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Stress Analysis of Ramberg-Osgood and Hollomon 1-D Axial Rods

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Stress Analysis of Ramberg-Osgood and Hollomon 1-D Axial Rods

A Thesis

Submitted to the Graduate Faculty of the
University of New Orleans
in partial fulfillment of the
requirements for the degree of

Masters of Science
in
Applied Mathematics

by
Ronald Joseph Giardina, Jr.
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I am grateful for every encouraging word and helpful hand.
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Abstract

In this paper we present novel analytic and finite element solutions to 1-D straight rods made of Ramberg-Osgood and Hollomon type materials. These material models are studied because they are a more accurate representation of the material properties of certain metals used often in manufacturing than the simpler composite linear types of stress/strain models. Here, various types of loads are considered and solutions are compared against some linear models. It is shown that the nonlinear models do have manageable solutions, which produce important differences in the results - attributes which suggest that these models should take a more prominent place in engineering analysis.
Chapter 1

Introduction

It is well known that many modern high strength alloys and steels can be modeled by simple power-law functions between stress, $\sigma$, and strain, $\varepsilon$. The two most widely used such power-laws are the Ramberg-Osgood power-law, $\varepsilon = \frac{\sigma}{E} + k\left(\frac{\sigma}{E}\right)^m$, where $m$ and $k$ are material constants and $E$ is Young’s modulus, first noted by W. Ramberg and W. R. Osgood in 1943 [1], and the Hollomon power-law, $\sigma = Ke^n$, where $K$ and $n$ are material constants with $K$ in Hollomon not to be confused with $k$ in Ramberg-Osgood, noted by J. H. Hollomon in 1945 [2]. Some of the more common of these Ramberg-Osgood type materials are Titanium and various Aluminum alloys, both frequently used where strength coupled with light weight are design requirements, whereas a Hollomon power-law can be used to model many heat treated, or annealed, metals. Currently, these materials are modeled in many commercial software packages using a composition of linear models. This is sufficient for very small amounts of strain where Hooke’s Law is still relevant, but when it is necessary to model the behavior of these materials outside the comfort of the elastic range, these models rely upon an extension of the linear Hooke’s-type relationship to approximate the plastic behavior of the material. For properly chosen material constants the Ramberg-Osgood and Hollomon relationships can provide much more accurate models of the material under consideration. We note that in the Ramberg-Osgood relationship we will use $A$ to represent the constant coefficient of the linear $\sigma$ term and $B$ to represent the constant coefficient of the nonlinear $\sigma$ term and keeping $m$ the same in order to simplify our expressions. In chapter two we provide analytic solutions to straight rods subject to constant, point, and stepped load axial forces. In chapter three we provide finite element solutions for these same loads. In chapter four we discuss pointwise convergence of the finite element models and some error-estimation. In chapter five we will compare these results to some linear models and in chapter six we briefly discuss the implication of the results.

1.1 Static Equilibrium Equation

At equilibrium the steady state momentum equation for a straight rod subject to axial force is given by

$$\frac{d}{dx}\sigma(x) + f(x) = 0$$

If we presume the rod to be fixed at one end and free at the other then the rod would experience zero displacement at the point where it is fixed and zero strain at the point where it is free. These will generate our boundary conditions. We note $\varepsilon = \frac{du}{dx}$.

1.1.1 Ramberg-Osgood

We employ the absolute value in the nonlinear term of the Ramberg-Osgood equation to the $m - 1$ power in order to preserve the direction of force. We define $\phi(\sigma) = \varepsilon = A\sigma + B|\sigma|^{m-1}\sigma$ and subsequently express $\sigma$ in the static equilibrium equation with the inverse of $\phi$.

$$\begin{cases}
\frac{d}{dx}\phi^{-1}\left(\frac{du}{dx}\right) + f(x) = 0 \\
u(0) = 0 \\
\frac{du}{dx}(L) = 0
\end{cases}$$

(1.1)
1.1.2 Hollomon

Similarly to Ramberg-Osgood, we shall take the absolute value of the $\varepsilon$ term to the $n-1$ power times the $\varepsilon$ term to preserve the direction of force.

