poisson jump component, where the jump size (*J*) is assumed to be log-normally distributed with mean log-jump κ and standard deviation δ. The stock price follows the SDE

$$
dS = \mu S dt + \sigma S dZ + (e^{\kappa + \delta\varepsilon} - 1) S dq \quad \text{with } \varepsilon \sim N(0,1) \tag{49}
$$

Then, Lévy measure is derived as $\pi(dx) = \lambda dF(x) = \frac{\lambda}{\sqrt{2\pi\delta^2}} \exp\left\{-\frac{(x-\alpha)^2}{2\delta^2}\right\}$ since the compound Poisson jump process of Merton (1976) exhibits a finite number of jumps within any finite time interval, $\int_{\mathbb{R}_0} \pi(dx) = \lambda < \infty$ where λ is the Poisson intensity,[13] defined as the number of jumps per year of the Poisson process.

By applying the Levy-Khintchine representation (2)-(3), the closed-form characteristic function is given by

$$
\begin{aligned}
\phi_X(u) &= \exp\left\{iu\omega t - \frac{1}{2}u^2\sigma^2 t + t\int [e^{iux} - 1]\pi(dx)dx\right\} \\
&= \exp\left\{iu\omega t - \frac{1}{2}u^2\sigma^2 t + t\int [e^{iux} - 1]\frac{\lambda}{\sqrt{2\pi\delta^2}} \exp\left\{-\frac{(x-\kappa)^2}{2\delta^2}\right\}dx\right\} \\
&= \exp\left\{iu\omega t - \frac{1}{2}u^2\sigma^2 t + \lambda t\left(e^{iu\kappa - u^2\delta^2/2} - 1\right)\right\}
\end{aligned}
\tag{50}
$$

and convexity correction factor, $\omega = -\frac{1}{2}\sigma^2 - \lambda\left(e^{\kappa+\delta^2/2} - 1\right)$, can be obtained by imposing $\phi_T(-i) = 1$ so that $\exp\left\{wt + \frac{1}{2}\sigma^2 t + \lambda t\left(e^{\kappa+\delta^2/2} - 1\right)\right\} = 1$.

[13] As Carr and Wu (2004) explain, it is possible to think of any distribution, $F(x)$, for the jump size under the compound Merton (1976)'s Poisson specification. For example, the Lévy measure of Kou (2002)'s a double-exponential conditional distribution for the jump size can be derived by

$$\pi(dx) = \lambda dF(x) = \lambda \frac{1}{2\eta} \exp\left(-\frac{|x-k|}{\eta}\right)dx$$

where the jump size has asymmetric double exponential distribution

$$\psi(J) = p\frac{1}{\eta_u}e^{-\frac{1}{\eta_u}J}\mathbf{1}_{\{J\geq 0\}} + q\frac{1}{\eta_d}e^{\frac{1}{\eta_d}J}\mathbf{1}_{\{J<0\}}$$

with $0 < \eta_u < 1$, $\eta_d > 0$ are means of positive and negative jumps, respectively. p and q represent the probabilities of positive and negative jumps satisfying $p,\ q \geq 0$ and $p + q = 1$.

2.2.2. Stochastic Volatility with Jumps (SVJ) Model

The SVJ model (Bates (1996); Bakshi et al. (1997)) can be derived by adding a Merton's lognormally distributed jump process to the Heston SV model specification. The characteristic function for this SVJ model is simply the product of Heston's SV and Merton's jump characteristic functions.

$$\begin{aligned}\phi(r(V,t),t;u) &= \exp\left[C(u,t)+D(u,t)V(0)\right] \\ &\times \exp\left\{\left[-\lambda iu\left(e^{\kappa+\delta^2/2}-1\right)+\lambda\left(e^{iu\kappa-u^2\delta^2/2}-1\right)\right]t\right\}\end{aligned} \tag{51}$$

where $C(u,t)$ and $D(u,t)$ are defined in (6).

2.3. Infinite Activity Pure Jumps Process with/without Diffusions and Stochastic Volatility Models

2.3.1. Variance-Gamma (VG) Model[14]

A VG process $X\left(t;\sigma_j,\nu,\theta\right)$ is a random time changing Brownian motion with tempered 0-stable subordinator with unit mean rate which is a pure jump process. A VG model for stock movements leads to an incomplete market and therefore the existence of many equivalent martingale measures since perfect hedging is impossible in the presence of a large number of very small jumps of varying sizes. Therefore, I need correction term, w, to obtain an equivalent martingale measure by imposing $\mu = r + w$ where μ is a drift parameter and r is a

[14] For robust check, an additional infinite-activity pure Lévy process such as the normal inverse Gaussian (NIG) is also considered. The characteristic function for NIG process is defined as

$$\phi_X(u) = \exp t\left(iuw + \frac{1}{\kappa} - \frac{1}{\kappa}\sqrt{1+\kappa\sigma^2u^2-2i\theta\kappa u}\right)$$

where convexity correction term to be martingale is $w = \frac{1}{\kappa}\left\{\sqrt{1-\sigma^2\kappa-2\theta\kappa}-1\right\}$. The diffusion and stochastic volatility parameters can be easily incorporated into this characteristic function specification of NIG process.

continuously-compounding risk-free interest rate assumed to be zero in this paper. Therefore, VG model can be defined as:

$$S(t) = S(0)\exp\left[wt + X\left(t;\sigma_j,\nu,\theta\right)\right] \tag{52}$$

where wt is the negative of the logarithm of the VG characteristic function, $\phi_X(u,t)$, evaluated at $1/i$.

$$w = -\frac{1}{t}\ln\left[\phi_X\left(u,t\right)\big|_{u=1/i}\right] \triangleq \frac{1}{\nu}\ln\left(1-\theta\nu-\sigma_j^2\nu/2\right) \tag{53}$$

where $i=\sqrt{-i}$ is an imaginary number and $\phi_X(u,t)$ is given by

$$\phi_X(u,t) = E\left\{\exp\left[iuX(t)\right]\right\} = \left(\frac{1}{1-i\theta\nu u+\sigma_j^2u^2\nu/2}\right)^{\frac{t}{\nu}} \tag{54}$$

and $X(t)\equiv X(t;\sigma_j,\nu,\theta)$. For the estimation purpose, the log difference is used to be consistent with most previous literature. Therefore, characteristic function specification for de-meaned returns should be transformed as in (16).

$$\begin{gathered} r(t) = \ln\left[\frac{S(t)}{S(0)}\right] \\ \phi_r(r(t),t;u) = E_0\left[e^{iur}\right] = \phi_X(u,t)\exp\left[iu\cdot wt\right] \end{gathered} \tag{55}$$

2.3.2. Extended Variance-Gamma (EVG) Model

Daal and Madan (2005) extend VG model for the exchange rate process by adding a separate diffusion component. Examples of this line of research include Carr et al. (2002) and

Carr and Wu (2003). They showed empirically that the addition of a diffusion component to the CGMY and FMLS process adds little to the explanatory power.

$$S(t) = S(0)\exp\left[\left(w - \frac{1}{2}\sigma^2\right)t + X\left(t;\sigma_j,\nu,\theta\right) + \sigma Z(t)\right] \tag{56}$$

where σ and $Z(t)$ are the instantaneous volatility of the added diffusion parameter and a standard Brownian motion, respectively. From equations (1) and (15), the explicit characteristic function for the de-meaned returns in the EVG model can be easily derived by

$$\phi_r(r(t),t;u) = E_0\left[e^{iur}\right] = \exp\left[iu\left(w - \frac{1}{2}\sigma^2\right)t\right]\cdot\phi_X(u,t)\cdot\exp\left[-\frac{1}{2}\sigma^2 u^2 t\right] \tag{57}$$

2.3.3. Extended Variance-Gamma with Stochastic "Diffusion" Volatility (EVGSV) Model

Unlike a diffusion parameter, to examine if stochastic volatility remains important even when incorporated with infinite activity pure Lévy jump process such as VG model, I allow a diffusion parameter to be stochastic across time. Therefore, EVGSV model is essential extensions of the paper written by Carr et al. (2002) who argue that index dynamics and the risk-neutral process should be free of a diffusion component. They find that the statistical and risk-neutral processes for equity prices are pure jump processes of infinite activity and finite variation.

$$\begin{aligned} S(t) &= S(0)\exp\left[wt + \int_0^t\left(-\frac{1}{2}V(s)\right)ds + X(t;\sigma_j,\nu,\theta) + \int_0^t \sqrt{V(s)}\,dW_s(s)\right] \\ dV_t &= k(\theta - V_t)dt + \sigma_V\sqrt{V_t}\,dZ_t \\ \rho dt &= E\left[dW_t dZ_t\right] \end{aligned} \tag{58}$$

Therefore, the characteristic function of $r(t) = \ln\left[\frac{S(t)}{S(0)}\right]$ is

$$\phi(r(V,t),t;u) = \exp\left[wt + C(u,t) + D(u,t)V(0)\right] \tag{59}$$

where wt is a convexity correction term and $C(u,t)$ and $D(u,t)$ are same as before.

2.3.4. The Variance Gamma with Stochastic "Jump" Volatility (VGSJV) Model

The VGSJV model proposed by Carr et al. (2003) is additionally considered to see if the source of stochastic volatility does matter. In particular, I study whether stochastic jump volatility does not have the same impact as stochastic diffusion volatility.

$$Z_{VG}(t) = X_{VG}\left(Y(t); \sigma_j, \nu, \theta\right) \tag{60}$$

where

$$\begin{aligned} Y(t) &= \int_0^t y(u)du \\ dy &= \kappa(\eta - y)dt + \lambda\sqrt{y}d\psi \end{aligned} \tag{61}$$

CGMY version of the characteristic function of $Y(t)$ is

$$\begin{aligned} \phi_Y(u,t) &= E\left\{\exp\left[iuY(t)\right]\right\} \\ &= A(i\phi,t)\exp\left[B(i\phi,t)Y(0)\right] \end{aligned} \tag{62}$$

where

$$A(t,u)=\frac{\exp\left(\frac{\kappa^2\eta t}{\lambda^2}\right)}{\left(\cosh\left(\frac{\gamma t}{2}\right)+\frac{\kappa}{\gamma}\sinh\left(\frac{\gamma t}{2}\right)\right)^{2\kappa\eta/\lambda^2}}$$

$$B(t,u)=\frac{2iu}{\kappa+\gamma\coth\left(\frac{\kappa t}{2}\right)}$$

$$\gamma=\sqrt{\kappa^2-2\lambda^2 iu} \tag{63}$$

The characteristic function for $Z_{VG}(t)$ is given by

$$\begin{aligned}E\left[\exp(iuZ_{VG}(t))\right]&=\phi_Y(u,t)\Big|_{u=-i\phi_X}\\&=A(-i\phi_X,t)\exp\left[B(-i\phi_X,t)Y(0)\right]\end{aligned} \tag{64}$$

3. The Data and Sample Selections Criteria

The data used in this study should have the following minimum qualifications for returns and options.

3.1. Historical Time-Series Underlying Assets Returns

For historical time-series underlying assets returns, empirical results should be relatively stable and robust across distinct sample periods. Therefore, this paper examines daily, weekly, and monthly[15] SP500 index returns for the whole sample periods from January 1, 1994 through December 31, 2003. Then, the whole sample is divided into two sub-samples to verify robustness of my estimation results. The first sub-sample periods (January 1, 1994 – December 31, 1998) coincide with those of Carr et al. (2002) and are directly comparable with their

[15] In general, non-normality and the role of jumps decrease as time horizon increases from high-frequencies (intraday or daily) to low-frequencies (weekly, monthly, quarterly, and annually). Therefore, in this paper, weekly and monthly SP 500 index returns are also considered to check if my empirical results are affected by data frequencies for robustness.

findings. The second sub-sample periods consist of remaining sample periods (January 1, 1999 – December 31, 2003). Consistent with other studies, actual SP500 index returns have significant deviations from normal distribution, evidenced by leptokurtosis (high peaks around zero returns and fatter tails having extreme positive or negative returns) as illustrated in Panel A of Figure 1. For the sub-periods, the 2nd sub-period, in general, has more severe fluctuations and fatter tails compared to those of the 1st sub-period.

For further robustness check, I also considered different sample periods from July 3, 1995 to August 7, 2002, equity returns such as GE and IBM, and an additional infinite-activity pure Lévy process such as normal inverse Gaussian (NIG) model. However, empirical findings and main conclusions for returns remain exactly same. The estimation tables are not reported here to save the space and can be provided upon requests.