\[
\begin{align*}
\frac{d}{dx} \left( K \left| \frac{du}{dx} \right|^{n-1} \frac{du}{dx} \right) + f(x) & = 0 \\
u(0) & = 0 \\
\frac{du}{dx} (L) & = 0
\end{align*}
\]

(1.2)

1.2 Loads

There are three types of loads which we will consider for $f(x)$ in our steady state equation to be applied over our rod: a constant load, a point load, and the combination of these two, a stepped load. We will make use of the step function centered at some point, $H(x - x_0)$, and it will be important to note that the integral of the step function is $(x - x_0)H(x - x_0)$, which is referred to as the ramp function. We will take this function to be undefined at $x_0$ which will allow us to make the following observations: $(H(x - x_0))^a = H(x - x_0)$ and $(1 - H(x - x_0))^a = 1 - H(x - x_0)$, where $a$ is some exponent.

1.2.1 Constant Load

A constant load is a load which is applied evenly at every point along a rod. From this we might expect the stress induced upon a rod to be greater closer to the fixed point of the rod because those points experience the load placed directly upon them and the loads of every point behind them. We express a constant load as follows.

\[ f(x) = \frac{f_0}{L} \]

1.2.2 Point Load

A point load is a load applied at a single point somewhere along the rod. Every point of the rod between the point load and the fixed point will experience equal stress, as each point is being pulled upon with the same load generated by the point at which the load is applied. Every point behind the point load will experience no stress at all. We express a point load as follows.

\[ f(x) = f_0 \delta(x - x_0), \quad 0 \leq x_0 \leq L \]

1.2.3 Stepped Load

A stepped load is a load which is applied over some interval along the rod. We could expect all points in between the interval and the fixed point to behave like a point load experiencing the sum total of the force applied over the step. Within the interval the load would behave like a constant load varying over the length of the interval. Behind the interval, there will be no stress at all, again like the point load. We express the stepped load as follows.

\[ f(x) = \frac{f_0}{b-a} [H(x-a) - H(x-b)], \quad 0 \leq a < b \leq L \]

where $a$ and $b$ define the ends of our interval on the rod.
Chapter 2

Analytic Solutions

2.1 Ramberg-Osgood

2.1.1 Constant Load

We begin by integrating (1.1).

\[ \phi^{-1} \left( \frac{du}{dx} \right) + \frac{f_0}{L} x + c_1 = 0 \]

With our boundary conditions we can easily solve for the integration constant, where we shall find that \( c_1 = -f_0 \).

With a bit of rearranging and subsequently taking \( \phi \) of both sides we find the following.

\[ \frac{du}{dx} = \phi \left( f_0 \left( \frac{1}{L} (L-x) \right) \right) = A f_0 \left( \frac{1}{L} (L-x) \right) + B \left| f_0 \left( \frac{1}{L} (L-x) \right)^{m-1} \right| \]

To solve for \( u \) we integrate once more.

\[ u(x) = -\frac{1}{2} A f_0 \left( \frac{1}{L} (L-x) \right)^2 - \frac{1}{m+1} B \left| f_0 \left( \frac{1}{L} (L-x) \right)^{m-1} \right| \]

Again, we can easily find our integration constant here using our boundary conditions.

\[ c_2 = \frac{1}{2} A f_0 \left( \frac{1}{L} (L-x) \right)^2 + \frac{1}{m+1} B \left| f_0 \left( \frac{1}{L} (L-x) \right)^{m-1} \right| \]

This gives the full solution for the displacement of our rod, \( u \).

\[ u(x) = \frac{1}{2} A f_0 \left[ L^2 - (L-x)^2 \right] + \frac{1}{m+1} B \left| f_0 \left( \frac{1}{L} (L-x) \right)^{m-1} \right| \left[ L^{m+1} - (L-x)^{m+1} \right] \]

2.1.2 Point Load

\[ \phi^{-1} \left( \frac{du}{dx} \right) + f_0 H(x-x_0) + c_1 = 0 \]

We find the constant of integration to be the same as with the constant load, and rewrite our expression to solve for \( \epsilon \).

\[ \frac{du}{dx} = \phi \left( f_0 [1 - H(x-x_0)] \right) = A f_0 [1 - H(x-x_0)] + B \left| f_0 [1 - H(x-x_0)] \right| \]

which can be represented as follows.
\[
\frac{du}{dx} = \begin{cases} 
A f_0 + B |f_0|^{m-1} f_0, & x < x_0 \\
0, & x > x_0 
\end{cases}
\]

We integrate once more.

\[
u(x) = A f_0 [x - (x - x_0) H(x - x_0)] + B |f_0|^{m-1} f_0 [x - (x - x_0) H(x - x_0)] + c_2
\]

When we solve for the integration constant here we find that it is zero. So, we have for our full solution for \(u\) the following.

\[
u(x) = A f_0 [x - (x - x_0) H(x - x_0)] + B |f_0|^{m-1} f_0 [x - (x - x_0) H(x - x_0)]
\]

which gives the following relationship.

\[
u(x) = \begin{cases} 
A f_0 x + B |f_0|^{m-1} f_0 x, & x < x_0 \\
A f_0 x_0 + B |f_0|^{m-1} f_0 x_0, & x > x_0 
\end{cases}
\]

2.1.3 Stepped Load

\[
\phi^{-1} \left( \frac{du}{dx} \right) + \frac{f_0}{b - a} [(x - a) H(x - a) - (x - b) H(x - b)] + c_1 = 0
\]

The constant of integration is once again \(-f_0\) and we set our expression to solve for \(du/dx\).

\[
\frac{du}{dx} = \phi \left( f_0 \left( 1 - \frac{1}{b - a} [(x - a) H(x - a) - (x - b) H(x - b)] \right) \right)
= A \left( f_0 \left( 1 - \frac{1}{b - a} [(x - a) H(x - a) - (x - b) H(x - b)] \right) \right)
+ B \left| f_0 \left( 1 - \frac{1}{b - a} [(x - a) H(x - a) - (x - b) H(x - b)] \right) \right|^{m-1}
+ f_0 \left( 1 - \frac{1}{b - a} [(x - a) H(x - a) - (x - b) H(x - b)] \right)
\]

which can be represented as follows.

\[
\frac{du}{dx} = \begin{cases} 
A f_0 + B |f_0|^{m-1} f_0, & x < a \\
A f_0 \left( \frac{b - x}{b - a} \right) + B |f_0|^{m-1} f_0 \left( \frac{b - x}{b - a} \right)^m, & a < x < b \\
0, & x > b 
\end{cases}
\]

To find \(u\) we will integrate each expression for \(du/dx\) over their respective intervals rather than attempt to integrate the general expression for \(du/dx\). Taking the integral over the first interval, \([0, a)\), we find the following.

\[
u(x) = A f_0 x + B |f_0|^{m-1} f_0 x + c_2, \quad 0 \leq x < a
\]

where, using our boundary conditions, we find that our integration constant here is zero. When solving for \(u\) over the next interval we can take \(u(a)\) from the expression over the first interval to be a boundary condition for the second interval, \((a, b)\), to solve for the subsequent integration constant. We can employ the same for the subsequent interval, \((b, L]\). We find, then, the following for \(u\).

\[
u(x) = A \frac{f_0}{b - a} \left( b x - \frac{1}{2} x^2 \right) - B |f_0|^{m-1} f_0 \left( \frac{1}{b - a} \right)^m \left( \frac{1}{m + 1} (b - x)^{m+1} \right) + c_3, \quad a < x < b
\]

Solving for our constant of integration using \(u(a)\) as the boundary condition on this interval we find the following.

\[
c_3 = -A f_0 \left( \frac{a^2}{2(b - a)} \right) + B |f_0|^{m-1} f_0 \left( \frac{b + am}{m + 1} \right)
\]
This gives the solution for \( u \) over the interval \((a, b)\) as follows.

\[
\begin{align*}
  u(x) &= \frac{A f_0}{2(b-a)} (2bx - a^2 - x^2) \\
  &+ \frac{B}{m+1} |f_0|^{m-1} f_0 \left( \frac{b + am - (b-x) \left( \frac{b-x}{b-a} \right)^m}{m} \right), \quad a < x < b
\end{align*}
\]