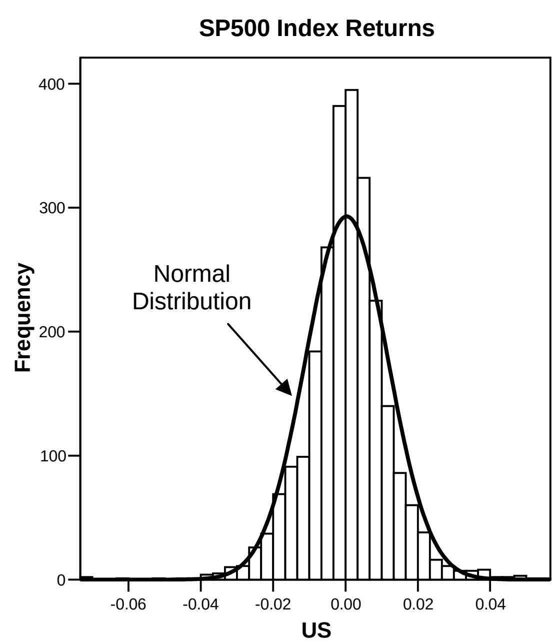
3.2. Options

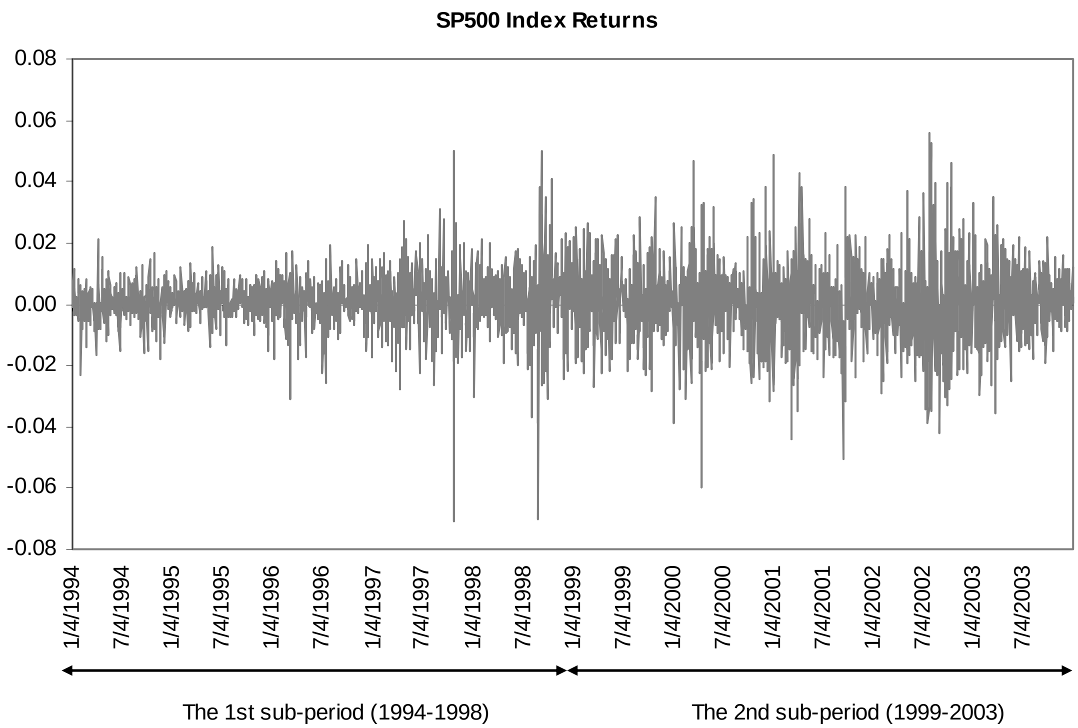
For calibrations, I use SP500 index option prices collected from OptionMetrics to obtain robust parameter values. Then, I observe how Lévy jumps and diffusion parameters are affected when stochastic volatility is considered. For options quotes, I use average prices of bid and ask quotes for each Wednesday call options covering from October 2, 2002 through December 18, 2002. Recent studies have preferred OTM options in parameter estimations because of greater liquidity, insensitivity of ITM with positive intrinsic value to model specification, and a cheaper way to speculate on or hedge (Carr and Wu (2003)). For example, Carr et al. (2002) only considered short-term OTM options with maturities between 1 and 2 months. Carr and Wu (2003) only analyzed OTM SP500 index options across all strikes and maturities from April 6, 1999 to May 31, 2000.

However, for complete analysis, I include a wider collection of options with maturities for short-term (<30 days), mid-term (30 – 120 days), and long-term (>120 days) and with moneynesses defined as K/S for ITM (0.9 – 0.97), ATM (0.97 – 1.03), and OTM (1.03 – 1.1). Due to illiquidity concerns, I exclude deep-ITM, deep-OTM options, very long-term options over 240 days, options less than $3/8, and zero-volume options. Most zero-volume options are very cheap (<$3/8) and/or expensive options (ITM).

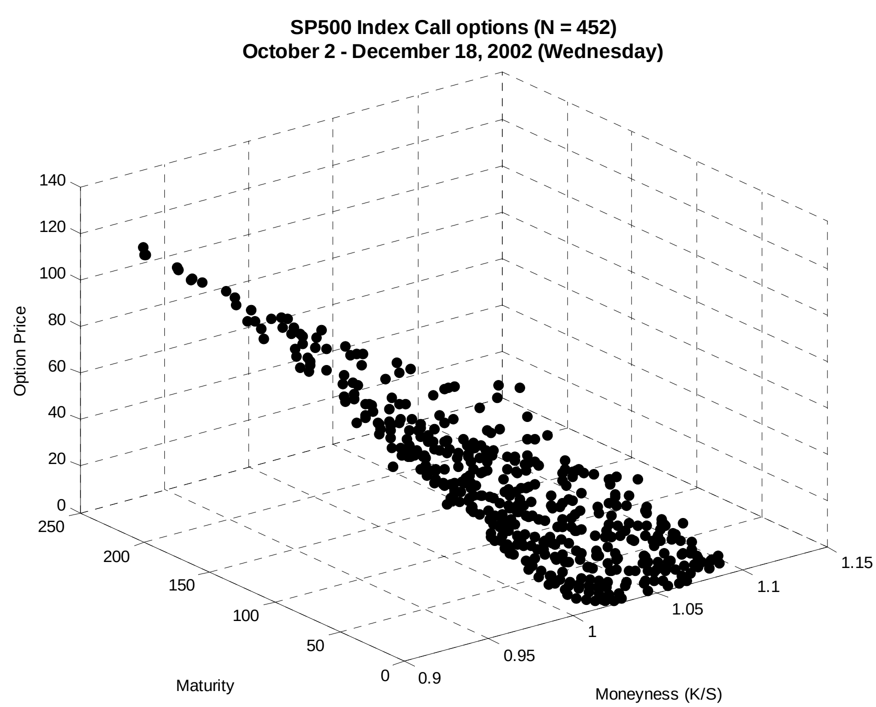
As in Table I, short-term, mid-term, and long-term options consist of about 34.51%, 49.78% and 15.71% of the total number of options considered in this paper, respectively. Unlike call options with different days to expiration, options with different moneynesses are relatively evenly distributed in my samples (e.g., ITM: 26.33%, ATM: 38.72%, and OTM: 34.96%). Panel B of Figure 1 further illustrates data descriptions by detailed 2D and 3D graphs on how many options are included in each category.

Panel A: SP500 Index Returns (January 1, 1994 – December 31, 2003, N = 2,518)





Panel B: SP500 Index Call Option Prices (Each Wednesday Call Option Covering from October 2, 2002 through December 18, 2002, N = 452)



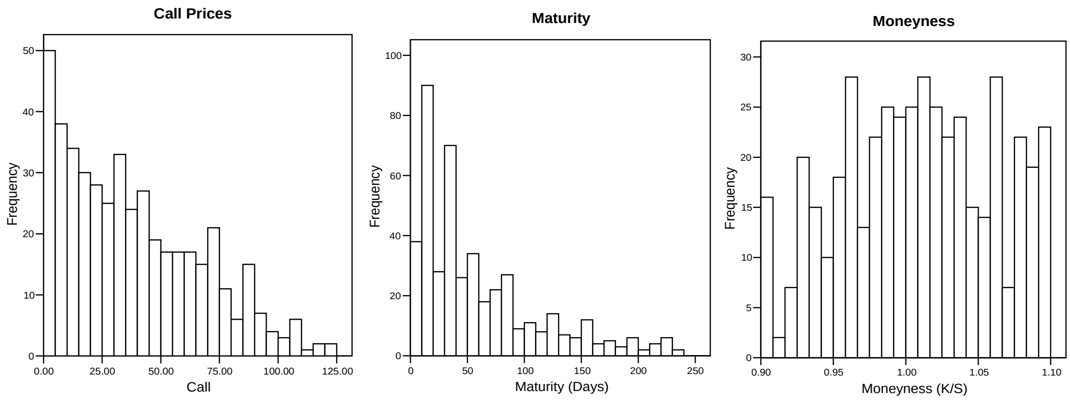


Figure 1. SP500 Index Returns and Call Option Prices. For SP500 index returns of Panel A, the whole sample periods cover from January 1, 1994 through December 31, 2003. The first sub-sample periods (January 1, 1994 – December 31, 1998) coincide with those of Carr et al. (2002) and the second sub-sample periods consist of remaining sample periods (January 1, 1999 – December 31, 2003). Panel B illustrates SP500 index options by detailed 2D and 3D graphs on how many options are included in each category.

Table I

Categorizations for SP500 Index Call Options across each Moneyness and Maturity

This table categorizes SP500 index options collected from OptionMetrics across different moneynesses and maturities. For options quotes, I use average prices of bid and ask quotes for each Wednesday call options covering from October 2, 2002 through December 18, 2002. I include a wide collection of options with maturities for short-term (<30 days), mid-term (30 – 120 days), and long-term (>120 days) and with moneynesses defined as K/S for ITM (0.9 – 0.97), ATM (0.97 – 1.03), and OTM (1.03 – 1.1). Due to illiquidity concerns, I exclude deep-ITM, deep-OTM options, very long-term options over 240 days, options less than $3/8, and zero-volume options. Most zero-volume options are very cheap (<$3/8) and/or expensive options (ITM). Median values of SP 500 index call prices are reported in square brackets.

Moneyness (K/S)	Call Options Days-to-Expiration (Maturity)						Sub-total
	ST (< 30 days)		MT (30 – 120 days)		LT (>120 days)		
	Call	N	Call	N	Call	N	
ITM (< 0.97)	58.45 [59.88]	40	74.83 [72.70]	55	98.66 [94.30]	24	119 (26.33%)
ATM (0.97 – 1.03)	17.72 [16.55]	68	40.51 [41.50]	80	63.57 [62.60]	27	175 (38.72%)
OTM (> 1.03)	3.77 [2.80]	48	16.27 [14.60]	90	34.88 [35.75]	20	158 (34.96%)
Sub-total		156 (34.51%)		225 (49.78%)		71 (15.71%)	452 (100%)

4. Econometric Methodology

Although it is possible to estimate parameters of returns and price options through probability density functions (PDF), I will fully utilize the advantages of using characteristic functions in this chapter since most models considered in this chapter do not have closed-form density functions for SV models[16] or, if exist, they are very computationally cumbersome for VG density function[17], which includes a special function such as modified Bessel functions of the second kind ($\mathrm{K}_v(z)$, also called Basset functions)

$$\begin{aligned} \mathrm{K}_v(z) &= \frac{\pi}{2}\frac{\mathrm{I}_v(z)-\mathrm{I}_{-v}(z)}{\sin(v\pi)} \\ &= \frac{1}{2}\int_0^\infty u^{v-1}\exp\left(-\frac{1}{2}z\left(u+u^{-1}\right)\right)du \end{aligned} \tag{65}$$

where $\mathrm{I}_v(z)$ is the modified Bessel function of the first kind defined as

$$\mathrm{I}_v(z) = \left(z/2\right)^v \sum_{k=0}^{\infty}\frac{\left(z^2/4\right)^k}{k!\Gamma\left(v+k+1\right)} \tag{66}$$

Therefore, for SP500 index returns, I rely on the binned maximum likelihood estimation to estimate parameter vectors, Θ , under the measure $\mathbb{P}$ by inverting characteristic functions via

---

[16] Indirect inference, Simulated Method of Moments (SMM), Efficient Method of Moments (EMM), Monte Carlo Markov Chain (MCMC), and Nonparametric methods have been often suggested as appealing alternatives to maximum likelihood estimation, when the likelihood function becomes intractable due to some unobserved state variables such as stochastic volatility (For more details, refer to Garcia et al. 2004).

[17] Daal and Madan (2005) derive the following VG probability density function for exchange rate process, $z(t)$.

$$f_{VG}\left(z(t)\right) = \frac{2\exp\left[\frac{\theta x}{\sigma_J^2}\right]}{v^{t/v}\sqrt{2\pi\sigma_J^2}\,\Gamma\left(\frac{t}{v}\right)}\left(\frac{x^2}{2\sigma_J^2/v+\theta^2}\right)^{\frac{1}{2v}-\frac{1}{4}} \times \mathrm{K}_{\frac{1}{v}-\frac{1}{2}}\left(\frac{1}{\sigma_J^2}\sqrt{x^2\left(2\sigma_J^2/v+\theta^2\right)}\right)$$

where K is the modified Bessel function of the second kind and the variable x is given by

$$x = z(t) - \mu t - \frac{t}{v}\left(1-\theta v-\sigma_J^2 v/2\right)$$

fast Fourier transform (FFT) initiated by Carr et al. (2002), which compute the level of the probability density at a prespecified set of values for returns. I used $N = 2^{14} = 16,384$ as a power of 2 used in the fast Fourier discrete transform, integration spacing of 0.25, and a return spacing of ($8\pi / N = 0.001534$) to minimize discretization errors. Then, I compare the performance of each model by likelihood ratio (LR) test for returns. *LR* test is performed by obtaining

$$LR = 2\left(\ln\left(L_{f_2}\left(r\middle|\Theta_{d_2}\right)\right) - \ln\left(L_{f_1}\left(r\middle|\Theta_{d_1}\right)\right)\right) \sim \chi^2_{d_2-d_1} \tag{67}$$

where L_{f_1} and L_{f_2} are the likelihood functions; d_1 and d_2 are the respective parameter spaces and r is the data. LR follows asymptotically χ^2 distribution with $\left(d_2 - d_1\right)$ degrees of freedom.

In addition, since past research has not provided common single conventional test criteria, I also perform the goodness-of-fit tests for the models using the following three information criteria; Akaike information criteria (AIC), Schwarz criteria (SC), and Hanna-Quinn criteria (HQ). All of them include an adjustment for the degrees of freedom that depends on both the number of parameters (*k*) and the sample size (*T*). Therefore, the model that minimizes the following information criteria values provides the best fit, since it is the model that has the highest likelihood value while controlling for the number of parameters in the model and the sample size.

$$\begin{aligned} AIC &= -2(l/T) + 2(k/T) \\ SC &= -2(l/T) + k\log(T)/T \\ HQ &= -2(l/T) + 2k\log(\log(T))/T \end{aligned} \tag{68}$$

For SP500 index options, one of intuitive ways to compute option prices using characteristic functions come from Bakshi and Madan (2000). They showed that option prices can be obtained by computing delta[18] of the option, Π_1, and the probability of finishing in the money, Π_2.

$$\begin{aligned} C(K,T) &= e^{-rT} E_0^{\mathbb{Q}}\left[\max\left(S_T - K, 0\right)\right] \\ &= S_0 \Pi_1 - K\exp(-rT)\Pi_2 \end{aligned} \tag{69}$$

where

$$\begin{aligned} \Pi_1 &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{\exp(-iu\log K) E\left[\exp(i(u-i)\log S_T)\right]}{iuE\left[S_T\right]}\right) du \\ &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{\exp(-iu\log K)\phi(u-i)}{iu\phi(-i)}\right) du \\ \Pi_2 &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{\exp(-iu\log K) E\left[\exp(iu\log S_T)\right]}{iu}\right) du \\ &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{\exp(-iu\log K)\phi(u)}{iu}\right) du \end{aligned} \tag{70}$$

According to Lewis (2000), I can also compute option prices from the characteristic functions by inverting the closed-form characteristic functions of various stochastic processes considered in this chapter.

$$C(S,K,T) = S - \sqrt{SK}\,\frac{1}{\pi}\int_0^\infty \frac{du}{u^2 + \frac{1}{4}} \mathrm{Re}\left[e^{-iuk}\phi\left(u - i/2\right)\right] \tag{71}$$

with $k = \ln\left(\frac{K}{S}\right)$.

[18] The delta ($\Delta = \frac{\partial C}{\partial S}$) is defined as the change in the value of the option compared with the change in the value of the underlying asset.

However, the most efficient way to compute option prices has been proposed by Carr and Madan (1999) by inverting the modified call price dampened by exponential factor (e.g., $\alpha = 1.5$) through fast Fourier transform (FFT). The FFT is used to invert the generalized Fourier transform of the call price, which is applicable to other types of options as well. Carr and Madan (1999) showed that

$$C(K,T) = \frac{\exp\left(-\alpha \log\left(K\right)\right)}{\pi} \int_0^\infty \exp\left(-iu \log\left(K\right)\right) \zeta\left(u\right) du \tag{72}$$

where

$$\begin{aligned} \zeta\left(u\right) &= \frac{e^{-rt} E\left[\exp\left(i\left(u - i(\alpha+1)\right)\log\left(S_T\right)\right)\right]}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u} \\ &= \frac{e^{-rt} \phi_T\left(u - i(\alpha+1)\right)}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u} \end{aligned} \tag{73}$$

where $\phi_T(u,t)$ denotes the characteristic function for the log of the stock price. Then, $C(K,T)$ is obtained by approximating the equation (34)-(35) using discrete Fourier transform (DFT), which replaces an infinite integral of Fourier transform with summation of N points.

$$\begin{aligned} C_T(k_n) \approx &\frac{\exp(-\alpha k_n)\exp(i\pi n)\exp(-i\pi N/2)}{\Delta k} \\ &\times \frac{1}{N}\sum_{j=0}^{N-1} u_j \left\{\exp(i\pi j)\psi_T(u_j)\right\} \exp(-i2\pi jn/N) \end{aligned} \tag{74}$$

which improves significant amount of computational time. To further obtain an accurate integration, I also apply Simpson's rule weightings. Since the FFT option pricing algorithm only returns option prices at discrete moneyness levels $k \equiv \ln K/S$ of a constant fine grid step Δk with grid size a power of 2 (e.g., $N = 2^{12}$ = 4,096), to add more preciseness of parameter estimations, I compute option prices via improved Fast Fourier Transform (FFT) algorithm to match arbitrary log-strike grids with equal intervals with each moneyness and maturity of actual market option prices.

Although Carr and Madan (1999) suggest using the following FFT formula for the short-term OTM options which makes the inverse Fourier transform integrand less oscillatory and facilitates the numerical integration problem, I find that using (34)-(35) to compute option prices for all the moneynesses and maturities does not alter my main empirical results[19].

$$C(K,T) = \frac{1}{\sinh(\alpha k)} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \gamma_T(u) du \tag{75}$$

where

$$\gamma_T(u) = \frac{\zeta_T(u - i\alpha) - \zeta_T(u + i\alpha)}{2} \tag{76}$$

with

$$\zeta_T(u) = e^{-rT} \left(\frac{1}{1+iu} - \frac{e^{rT}}{iu} - \frac{\phi_T(u-i)}{u^2 - iu} \right) \tag{77}$$

For estimation of risk-neutral parameter vectors, $\tilde{\Theta}$, under the measure $\mathbb{Q}$ for SP500 index options, I use nonlinear least squares method by minimizing the average of squared pricing errors (MSE) of matching option prices across both strikes and maturities by weekly calibrations and by pooled calibrations as in Bakshi et al. (1997).

---

[19] Very recently, the fractional fast Fourier transform (FRFT) algorithm for option pricing has been proposed by Chourdakis (2005) since the vector of option prices computed from FFT will extend well beyond the range of log-strike prices that are actually required, thus only small faction of the output vector may be of interest. While this method enables me to compute option prices more efficiently with less function evaluations, I still utilize FFT algorithm proposed by Carr and Madan (1999) to be consistent with previous literature. The advantage of computation time in FRFT is not critical because of the moderate sizes of options data (N = 452) I have used in this study. In addition, Chourdakis (2005) shows that FFT and FRFT provide identical calibration results based on the experiment using VG model.

$$\tilde{\Theta} = \arg\min_{\tilde{\Theta}} \frac{1}{N} \sum_{i=1}^{N} \left| C_i - \hat{C}_i\left(\tilde{\Theta}\right) \right|^2 \tag{78}$$

where N represents the number of options at date t, C_i denotes the observed market price of SP500 index option at a given strike K and maturity $\tau = T - t$, and $\hat{C}_i\left(\tilde{\Theta}\right)$ denotes the model-implied option prices obtained from FFT as a function of the parameter vector $\tilde{\Theta}$.

However, as Lehar et al. (2002) and Lehnert (2003) point out, the squared pricing errors tend to put too much weight on options with a high market price in the parameter estimation. If so, the optimization procedure mainly minimizes the pricing errors of deep in-the-money call options. For that reason, I also estimate parameters by minimizing the average of squared relative pricing errors (MSRE), defined as $\frac{1}{N} \sum_{i=1}^{N} \left(\frac{C_i - \hat{C}_i\left(\tilde{\Theta}\right)}{C_i} \right)^2$, for comparison (results not reported). Nonetheless, I find that my empirical findings and main conclusions are robust in the specifications of loss functions.

For weekly calibrations, the aforementioned procedures should be taken at each Wednesday to obtain time series patterns of parameter estimates and pricing errors. However, for pooled calibrations, only one step of estimation is required since N is simply total number of options during the sample periods. Then, I compare the performance of each model by overall measures of the quality of the calibration such as MSE, MAE, MARE, and RMSE for SP500 index options. Pricing errors to measure the quality of fit can be computed in the appropriate way listed below. Among them, MAE is an indication of the actual mispricing in dollars per option premium, whereas MARE takes into account the magnitude of the option value.

$$
\begin{aligned}
MSE &= \frac{1}{N}\sum_{i=1}^{N}\left(C_i - \hat{C}_i\left(\tilde{\Theta}\right)\right)^2 \\
MAE &= \frac{1}{N}\sum_{i=1}^{N}\left|C_i - \hat{C}_i\left(\tilde{\Theta}\right)\right| \\
MARE &= \frac{1}{N}\sum_{i=1}^{N}\left|\frac{C_i - \hat{C}_i\left(\tilde{\Theta}\right)}{C_i}\right| \\
RMSE &= \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(C_i - \hat{C}_i\left(\tilde{\Theta}\right)\right)^2}
\end{aligned}
\qquad (79)
$$

5. Analysis of Statistical and Risk-Neutral Estimation Results and Comparison of Models Performance

Huang and Wu (2004) compare the empirical performance of different jump and stochastic volatility specifications in pricing SP500 index options based on time-changed Lévy processes. Although Huang and Wu (2004)'s approaches to generate stochastic volatility are slightly different from ours, the empirical results on options are consistent with my evidences presented in this chapter. Allowing stochastic volatility to be generated separately from the jump component and the diffusion component significantly improves the pricing performance. In addition, I also agree with the Carr and Wu (2003)'s arguments that there are both continuous and jump components in the underlying index process by observing asymptotic behavior of short-maturity options.

However, Carr and Wu (2003) explicitly exclude analysis of underlying time-series of asset prices due to the following two reasons and identify the type of an asset price process by examining the convergence speeds of option prices as the time to maturity approaches to zero. First, the discretely sampled underlying assets' time-series returns paths from different stochastic processes look very similar, unless the sampling frequency is extremely high. Second, increasing the sampling frequency introduces market microstructure effects.

5.1. Model Performance Comparisons via Likelihood Ratio (LR) Tests and Information Criteria for SP500 Index Underlying Returns

5.1.1. The Role of Stochastic Diffusion Volatility

Recent research studies have shown that GARCH (in a discrete-time framework) and/or mean-reverting stochastic volatility (in a continuous-time framework) provide an excellent fit for financial asset returns when combined with finite-activity Merton's type compound Poisson Jump-diffusion models. Consistent with previous studies, the density plot drawn by FFT using characteristic functions for finite activity Lévy jump models such as Merton's type jump-diffusion (JD) and stochastic volatility with jump process (SVJ) of Bates (1996, 2000) and Bakshi et al. (1997) shows that SVJ model significantly improves overall fits of actual data as I can observe from Panel A of Figure 2. Unlike JD model, the MLE density plot of SVJ model passes through the middle of circles of actual returns points and high peaks around zero returns almost perfectly, which result in a much higher likelihood value (*LL*) for SVJ model (*LL*=7832.50) compared to that of JD model (*LL*=7793.99) for the whole sample periods. These empirical results fully coincide with previous literatures emphasizing the importance of stochastic volatility in returns.

However, most importantly, this conclusion does not hold when stochastic volatility is incorporated with an infinite activity pure Lévy jump model such as extended variance-gamma (EVG) process by allowing a drift parameter to be time-varying using mean-reverting stochastic volatility process. I observe that stochastic volatility does not play an important role in this scenario as in a diffusion parameter. For example, likelihood ratio test shows that the likelihood value (*LL*=7832.60) for EVG model is not improved by addition of a diffusion parameter into VG model (*LL*=7831.64), which is statistically insignificant at any confidence intervals. This finding is also consistent with Carr et al. (2002) who showed redundancy of a diffusion parameter in the case of CGMYe model for index returns.

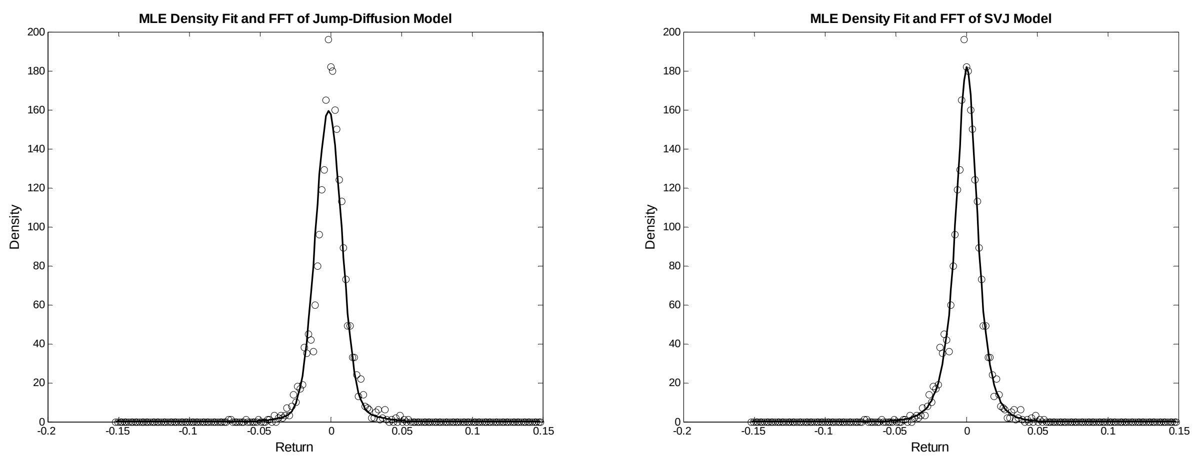
Surprisingly, likelihood value (*LL*=7832.67) of EVGSV model, which assumes stochastic diffusion volatility, is not high enough to compensate for additional stochastic volatility parameters compared to that of EVG (*LL*=7832.60) model. This result remains in the 2nd sub-period, even though the likelihood ratio between EVG and EVGSV models is statistically significant for the 1st sub-period. The likelihood values of EVG and EVGSV models are 4264.47 and 4275.11 for the 1st sub-period and 3657.28 and 3658.32 for the 2nd sub-period, respectively[20]. *LR* test confirms that the difference between likelihood values of EVG and EVGSV models for the 2nd period is almost negligible since EVGSV model requires me to include four additional stochastic volatility parameters.