Integrating over our final interval we get a constant, which will be the value of \( u(b) \) from the previous interval. We have the following relationship for the full expression for \( u \).

\[
\begin{align*}
  u(x) &= \begin{cases} 
    A f_0 x + B |f_0|^{m-1} f_0 x, & 0 \leq x < a \\
    \frac{A f_0}{2(b-a)} (2bx - a^2 - x^2) + \frac{B}{m+1} |f_0|^{m-1} f_0 \left( b + am - (b-x) \left( \frac{b-x}{b-a} \right)^m \right), & a < x < b \\
    A f_0 \left( \frac{b^2-a^2}{2(b-a)} \right) + \frac{B}{m+1} |f_0|^{m-1} f_0 (b + am), & b < x \leq L
  \end{cases}
\end{align*}
\]

### 2.2 Hollomon

#### 2.2.1 Constant Load

We begin by integrating (1.2).

\[
K \left| \frac{du}{dx} \right|^{n-1} \frac{du}{dx} + f_0 L + c_1 = 0
\]

From our boundary condition we find the constant of integration is \(- f_0\) which gives the solution for \( \frac{du}{dx} \) as follows.

\[
\frac{du}{dx} = \left| f_0 K \right|^{\frac{1}{n+1}} f_0 K \left( L - x \right)^{\frac{1}{n}}
\]

We integrate once more to find \( u \).

\[
\begin{align*}
  u(x) &= -\frac{n}{n+1} \left| f_0 K \right|^{\frac{1}{n+1}} f_0 K \left( L - x \right)^{\frac{1}{n}} + c_2
\end{align*}
\]

We again solve for our integration constant, which gives the following.

\[
\begin{align*}
  c_2 &= \frac{n}{n+1} \left| f_0 K \right|^{\frac{1}{n+1}} f_0 K \left( L^{\frac{n+1}{n}} - \left( L - x \right)^{\frac{n+1}{n}} \right)
\end{align*}
\]

which gives the full expression for \( u \) as follows.

\[
\begin{align*}
  u(x) &= \frac{n}{n+1} \left| f_0 K \right|^{\frac{1}{n+1}} f_0 K \left[ L^{\frac{n+1}{n}} - \left( L - x \right)^{\frac{n+1}{n}} \right]
\end{align*}
\]

#### 2.2.2 Point Load

\[
K \left| \frac{du}{dx} \right|^{n-1} \frac{du}{dx} + f_0 H(x-x_0) + c_1 = 0
\]

The constant of integration is once again \(- f_0\) and we rewrite to solve for \( \frac{du}{dx} \).

\[
\begin{align*}
  \frac{du}{dx} &= \left| f_0 K \right|^{\frac{1}{n+1}} f_0 K \left[ 1 - H(x-x_0) \right]
\end{align*}
\]

which can be represented as follows.
\[
\frac{du}{dx} = \begin{cases} 
\frac{f_0}{K} \frac{1}{n} \frac{f_0}{K} x, & x < x_0 \\
0, & x > x_0 
\end{cases}
\]

We integrate once more.

\[
u(x) = \left| \frac{f_0}{K} \right|^\frac{1}{n} \int_0^x \frac{f_0}{K} [x - (x - x_0)H(x - x_0)] + c_2
\]

We will find that our constant of integration here is zero, so we have the full expression for \( u \) as follows.

\[
u(x) = \left| \frac{f_0}{K} \right|^\frac{1}{n} \int_0^x \frac{f_0}{K} [x - (x - x_0)H(x - x_0)]
\]

which gives the following relationship.

\[
u(x) = \begin{cases} 
\left| \frac{f_0}{K} \right|^\frac{1}{n} \frac{f_0}{K} x, & x < x_0 \\
0, & x > x_0 
\end{cases}
\]

### 2.2.3 Stepped Load

\[
\frac{K}{dx} \left| \frac{du}{dx} \right|^{n-1} \frac{du}{dx} + \frac{f_0}{b-a} [(x-a)H(x-a) - (x-b)H(x-b)] + c_1 = 0
\]

With the constant of integration again \( -f_0 \), we solve for \( \frac{du}{dx} \).

\[
\frac{du}{dx} = \left| \frac{f_0}{K} \right|^\frac{1}{n} \frac{f_0}{K} \left( 1 - \frac{1}{b-a} [(x-a)H(x-a) - (x-b)H(x-b)] \right)^\frac{1}{n}
\]

which can be represented as follows.

\[
\frac{du}{dx} = \begin{cases} 
\left| \frac{f_0}{K} \right|^\frac{1}{n} \frac{f_0}{K} x, & x < a \\
\left| \frac{f_0}{K} \right|^\frac{1}{n} \frac{f_0}{K} (\frac{b-a}{b-a}) x, & a < x < b \\
0, & x > b 
\end{cases}
\]

Just as with Ramberg-Osgood before, we will integrate over each interval using for boundary conditions, \( u(0) \), \( u(a) \), and \( u(b) \) where appropriate. Over the first interval, \([0,a)\), we find the following.

\[
u(x) = \left| \frac{f_0}{K} \right|^\frac{1}{n} \frac{f_0}{K} x + c_2, \ 0 < x < a
\]

where we will find the constant of integration to be zero. For the subsequent interval, \((a,b)\), we find the following.

\[
u(x) = -\frac{n}{n+1} \left| \frac{f_0}{K} \right|^\frac{1}{n} \frac{f_0}{K} x + c_3, \ a < x < b
\]

where we use \( u(a) \) to solve for the constant of integration.

\[
c_3 = \frac{n}{n+1} \left| \frac{f_0}{K} \right|^\frac{1}{n} \frac{f_0}{K} \left( \frac{b-a}{n} \right)
\]

which gives the solution for \( u \) over the interval \((a,b)\) as follows.
\[
\begin{align*}
    u(x) &= \frac{n}{n+1} \left[ \frac{f_0}{K} \frac{1}{n+1} \frac{f_0}{K} \left( b + \frac{a}{n} - (b-x) \frac{b-x}{b-a} \right) \right], \quad a < x < b
\end{align*}
\]

Our final interval yields a constant after integration which will be the value of \( u(b) \) from the previous interval. We have then the following relationship for the full expression for \( u \).

\[
\begin{align*}
    u(x) &= \begin{cases} 
        \frac{f_0}{K} \frac{1}{n+1} \frac{f_0}{K} x, & 0 \leq x < a \\
        \frac{n}{n+1} \frac{f_0}{K} \frac{1}{n} \frac{f_0}{K} \left( b + \frac{a}{n} - (b-x) \left( \frac{b-x}{b-a} \right)^{\frac{1}{n}} \right), & a < x < b \\
        \frac{n}{n+1} \frac{f_0}{K} \frac{1}{n} \frac{f_0}{K} \left( b + \frac{a}{n} \right), & b < x \leq L
    \end{cases}
\end{align*}
\]
Chapter 3

Finite Element Solutions

We will employ the well known Galerkin method to (1.1) and (1.2) to construct a finite element model of these two systems. We note that \( \psi_1^{(e)} \) and \( \psi_2^{(e)} \) are the shape functions over some given interval \( e \).

We will make use of the following.

\[
\chi_i(x_0) = \begin{cases} 
0 & x_0 \in [x_1^{(i-1)}, x_2^{(i-1)}] \\
1 & x_0 \in [x_1^{(i)}, x_2^{(i)}] \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
\Gamma_{a,b}(x_0) = \begin{cases} 
1 & x_0 \in (a, b) \\
0 & x_0 \not\in (a, b)
\end{cases}
\]

3.1 Ramberg-Osgood

We apply Galerkin first to (1.1) for our three loads.

3.1.1 Constant Load

\[
\int_{x_1^{(e)}}^{x_2^{(e)}} \left( \frac{d}{dx} \phi^{-1} \left( \frac{du}{dx} \right) + \frac{f_0}{L} \right) \left[ \begin{array}{c} \psi_1^{(e)} \\ \psi_2^{(e)} \end{array} \right] dx = 0
\]

We next integrate the left-most term by parts and then take the remaining integrals and simplify our expression.