It seems that infinite numbers of frequent large jumps dominated the effect of stochastic volatility due to higher fluctuations and fatter-tailedness of SP500 index returns for the 2nd sub-period compared to the 1st sub-period, as already illustrated in Figure 1. Therefore, the estimation results of the whole sample and the 2nd sub-period strongly support my argument of the role of stochastic volatility combined with infinite activity pure Lévy jump processes. This conclusion is also reconfirmed by the smallest values of three information criteria. As shown in Table II, AIC (-6.2181), SC (-6.2112), and HQ (-6.2156) measures are minimized for VG model. Although EVGSV model has slightly higher likelihood value than that of VG model, this marginal improvement are severely penalized by excessive numbers of parameters, which results

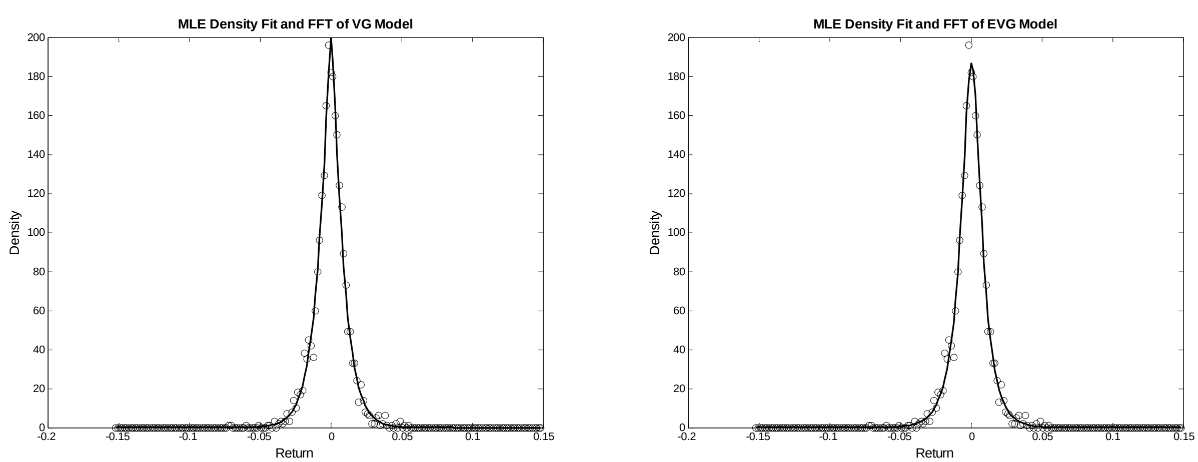
[20] Estimation results for each sub-period are not reported here for brevity and can be provided upon requests.

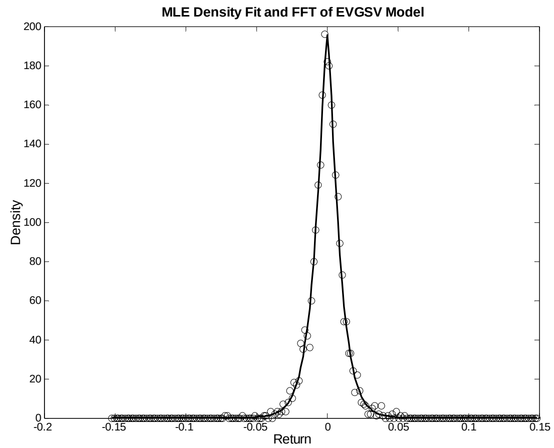
in higher values of AIC (-6.2150), SC (-6.1965), and HQ (-6.2083) measures. Consequently, it appears that stochastic volatility is not required as VG model is flexible enough to fully capture randomness and persistence in the volatility of SP500 index returns.

Panel A: Finite Activity Lévy Jump Models with/without Stochastic Volatility



Panel B: Infinite Activity Lévy Jump Models with/without Stochastic Volatility





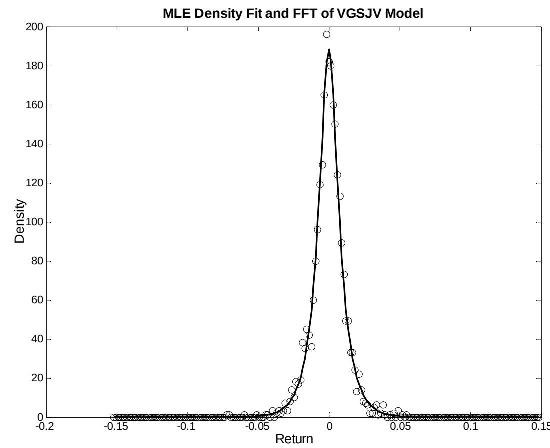


Figure 2. MLE Density Fits via FFT using Characteristic Functions. The figures illustrate the density plot drawn by FFT using characteristic functions of finite/infinite activity Lévy Jump Models with/without stochastic volatility.

Table II

Parameter Estimations using Historical Time-Series SP500 Index Returns via Fast Fourier Transformations (FFT)

For SP500 index returns, I rely on the binned maximum likelihood estimation to estimate parameter vectors, Θ, under the measure $\mathbb{P}$ by inverting characteristic functions via fast Fourier transform (FFT) initiated by Carr et al. (2002), which compute the level of the probability density at a prespecified set of values for returns. I used $N = 2^{14} = 16,384$ as a power of 2 used in the fast Fourier discrete transform, integration spacing of 0.25, and a return spacing of ($8\pi / N =$ 0.001534) to minimize discretization errors. *LL* is log-likelihood values and the following three information criteria (AIC, SC, and HQ) are also computed to measure the goodness-of-fit for the models. The *p*-values are reported in square brackets.

Parameters	Pure Diffusions	Finite Activity Pure Lévy Jumps with/without SV		Finite Activity Pure Lévy Jumps with/without Diffusions and SV			
	BS	JD	SVJ	VG	EVG	EVGSV	VGSJV
σ	0.0114	0.0090			0.0034		
	[0.0000]	[0.0000]			[0.0001]		
θ				-0.0003	-0.0005	-0.0002	-0.0057
				0.1202]	[0.0388]	[0.2221]	[0.4851]
ν				0.7661	1.1281	0.7313	0.0000
				[0.0000]	[0.0000]	[0.0000]	[0.4477]
σ_J				0.0113	0.0108	0.0109	0.2872
				[0.0000]	[0.0000]	[0.0000]	[0.0000]

Table II, cont.

η							0.0027
							[0.0000]
λ		0.1104	0.0824				0.0709
		[0.0000]	[0.0691]				[0.0000]
κ		0.0023	0.4201				0.2416
		[0.0757]	[0.0000]				[0.0000]
δ		0.0195	0.1051				
		[0.0000]	[0.0000]				
θ_J			0.0022			0.0003	
			[0.0000]			[0.0424]	
κ_J			16.5908			24.9279	
			[0.0000]			[0.0000]	
σ_v			0.2199			1.8323	
			[0.0000]			[0.0000]	
ρ			-0.0500			-0.3183	
			[0.0682]			[0.0399]	
V_{init}			0.0001			0.0001	0.0014
			[0.2072]			[0.4883]	[0.0000]
LL	7689.24	7793.99	7832.50	7831.64	7832.60	7832.67	7832.07
AIC	-6.1066	-6.1874	-6.2149	-6.2181	-6.2181	-6.2150	-6.2153
SC	-6.1043	-6.1782	-6.1963	-6.2112	-6.2088	-6.1965	-6.1991
HQ	-6.1058	-6.1841	-6.2081	-6.2156	-6.2147	-6.2083	-6.2094

5.1.2. Do Sources of Stochastic Volatility Matter?

To examine the importance of the source of stochastic volatility, I also consider VGSJV model, which assumes stochastic jump volatility. For SP500 index returns, I find that this specification does not help to improve fits of underlying index returns compared to simple VG model. As Table II shows, I observe that the estimate of jump volatility parameter, σ_J, for VGSJV model is higher than those of other constant jump volatility models such as VG, EVG and EVGSV, which mirror dynamic feature of jump volatility captured only in VGSJV models. Therefore, it seems that unlike Merton's finite activity jump-diffusion model, infinite-activity pure Lévy jumps models such as VG are very flexible and powerful enough to capture large jumps, small movements mimicking diffusions, and stochastic volatility features for returns simultaneously. Panel B of Figure 2 illustrates MLE density fits for infinite activity pure Lévy jumps with/without stochastic volatility. As expected, most density plots look very similar and simple VG model performs very well even without including diffusion and stochastic volatility parameters.

All of model specifications considerably improve fits of SP500 index returns compared to BS model and, interestingly, SVJ and EVGSV models perform equally well, which implies that jumps specifications do not enhance model performances once stochastic volatility is incorporated. Therefore, I am now interested in investigating if these empirical results observed in SP500 index returns can be valid to the SP 500 index options as well.

5.2. In-Sample Weekly Calibrations of SP500 Index Call Options

5.2.1. Empirical Examinations between Statistical and Risk-Neutral Processes

Table III reveals that the average values of weekly parameter estimates of the diffusion volatility parameters, σ, differ noticeably across statistical and risk-neutral processes. For example, the parameter values of $\tilde{\sigma}$ for risk-neutral estimations are 0.2345 and 0.1462 for BS and JD models, respectively. However, parameter estimates of σ for statistical process are as small as 0.0114 and 0.0090 for the corresponding models. The drift parameters, θ, to capture skewness become also more distinguishable for risk-neutral parameters compared to those of statistical ones as shown in Table II and Table III. Although the statistical skewness parameters, θ, are only -0.0003 and -0.0002 for VG and EVGSV models, the risk-neutral skewness parameters, $\tilde{\theta}$ are -1.1698 and -0.1612 for VG and EVGSV models with negative signs for statistical and risk-neutral estimations.

Table III
Calibrated Risk-Neutral Parameters using SP500 Index Option Prices via Fast Fourier Transformations (FFT)

For estimation of risk-neutral parameter vectors, $\tilde{\Theta}$, under the measure $\mathbb{Q}$ for SP500 index options, I use nonlinear least squares method by minimizing the average of squared pricing errors (MSE) of matching option prices across both strikes and maturities by weekly calibrations through fast Fourier transform (FFT).

$$\tilde{\Theta} = \arg\min_{\tilde{\Theta}} \frac{1}{N}\sum_{i=1}^{N}\left|C_i - \hat{C}_i\left(\tilde{\Theta}\right)\right|^2$$

The average values of parameter estimates are obtained and standard deviations are reported in square brackets for each model.

Parameters	Pure Diffusions and Stochastic Volatility		Finite Activity Lévy Jumps with/without SV		Infinite Activity Pure Lévy Jumps with/without Diffusions and SV	
	BS	SV	JD	SVJ	VG	EVGSV
$\tilde{\sigma}$	0.2345		0.1462			
	[0.0360]		[0.0320]			
$\tilde{\theta}$					-1.1698	-0.1612
					[0.6073]	[0.4977]
$\tilde{\nu}$					0.0290	0.2688
					[0.0193]	[0.2325]
$\tilde{\sigma}_J$					0.1369	0.1387
					[0.0674]	[0.0317]
$\tilde{\lambda}$			2.7223	1.1671		
			[1.4365]	[0.9951]		
$\tilde{\kappa}$			-0.1582	-0.1519		
			[0.0734]	[0.0885]		
$\tilde{\delta}$			0.0442	0.0812		
			[0.0656]	[0.0569]		
$\tilde{\theta}_J$		0.0572		0.0389		0.0148
		[0.0205]		[0.0382]		[0.0190]
$\tilde{\kappa}_J$		6.1556		4.6367		6.7837
		[9.4338]		[2.8166]		[9.4449]
$\tilde{\sigma}_v$		3.4006		0.8918		2.3296
		[9.2752]		[0.7382]		[2.9459]
$\tilde{\rho}$		-0.7001		-0.6121		-0.8589
		[0.8093]		[0.2588]		[0.5583]
$\tilde{V}_{init}$		0.1200		0.0620		0.0639
		[0.1508]		[0.0304]		[0.0447]

However, the size of the variance rate of the gamma process, ν, to controls for the excess kurtosis between statistical and risk-neutral processes are opposite. That is, the variance rate of the gamma process, ν, has larger value for statistical process compared to risk-neutral process for VG and EVGSV models. Therefore, to compute overall levels of model-implied skewness and kurtosis of VG models along with JD and SVJ models, one should refer to the formula in Madan et al. (1998) and Das and Sundaram (1999)[21].

For the JD and SVJ models, the numbers of jumps per year, denoted as Poisson jump intensity, λ, for statistical processes are much smaller than those of risk-neutral processes. For example, although the numbers of jumps per year are only 0.1104 and 0.0824 for JD and SVJ models in statistical processes, risk-neutral estimations show that Poisson jump intensity, $\tilde{\lambda}$, increases up to, on weekly average, 2.7223 and 1.1671, respectively. Comparably, the volatility of volatility parameter, σ_v, also presents distinct features between statistical and risk-neutral processes. It tends to have inconsistent parameter values across different calibration weeks for SP500 index options, evidenced by high standard deviations. Similar instability is also found in the speed of mean reversion parameter, $\tilde{\kappa}_J$, for risk-neutral calibrations as reported in Table III.

The non-zero correlation between spot returns and volatility is necessary to generate negative skewness and leptokurtosis in the risk-neutral distribution of SP500 index options, which leads to substantial improvements in pricing options by reducing the implied volatility smile or skew. (Heston (1993); Schöbel and Zhu (1999)). The estimation results of correlation parameter, ρ, to capture leverage effects and volatility feedback effects between returns and

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[21] Madan et al. (1998): $Sk(\bullet) = \left(2\theta^3\nu^2 + 3\sigma_J\theta\nu\right)t$ and

$$Ku(\bullet) = \left(3\sigma_J^4\nu + 12\sigma_J^2\theta^2\nu^2 + 6\theta^4\nu^3\right)t + \left(3\sigma_J^4 + 6\sigma_J^2\theta^2\nu + 3\theta^4\nu^2\right)t^2$$

Das and Sundaram (1999): $Sk(\bullet) = \dfrac{\lambda(\kappa^3 + 3\kappa\delta^2)}{(\sigma + \sigma\delta^2 + \sigma\kappa^2)^{3/2}}$ and $Ku(\bullet) = 3 + \dfrac{\lambda(\kappa^4 + 6\kappa^2\delta^2 + 3\delta^4)}{(\sigma + \lambda\delta^2 + \lambda\kappa^2)^2}$

volatility, defined as $E\left[dW_t dZ_t\right] = \rho dt$ where $\rho < 0$, are consistent with several widespread beliefs among researchers. In other words, the volatility of equity returns is increased by lower overall firm values (leverage effects) and the equity price is decreased by heavier discounting of future expected dividends stemming from higher volatility assessments (volatility feedback effects). These phenomena also tend to be valid in index options and currency options as commonly founded in equity options[22].

In my calibrations of SP500 index call options, all of stochastic volatility models show strongly negative risk-neutral correlations, $\tilde{\rho}$, between SP500 index returns and volatility, which are -0.7001, -0.6121, and -0.8589 for SV, SVJ, and EVGSV models, respectively. Compared to the statistical estimation results of Table II, it seems that negative correlations become reinforced for risk-neutral case, which implies that leverage effects between returns and volatility process under the risk-neutral measure $\mathbb{Q}$ is substantially larger than under the objective measure $\mathbb{P}$.

However, it is likely that these differences among magnitudes of parameter values are mainly due to diverse time horizons of options prices compared to those of SP500 index returns. More specifically, the risk-neutral parameters are inferred from calibrations of market option prices with multiple days-to-expirations from short-term (<30 days), mid-term (30 – 120 days) to long-term (>120 days) maturities. On the other hand, the parameter values for statistical processes are generally estimated by using high-frequency financial data such as daily or weekly, at most, monthly SP500 index returns. Therefore, the risk-neutral calibrations tend to overplay the parameter values obtained by minimizing MSE between market and theoretical option prices.

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[22] Some examples include SP500 index options (Bakshi et al. (1997)), European currency call options written on British pound (Sarwar and Krehbiel (2000)), and KOSPI 200 index options (Kim and Kim (2004)), among others.

5.2.2. Model Performance Comparisons via Pricing Errors

Since the jumps models (JD or VG) do not handle very long-term maturities properly, introducing stochastic volatility improves the overall quality of calibration. As a result, when stochastic volatility is considered with finite or infinite activity of jumps, various measures of pricing errors such as MSE, MAE, and RMSE noticeably decrease. For example, MAE for SVJ and EVGSV models are only 0.9733 and 1.0039 and significantly lower than those (1.9782 and 2.1134) of JD and VG models.

Formally, to judge the statistical significance of differences between pricing errors, I compute the pair-wise *t*-statistics,

$$t = \frac{\text{Average}\left(\text{PE}^{i} - \text{PE}^{j}\right)}{\text{Stdev}(\text{PE}^{i} - \text{PE}^{j})} \tag{80}$$

which is the mean difference between the weekly pricing errors of the models divided by the standard deviation of the differences scaled by root of the number of weeks ($\sqrt{12}$ in the samples) as in Huang and Wu (2004) and Daal and Madan (2005). For further analysis, I also apply the nonparametric Wilcoxson sign rank test to check the statistical significance of the reduction in mispricing as in Chu and Freund (1996) and Fofana and Brorsen (2001). For the sign rank test, the null and alternative hypotheses of the mispricing are

$$\begin{aligned} H_0 &: \text{PE}^{i} \leq \text{PE}^{j} \\ H_\alpha &: \text{PE}^{i} > \text{PE}^{j} \end{aligned} \tag{81}$$

where i and j are different model specifications compared and PE is the pricing error measures.

As reported in Table IV, SVJ and EVGSV models obviously improve in-sample pricing performance over JD and VG models, evidenced by statistically significant *t*-test values and near-zero values of *p*-probability obtained from the Wilcoxson sign rank test. All measures of pricing errors consistently substantiate superiority of JD and VG models over BS model as well. Besides, it seems that JD model slightly fits better than VG model in the samples based on *t*- and sign rank tests. Remarkably, *t*- and sign rank tests between SVJ and EVGSV models confirm that these differences are statistically insignificant, implying that these two stochastic volatility models with finite/infinite activity of Lévy jump models have similar pricing errors. Other measures of pricing errors reported in Table IV show similar pricing improvements when stochastic volatility is incorporated with Lévy jump models. As such, stochastic volatility significantly contributes to reducing in-sample pricing errors and helping model-implied option prices to be closer to market option prices.

Table IV

Pricing Errors, Pair-Wise *t*-statistics and Nonparametric Wilcoxson Sign Rank Tests for In-Sample Model Performance Comparisons

I evaluate overall measures of the quality of the calibration such as MSE, MAE, and RMSE for SP500 index options. Then, to judge the statistical significance of differences between pricing errors in formal way, I perform the pair-wise *t*-test and the nonparametric Wilcoxson sign rank test to check the statistical significance of the reduction in mispricing. In Panel B, the *p*-values computed from sign rank test are reported in square brackets.

Panel A: Pricing Errors of In-Sample Risk-Neutral Calibrations

Pricing Errors	Pure Diffusions and Stochastic Volatility		Finite Activity Lévy Jumps with/without SV		Infinite Activity Pure Lévy Jumps with/without Diffusions and SV	
	BS	SV	JD	SVJ	VG	EVGSV
MSE	10.9699	1.9544	6.4131	1.6046	7.4394	1.6033
MAE	2.5305	1.0515	1.9782	0.9733	2.1134	1.0039
RMSE	3.1968	1.3549	2.4274	1.2426	2.5933	1.2484

Panel B: The *t*- and Wilcoxson Sign Rank Tests Statistics

Do Lévy jumps models outperform a pure diffusion Model?

Pricing Errors	BS – JD	BS – VG
MSE	6.1749 [0.0005]	4.7510 [0.0005]
MAE	6.0915 [0.0005]	3.8535 [0.0005]
RMSE	7.4353 [0.0005]	4.9776 [0.0005]

Does stochastic volatility play an important role for option pricing?

Pricing Errors	JD – SVJ	VG – EVGSV
MSE	4.6221 [0.0005]	4.4243 [0.0005]
MAE	5.8595 [0.0005]	5.4400 [0.0005]
RMSE	5.4571 [0.0005]	5.3464 [0.0005]

Which jump specifications are more relevant for option pricing with/without stochastic volatility?

Pricing Errors	JD – VG	SVJ – EVGSV
MSE	-2.2905 [0.0034]	0.0144 [0.7910]
MAE	-2.5508 [0.0068]	-0.9552 [0.9697]
RMSE	-2.4369 [0.0034]	-0.1841 [0.7910]

5.2.3. Pricing Errors in Moneyness/Maturity Categories

Carr and Wu (2003) point out that unlikely a purely continuous diffusion process such as BS model, the probability that the underlying asset price can jump into the money within any short interval of time is significantly larger if the process has jumps. Therefore, I expect that inclusion of jumps should help to enhance model performance for short-term OTM options. Similarly, incorporation of stochastic volatility should reduce the pricing errors of long-term ATM and OTM options. Hence, I also assume that having jump components addresses moneyness biases, while having stochastic volatility allows risk-neutral distributions to evolve stochastically over time as discussed by Bates (2003). For this reason, to scrutinize the role of jumps and stochastic volatility, I am now interested in moneyness/maturity categories in which make the most influential contributions of stochastic volatility.

Within each model, OTM options, especially options with a short-time to maturity, are priced worst by all models in percentage terms (ARPE) due to low market prices in this category, as shown in Table V. In the comparative study of different models, it is well-known that BS model outperforms other competing model specifications for short-term ATM options. In the sample, BS model performs better than and JD and VG models for short-term ATM and OTM options in-sample calibrations. In addition, JD and VG models only marginally improve pricing performance for long-term OTM options compared to BS model. As expected, pricing performance of VG model is superior to that of BS model for short-term ITM and mid-term OTM options.

Table V

In-Sample Pricing Errors in Moneyness/Maturity Categories

The tables report average values of absolute pricing errors (APE) and absolute relative pricing errors (ARPE) in moneyness/maturity category. Median values of pricing errors are reported in square brackets.

Panel A: Pure Diffusions and Pure Stochastic Volatility Models

Models	Moneyness (K/S)	Call Options Days-to-Expiration (Maturity)					
		ST (< 30 days)		MT (30 – 120 days)		LT (>120 days)	
		APE	ARPE	APE	ARPE	APE	ARPE
BS	ITM	3.6909	0.0660	4.6721	0.0619	2.8613	0.0293
	(< 0.97)	[3.2435]	[0.0622]	[4.9803]	[0.0658]	[2.7101]	[0.0315]
	ATM	1.7965	0.1265	1.8958	0.0469	3.0093	0.0486
	(0.97 – 1.03)	[1.1565]	[0.0969]	[1.4863]	[0.0403]	[2.1904]	[0.0350]
	OTM	0.4304	0.1794	2.0415	0.1998	5.6147	0.1707
	(> 1.03)	[0.3776]	[0.1412]	[2.0763]	[0.1323]	[5.2766]	[0.1719]
SV	ITM	1.8796	0.0346	1.3118	0.0179	1.0713	0.0113
	(< 0.97)	[1.5571]	[0.0253]	[1.1441]	[0.0147]	[0.8809]	[0.0092]
	ATM	1.2036	0.1030	1.0191]	0.0258	0.9084	0.0143
	(0.97 – 1.03)	[0.9768]	[0.0666]	[0.9501]	[0.0229]	[0.7565]	[0.0117]
	OTM	0.5034	0.2312	0.8104	0.0605	0.8234	0.0242
	(> 1.03)	[0.3368]	[0.1488]	[0.7749]	[0.0453]	[0.6242]	[0.0224]

Panel B: Finite Activity Lévy Jumps with/without SV

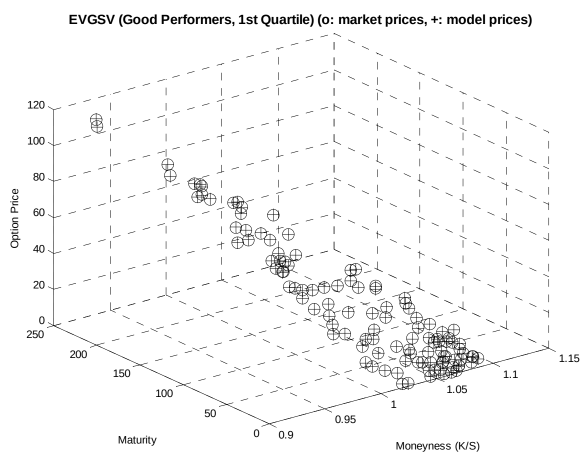
JD	ITM	2.1198	0.0400	2.5446	0.0331	2.4658	0.0248
	(< 0.97)	[1.6946]	[0.0287]	[2.4030]	[0.0347]	[2.1268]	[0.0224]
	ATM	2.3166	0.2028	1.3100	0.0358	2.4304	0.0402
	(0.97 – 1.03)	[2.0883]	[0.1470]	[1.0820]	[0.0272]	[2.2073]	[0.0422]
	OTM	1.6193	0.5529	1.2643	0.0894	4.8209	0.1427
	(> 1.03)	[1.2580]	[0.5250]	[0.9676]	[0.0729]	[4.4764]	[0.1540]
SVJ	ITM	1.6606	0.0316	1.2523	0.0177	0.8446	0.0088
	(< 0.97)	[1.4323]	[0.0241]	[1.0637]	[0.0135]	[0.5076]	[0.0043]
	ATM	1.1837	0.0937	1.0369	0.0264	0.8624	0.0137
	(0.97 – 1.03)	[0.9743]	[0.0763]	[0.9777]	[0.0262]	[0.6493]	[0.0107]
	OTM	0.3629	0.1770	0.7120	0.0530	0.8308	0.0239
	(> 1.03)	[0.2791]	[0.1138]	[0.6384]	[0.0424]	[0.7905]	[0.0239]

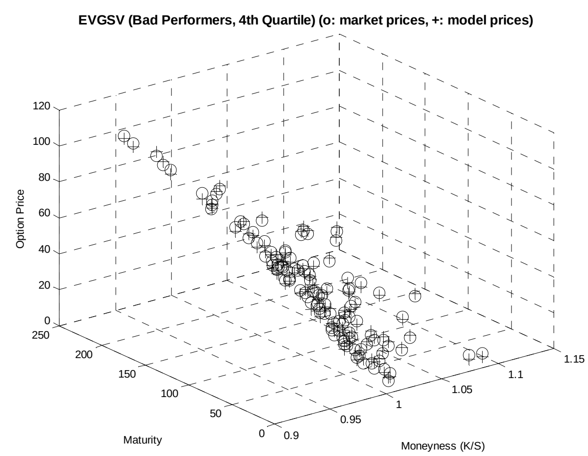
Panel C: Infinite Activity Pure Lévy Jumps with/without Diffusions and SV

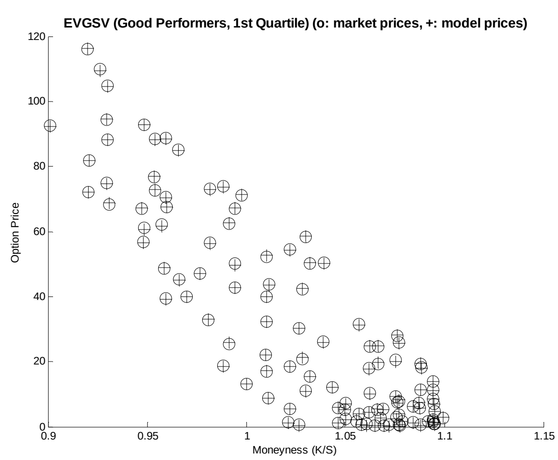
Models	Moneyness (K/S)	Call Options Days-to-Expiration (Maturity)					
		ST (< 30 days)		MT (30 – 120 days)		LT (>120 days)	
		APE	ARPE	APE	ARPE	APE	ARPE
VG	ITM (< 0.97)	2.6030 [2.3053]	0.0475 [0.0384]	3.0999 [2.9877]	0.0411 [0.0437]	1.8804 [1.4293]	0.0198 [0.0161]
	ATM (0.97 – 1.03)	2.1617 [2.0519]	0.2071 [0.1351]	1.8244 [1.5768]	0.0479 [0.0413]	3.1898 [2.3148]	0.0512 [0.0412]
	OTM (> 1.03)	1.5121 [1.2200]	0.5340 [0.5623]	1.0939 [1.0572]	0.0913 [0.0602]	4.6600 [4.2804]	0.1358 [0.1258]
EVGSV	ITM (< 0.97)	1.5091 [1.2735]	0.0299 [0.0209]	1.3718 [1.2182]	0.0195 [0.0153]	0.9344 [0.6070]	0.0097 [0.0062]
	ATM (0.97 – 1.03)	1.1335 [0.9367]	0.0902 [0.0709]	1.0647 [0.9807]	0.0272 [0.0267]	0.8504 [0.7457]	0.0132 [0.0122]
	OTM (> 1.03)	0.4844 [0.3473]	0.2142 [0.1594]	0.7435 [0.7339]	0.0545 [0.0504]	0.9946 [0.9160]	0.0280 [0.0244]

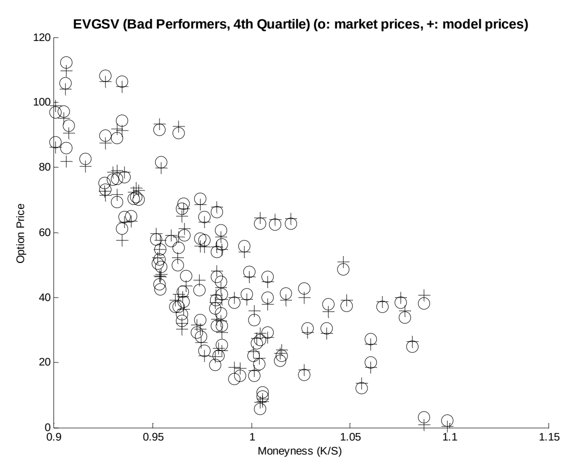
On the other hand, it is obvious that inclusion of stochastic volatility significantly reduces pricing errors for the long-term ATM and OTM options from Table V. For example, APE and ARPE for VG model ($3.1898 and 5.12%) decrease below to $0.8204 and 1.32% for long-term ATM options for EVGSV model. Similarly, for long-term OTM options, APE and ARPE for VG model ($4.6600 and 13.58%) decrease below to $0.9946 and 2.80% for EVGSV model. Similar analogy is also applied to JD and SVJ models. That is, APE and ARPE for JD model ($2.4304 and 4.02%) drop below to $0.8624 and 1.37% for long-term ATM options for SVJ model. For long-term OTM options, APE and ARPE for JD model ($4.8209 and 14.27%) drop below to $0.8308 and 2.39%.

Consistent with Kim and Kim (2004), I also observe that for all models OTM (ITM) options have highest (lowest) ARPE, whereas there are some inconsistencies for APE. In addition, it seems that the impact of stochastic diffusion volatility on OTM options does not differ significantly when compared to ATM options. Furthermore, pricing improvements for ITM options when stochastic volatility is taken into account are relatively moderate because of relative insensitivity of ITM options with positive intrinsic values to model specifications as pointed out by Carr and Wu (2003). In other words, ITM options approach the nonparametric lower boundary given by the difference between current index level and discounted strike price. Nonetheless, the impact of stochastic diffusion volatility on long-term options differs significantly when compared to short-term and mid-term options.









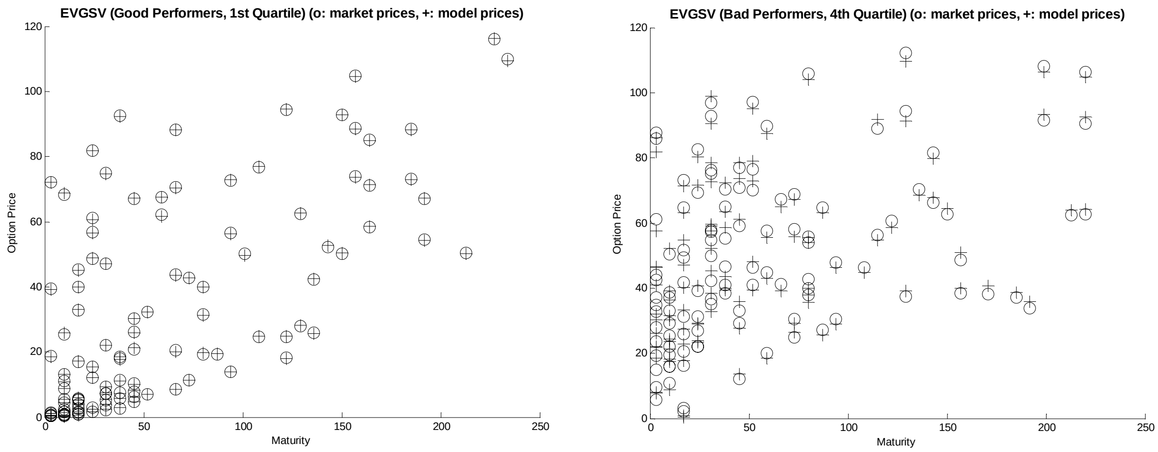


Figure 3. EVGSV Model Calibrations of SP500 Index Options and Pricing Errors. The figures illustrate calibrations results of EVGSV model for the SP500 index options. To examine sources of mispricing, I divide option data into 4 quartiles from the 1st quartile (denoted by good performers) to the 4th quartile (denoted by bad performers) sorted by APE defined as $\left|C_i - \hat{C}_i\left(\tilde{\Theta}\right)\right|$. In the scatterplots, circles are market prices and pluses are model prices.

5.2.4. Quartile Analysis and Price Differences for EVGSV Model Calibrations

Figure 3 illustrates calibrations results of EVGSV model for the SP500 index options. To examine sources of mispricing, I divide my option data into 4 quartiles from the 1st quartile (denoted by good performers) to the 4th quartile (denoted by bad performers) sorted by absolute pricing errors (APE) defined as $\left|C_i - \hat{C}_i\left(\tilde{\Theta}\right)\right|$. In Figure 3, circles are market prices and pluses are model prices. If I carefully observe "bad performers", which are right columns of Figure 3, there still remain small deviations between market options prices and model-implied option prices computed by FFT using characteristic functions.

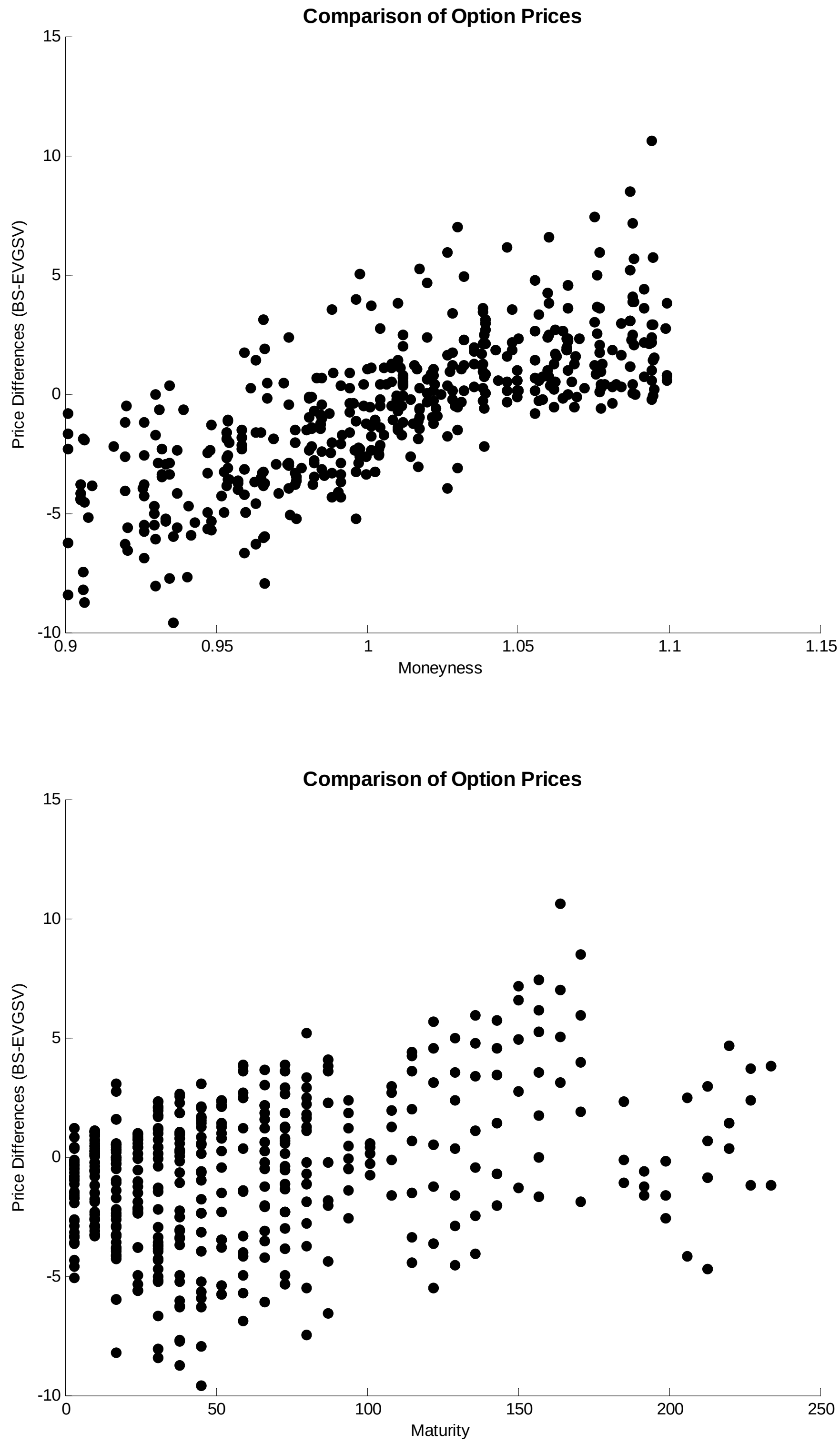


Figure 4. Comparison of Option Prices of BS and EVGSV models. The above graphs compare price differences between model-implied option prices obtained from BS and EVGSV models. Positive (negative) values imply overpricing (underpricing) of BS model compared to EVGSV model.

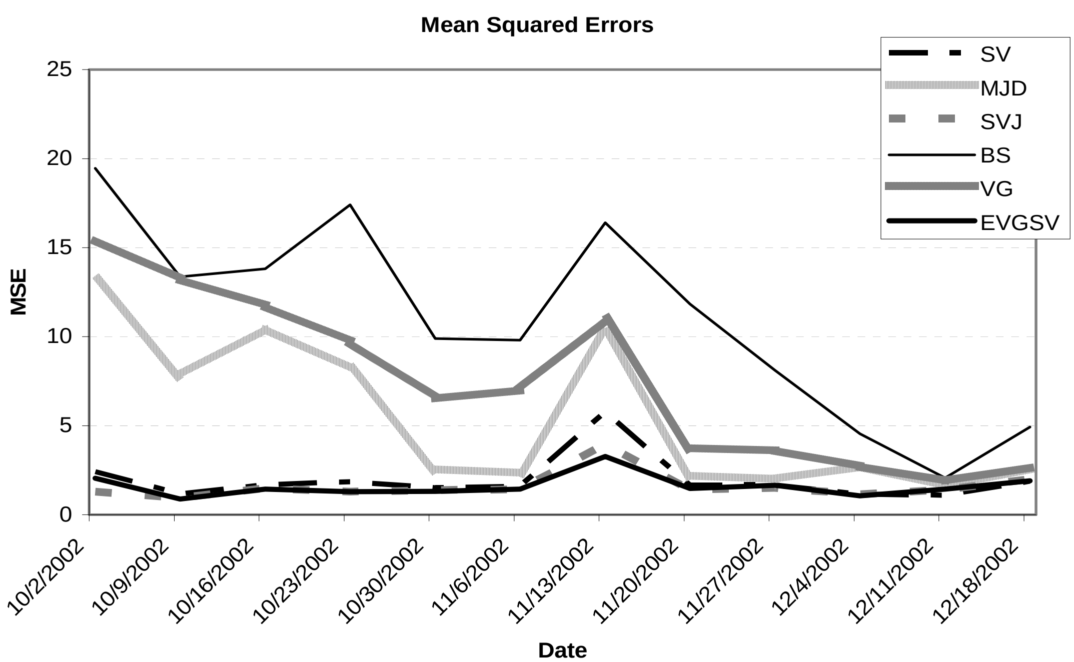
I find that most "good performers" are located in the category of OTM options ($1.03 < K / S < 1.1$) and their model-implied option prices are almost perfectly identical with markets option prices. Unlike "good performers", "bad performers" are scattered across each moneyness. It appears that EVGSV model also performs relatively well for the short-term options, which can be explained by VG components. However, although EVGSV model significantly reduces pricing errors for long-term options as found in Table IV, the empirical performance leaves still room for improvements to fit better for options with different moneynesses and maturities.

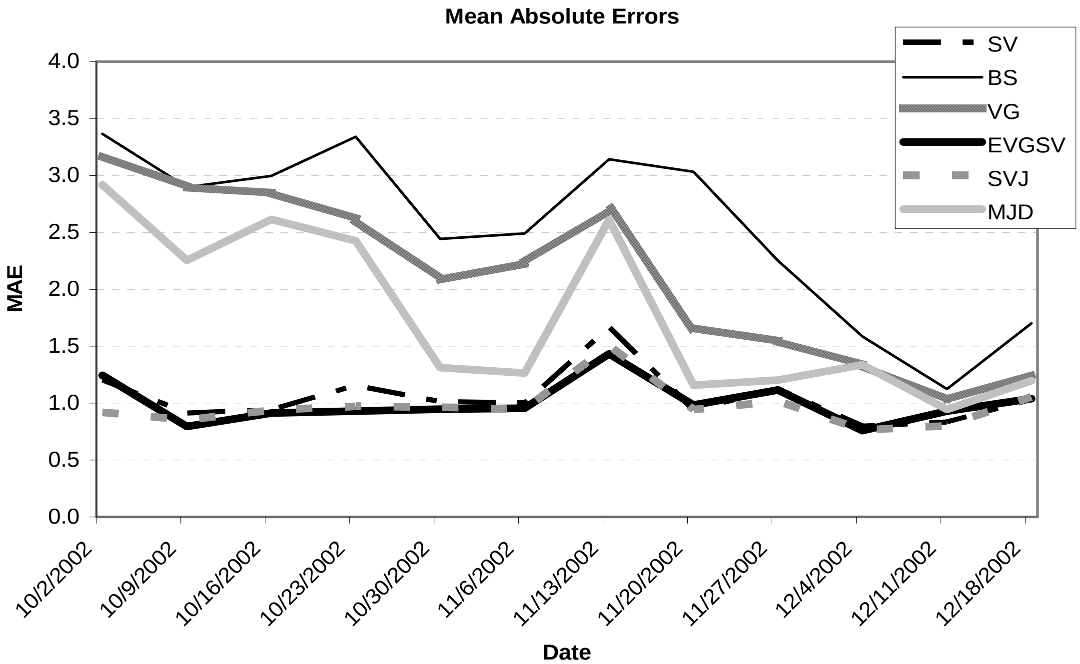
Figure 4 compares price differences between model-implied theoretical option prices obtained from BS and EVGSV models. In Figure 4, positive (negative) values imply overpricing (underpricing) of BS model compared to EVGSV model. When price differences are plotted across moneyness, I find that BS model, in general, overprices for OTM options and underprices for ITM options. For ATM options, price differences are relatively small. However, when price differences are plotted across maturity, I find that BS model, in general, underprices for short-term options and overprices for long-term options.

5.2.5. Time-Series Plots of In-Sample Pricing Errors and Pooled Calibrations for Model Performance Comparisons

Figure 5 plots time-series of pricing errors measures (MSE, MAE, and MARE) obtained by weekly calibrations for model comparisons. Interestingly, although JD model performs better than VG model in most cases except MARE, pricing performances among BS, VG, and JD models are sometimes changing dramatically across time. That is, pricing performances of model specifications without stochastic volatility are not stable and highly depend on calibrations weeks. However, Lévy jumps models with stochastic volatility such as SVJ and EVGSV models outperform other model specifications through the sample periods and minimize in-sample pricing errors as far as stochastic volatility is considered, which reconfirm the importance of stochastic volatility for option pricing, unlike that of underlying SP500 index returns.

To further check robustness of my results, I also analyze the results based on pooled calibrations by aggregating the whole options included in the samples. As such, unlike weekly calibrations of the models, pooled calibrations impose constant risk-neutral parameters over the full data interval. Although pricing errors for weekly calibrations are smaller than the corresponding ones for pooled calibration, I find that the pricing performances are quite similar to each other and all of my empirical findings and main conclusions remain valid in my analysis without regard to the way of calibrations.





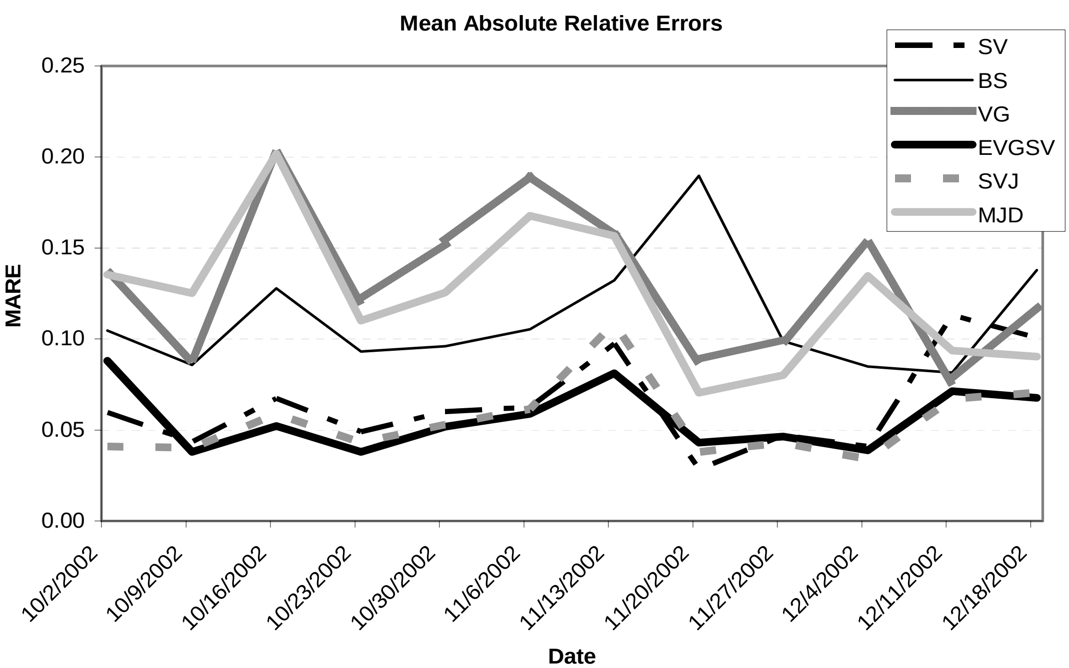


Figure 5. In-Sample Model Performance Comparisons by Pricing Errors Measures. The figures plot time-series patterns of measures of pricing errors such as MSE, MAE, and MARE obtained by in-sample weekly calibrations for 6 different model specifications.

5.2.6. Out-of-Sample Weekly Calibrations

Unlike statistical processes on SP500 index underlying returns, I have shown that in-sample risk-neutral calibration results of SP500 index call options clearly show the important economic and statistical roles of stochastic diffusion volatility incorporated with Lévy jumps models. Nevertheless, I am still concerned about controlling for in-sample over-fitting by complicated models with many free parameters. That is, pricing models with more parameters will, most likely, improve in-sample pricing performances simply due to excessive numbers of parameters.

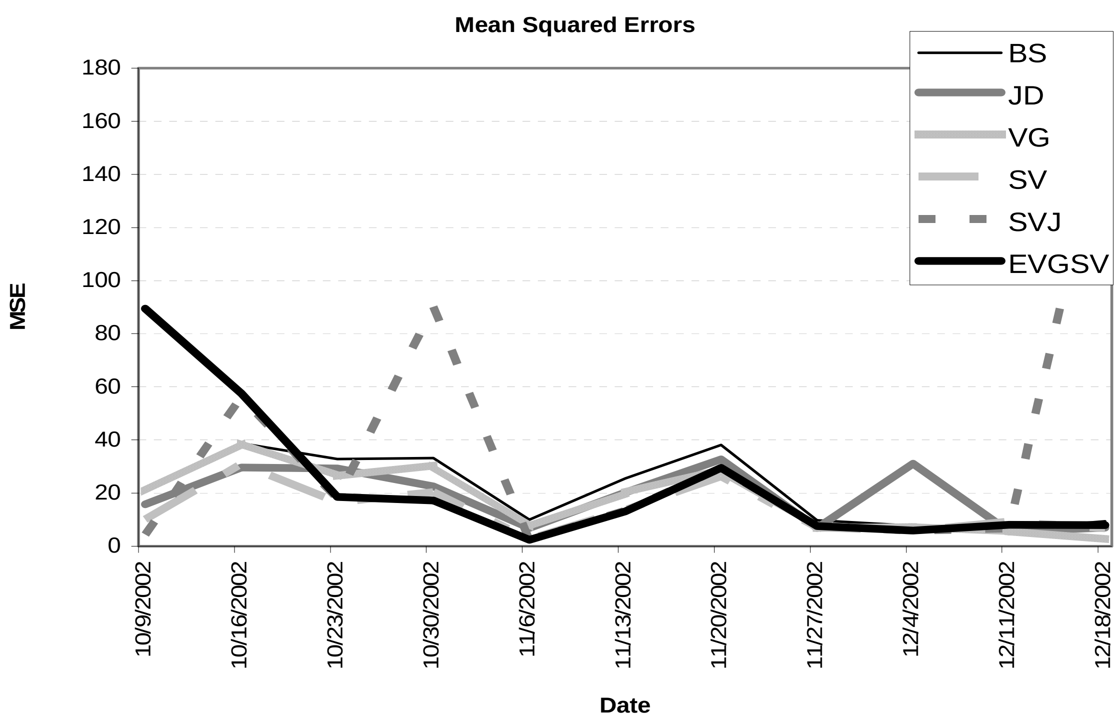
Therefore, it is highly recommended to perform out-of-sample tests to further check the stability of parameters and the robustness of my empirical results because misspecified models may achieve good in-sample results by over-fitting the data, but they have less predictive power for out-of-sample option valuation. Although stochastic volatility has mean-reverting properties and daily volatility surface is highly serially correlated, out-of-sample option pricing tests are still informative for my purpose since I am using weekly calibrations rather than daily calibrations[23] (See Bates (2003) and Lehnert (2003)).

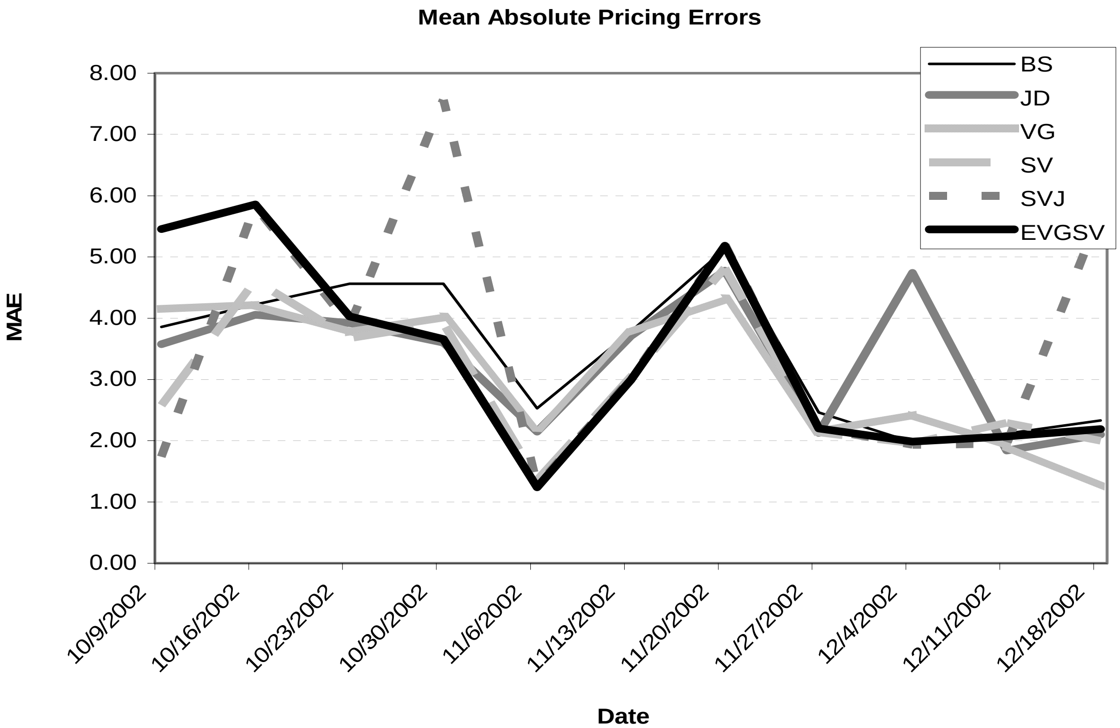
Using model-implied risk-neutral parameters obtained in the previous week, I compute out-of-sample pricing errors such as MSE, MAE, and MARE for each different model specification. The out-of-sample calibration results imply that there exist strong evidences of over-fitting for more complicated models including stochastic volatility. Figure 6 does not show superiority of stochastic volatility combined with Lévy jumps models such as SVJ and EVGSV over simple JD and VG models. If I examine estimated parameter values carefully, I can observe

[23] To overcome this drawback of daily calibrations, Lehar et al. (2002) uses the *sliding window technique* that re-estimates the models within relatively short time intervals to allow for changing parameters without re-estimating the option pricing models every day. Then, the models are tested out-of-sample on the next window of ten days. Finally, the time window is shifted by ten days and the models are reestimated.

that the stochastic volatility parameters such as $\tilde{\kappa}_J$, and $\tilde{\sigma}_v$ have relatively excessive variations depending on calibration weeks evidenced by larger values of standard deviations. Therefore, it seems that these unstable parameters across sample periods are hard to represent the overall levels of parameter values and result in excellent in-sample fits but poor out-of-sample performances.

MSE, MAE and MARE measures undermine the usefulness of stochastic volatility combined with Lévy jumps models since they generate a few excess pricing errors of SVJ or EVGSV coming from unsteady parameters, which dominate remaining pricing performances across calibration weeks. Therefore, I further classify out-of-sample pricing errors based on moneyness/maturity categories. For comparison purpose, I keep the same table formats as in-sample pricing errors performances.





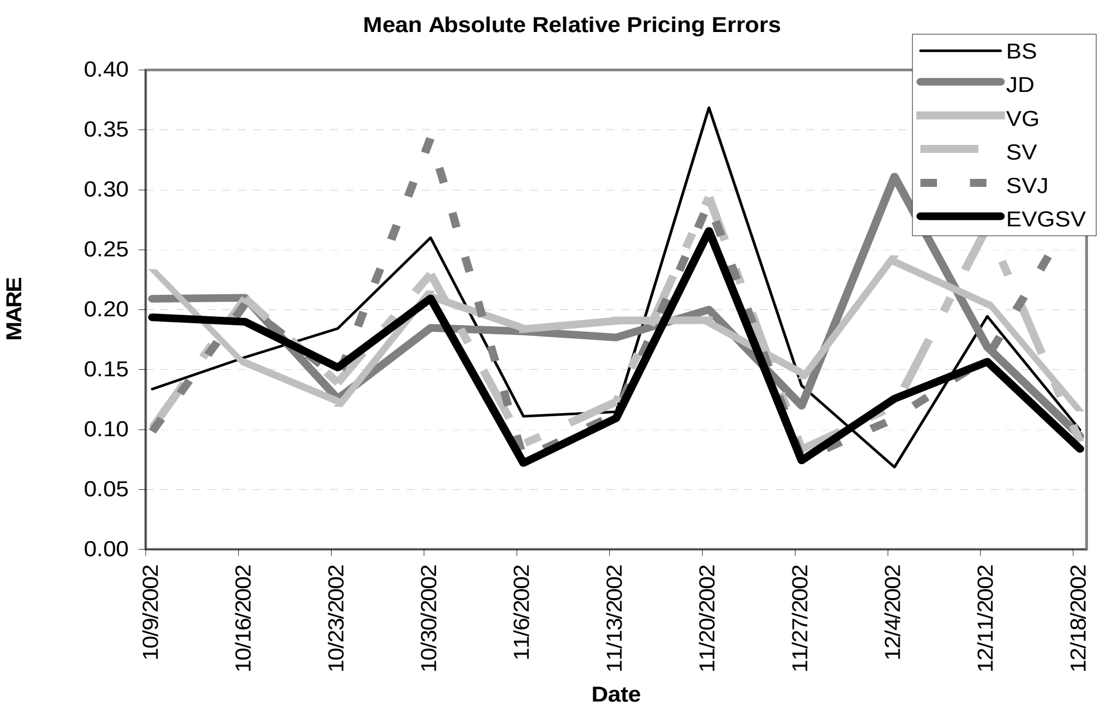


Figure 6. Out-of-Sample Model Performance Comparisons by Pricing Errors Measures. The figures plot time-series patterns of measures of pricing errors such as MSE, MAE, and MARE obtained by out-of-sample weekly calibrations for 6 different model specifications.

Table VI

Out-of-Sample Pricing Errors in Moneyness/Maturity Categories

The tables report average values of absolute pricing errors (APE) and absolute relative pricing errors (ARPE) in moneyness/maturity category. Median values of pricing errors are reported in square brackets.

Panel A: Pure Diffusions and Pure Stochastic Volatility Models

Models	Moneyness (K/S)	Call Options Days-to-Expiration (Maturity)					
		ST (< 30 days)		MT (30 – 120 days)		LT (>120 days)	
		APE	ARPE	APE	ARPE	APE	ARPE
BS	ITM	2.7746	0.1154	3.8850	0.1221	2.7864	0.1721
	(< 0.97)	[2.6321]	[0.0641]	[3.8255]	[0.0730]	[2.1815]	[0.1197]
	ATM	2.6918	0.1370	4.0502	0.1802	4.5010	0.2350
	(0.97 – 1.03)	[1.8964]	[0.0666]	[2.7638]	[0.0722]	[4.7791]	[0.1627]
	OTM	3.1371	0.1581	3.3034	0.1845	3.0406	0.2053
	(> 1.03)	[2.7022]	[0.1152]	[2.6799]	[0.0987]	[2.2441]	[0.1349]
SV	ITM	2.7435	0.1289	3.5942	0.0997	2.6229	0.1414
	(< 0.97)	[2.1909]	[0.0756]	[3.5171]	[0.0721]	[1.4466]	[0.0747]
	ATM	2.7007	0.1540	3.4729	0.1662	2.7576	0.2047
	(0.97 – 1.03)	[2.7151]	[0.0845]	[3.2309]	[0.1018]	[2.2825]	[0.0865]
	OTM	2.5818	0.2020	2.6827	0.1565	2.1741	0.1658
	(> 1.03)	[1.9379]	[0.1065]	[2.1687]	[0.0812]	[1.8292]	[0.0799]

Panel B: Finite Activity Lévy Jumps with/without SV

JD	ITM	2.5196	0.1398	4.1166	0.1238	2.9804	0.2659
	(< 0.97)	[2.0366]	[0.0523]	[3.7154]	[0.0880]	[2.9799]	[0.1667]
	ATM	2.7529	0.2045	3.9891	0.1962	3.8429	0.2023
	(0.97 – 1.03)	[2.5772]	[0.0875]	[2.9458]	[0.1152]	[3.3964]	[0.1318]
	OTM	2.9710	0.1824	3.0837	0.1708	2.8185	0.1859
	(> 1.03)	[3.0130]	[0.1095]	[2.5911]	[0.1216]	[2.2089]	[0.0857]
SVJ	ITM	2.9776	0.1294	4.3195	0.1141	2.4721	0.1271
	(< 0.97)	[1.9272]	[0.0830]	[3.5885]	[0.0758]	[1.2744]	[0.0907]
	ATM	5.9950	0.2886	3.8980	0.1644	2.4809	0.1767
	(0.97 – 1.03)	[3.2596]	[0.1217]	[3.0799]	[0.1016]	[1.8104]	[0.0864]
	OTM	2.8457	0.1563	2.9007	0.1568	2.1223	0.1799
	(> 1.03)	[1.9119]	[0.0991]	[2.1540]	[0.0789]	[1.7871]	[0.0637]

Panel C: Infinite Activity Pure Lévy Jumps with/without Diffusions and SV

Models	Moneyness (K/S)	Call Options Days-to-Expiration (Maturity)					
		ST (< 30 days)		MT (30 – 120 days)		LT (>120 days)	
		APE	ARPE	APE	ARPE	APE	ARPE
VG	ITM (< 0.97)	2.4209 [2.1390]	0.1221 [0.0687]	3.4317 [3.4364]	0.1466 [0.0710]	2.9795 [2.6365]	0.2407 [0.1632]
	ATM (0.97 – 1.03)	2.4959 [1.6740]	0.2021 [0.0781]	3.7399 [2.3979]	0.1900 [0.0859]	3.9130 [3.6349]	0.1918 [0.1127]
	OTM (> 1.03)	3.1477 [2.2690]	0.2105 [0.1201]	2.8911 [2.3168]	0.1689 [0.1112]	2.6830 [1.5316]	0.1911 [0.0753]
EVGSV	ITM (< 0.97)	3.9020 [2.5412]	0.1305 [0.0934]	4.3192 [3.5065]	0.1141 [0.0733]	3.2945 [1.6473]	0.1720 [0.1030]
	ATM (0.97 – 1.03)	3.7728 [3.1481]	0.1488 [0.1036]	4.0620 [3.7193]	0.1633 [0.0978]	2.6382 [2.4261]	0.1672 [0.0872]
	OTM (> 1.03)	2.5424 [1.9955]	0.1519 [0.1008]	2.5717 [2.3267]	0.1392 [0.0836]	1.9259 [1.5212]	0.1584 [0.0704]

From Table VI, I find that most excessive pricing errors for SVJ and EVGSV models in Figure 6 originate from ITM options. Stochastic volatility specifications combined with Lévy jumps models does not add any explanatory power for ITM options for out-of-sample calibrations. In most cases, the simplest BS model performs better than more complicated models do. These results are mainly comply with the rationale which some researchers exclude ITM options explicitly due to insensitivity of model specifications because of positive intrinsic values of ITM options (Carr et al. (2002) and Carr and Wu (2003)).

However, for the ATM and OTM options, out-of-sample pricing errors performances are, in general, consistent with those of in-sample calibrations. SV, SVJ and EVGSV models considerably reduce pricing errors for long-term ATM and OTM options compared to BS, JD, and VG models, strongly implying that stochastic volatility with/without inclusions with Lévy jumps models play an important role in improving empirical fits for SP500 index options. For example, APE and ARPE of BS model are \$4.5010 and 23.5% (\$3.0406 and 20.53%) for long-term ATM (long-term OTM) options. On the other hand, for SVJ and EVGSV models, APE and ARPE are only \$2.4809 and 17.67% (\$2.1223 and 17.99%) for long-term ATM (long-term OTM) options for SVJ model and \$2.6382 and 16.72% (\$1.9259 and 15.84%) for long term ATM (long-term OTM) options for EVGSV model. Similar pricing improvements are also found for mid-term ATM and OTM options for SVJ and EVGSV models. However, consistent with previous studies, the BS model outperforms more complicated model specifications for short-term ATM options even in the out-of-sample calibrations.

5.3. Why does stochastic volatility behave differently between statistical and risk-neutral processes?

Empirical results have confirmed that the role of stochastic volatility is different from between statistical process for underlying SP500 index returns and risk-neutral process for SP500 index options. It is striking that stochastic volatility does not add any explanatory powers for statistical process for SP500 index returns when it is incorporated with infinite-activity pure Lévy jumps models such as VG and NIG processes. However, stochastic volatility plays a major role in improving empirical fits once it is combined with a finite-activity Lévy jumps model such as Merton's JD model for SP500 index returns. Although Lévy jumps types are important for statistical process, it appears that stochastic volatility consistently contributes to enhancing model performances for SP500 index options without regard to finite or infinite activity Lévy jumps.

A finite activity Poisson jump-diffusion model consists of frequent small diffusions and infrequent large discontinuous jumps. Consequently, this model specification is difficult to mimic the role of stochastic volatility. However, it seems that infinite activity jumps models such as VG and NIG processes can generate infinite numbers of small jumps mimicking diffusions as well as finite numbers of large jumps by the definition of finite variation. Interesting feature unexploited yet is that I can show theoretically whether VG or NIG processes can mimic the role of stochastic volatility, at least, in high-frequency data such as daily or weekly time horizons. If so, I do not have to consider stochastic volatility explicitly for infinite activity Lévy jumps models since the most statistical processes for SP500 index time-series returns have daily or weekly frequencies. They have much shorter time horizons compared to the SP500 index options panel data with different option maturities, which may be longer than several months. Hence, stochastic volatility can readily contribute to improving pricing performances for long-term ATM and OTM options.

6. Conclusions and Suggestions of Future Research

In this chapter, I have shown a variety of empirical evidences to support redundancy of stochastic volatility for SP500 index returns when stochastic volatility is incorporated with infinite activity pure Lévy jumps models and importance to reduce pricing errors for SP500 index options without regard to jumps specifications. This chapter also has demonstrated that SVJ and EVGSV models perform almost equally well for option pricing, which strongly imply that whether pure Lévy jumps specifications are finite or infinite activity is not important factors to enhance model performances once stochastic volatility is incorporated. Therefore, theoretical justifications are necessary to complement these empirical findings along with further relevant empirical extensions.

These empirical findings are important for a number of financial applications including risk management, hedging, and derivative pricing. For instance, first, in the perspective of risk management, Lehar et al. (2002) shows that more complex GARCH or stochastic volatility option pricing models can improve on the BS model only for the purpose of pricing, but not for the risk management tool such as Value-at-Risk (VaR), the expected loss that will only be exceeded with probability α. These findings strongly imply a discrepancy between appropriateness for pricing and for risk management.

Second, in the point of view of hedging, as Belledin and Schlag (1999) point out, option traders have a tendency to give more weights on a model with superior hedging performance since changes in the value of their positions are main concerns to them, compared to pricing quality of a model measured by absolute or relative pricing errors. Therefore, the usefulness of a given model primarily relies on its ability to properly capture the price changes of options given a change in the value of the underlying assets. To perform hedging test, I may think of the procedure taken by Bakshi et al. (2000) to compare the hedging performance of each model.

Third, in the side of derivative pricing, unlike the objective distributions, risk-neutral distributions recovered from option prices suffer fundamental structural change evidenced by implied volatility smile/skew patterns before and after the crash of 1987 due to the large change in the risk aversion of the average investor. Consequently, put options tend to be frequently overpriced for hedging since the 1987 crash, which is termed as 'crashophobia'. Hence, it is also required to study if my empirical findings still hold for put options since put options can be readily obtained using the put-call parity. As a result, while the numerically intensive techniques have been proposed to price various types of derivatives, the continuous endeavors to find better stochastic volatility specifications are still expected to remain important subjects of future research to further improve the empirical fits of actual returns and option data.

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