\[
-\int_{x_1^{(e)}}^{x_2^{(e)}} \phi^{-1} \left( \frac{du}{dx} \right) \left[ \begin{array}{c} \psi_1^{(e)} \\ \psi_2^{(e)} \end{array} \right] dx + \phi^{-1} \left( \frac{du}{dx} \right) \left[ \begin{array}{c} \psi_1^{(e)} \\ \psi_2^{(e)} \end{array} \right] \bigg|_{x_1^{(e)}}^{x_2^{(e)}} + \int_{x_1^{(e)}}^{x_2^{(e)}} \frac{f_0}{L} \left[ \begin{array}{c} \psi_1^{(e)} \\ \psi_2^{(e)} \end{array} \right] dx = 0
\]

\[
\implies \phi^{-1} \left( \frac{u_2^{(e)} - u_1^{(e)}}{x_2^{(e)} - x_1^{(e)}} \right) \left[ \begin{array}{c} -1 \\ 1 \end{array} \right] = \frac{f_0}{2L} \Delta x^{(e)} \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] + \left[ Q_1^{(e)} \\ Q_2^{(e)} \right]
\]

From this result, we construct the global finite element system as follows.
3.1.2 Point Load

\[ \int x^2_i \left( \frac{d}{dx} \phi^{-1} \left( \frac{du}{dx} \right) + f_0 \delta(x-x_0) \right) \left[ \begin{array}{c} \psi_1^{(e)} \\ \psi_2^{(e)} \end{array} \right] dx = \int x^2_i \frac{d}{dx} \phi^{-1} \left( \frac{du}{dx} \right) \left[ \begin{array}{c} \psi_1^{(e)} \\ \psi_2^{(e)} \end{array} \right] dx + \int x^2_i f_0 \delta(x-x_0) \left[ \begin{array}{c} \psi_1^{(e)} \\ \psi_2^{(e)} \end{array} \right] dx = 0 \]

We now integrate and simplify our expression.

\[ \int x^2_i \phi^{-1} \left( \frac{du}{dx} \right) \frac{d}{dx} \left[ \begin{array}{c} \psi_1^{(e)} \\ \psi_2^{(e)} \end{array} \right] dx \]

where \( i \) is the index associated with the point load. We note that the point load lies upon an existing node, so the force is counted twice between the two elements as defined by \( \chi_i(x_0) \). We split the force, half over each element, to have the sum total of the force we are applying be one on the node rather than counting it twice.

From this we construct the global finite element system.

\[ \left[ \begin{array}{c} \phi^{-1} \left( \frac{u_1^{(1)}-u_1^{(2)}}{\Delta x^{(1)}} \right) \\ \vdots \\ \phi^{-1} \left( \frac{u_1^{(N-1)}-u_1^{(N)}}{\Delta x^{(N-1)}} \right) \end{array} \right] \left[ \begin{array}{c} \phi^{-1} \left( \frac{u_2^{(1)}-u_1^{(1)}}{\Delta x^{(1)}} \right) \\ \vdots \\ \phi^{-1} \left( \frac{u_2^{(N-1)}-u_1^{(N-1)}}{\Delta x^{(N-1)}} \right) \end{array} \right] \left[ \begin{array}{c} \psi_1^{(e)} \\ \psi_2^{(e)} \end{array} \right] = \left[ \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \right] + \left[ \begin{array}{c} Q_1^{(e)} \\ \vdots \\ Q_2^{(N-1)} \end{array} \right] \]

where again \( u_1^{(1)} \) and \( Q_2^{(N-1)} \) are both zero.

3.1.3 Stepped Load

\[ \int x^2_i \left( \frac{d}{dx} \phi^{-1} \left( \frac{du}{dx} \right) + \frac{f_0}{b-a} (H(x-a) - H(x-b)) \right) \left[ \begin{array}{c} \psi_1^{(e)} \\ \psi_2^{(e)} \end{array} \right] dx = \int x^2_i \frac{d}{dx} \phi^{-1} \left( \frac{du}{dx} \right) \left[ \begin{array}{c} \psi_1^{(e)} \\ \psi_2^{(e)} \end{array} \right] dx + \int x^2_i f_0 \frac{H(x-a) - H(x-b)}{b-a} \left[ \begin{array}{c} \psi_1^{(e)} \\ \psi_2^{(e)} \end{array} \right] dx = 0 \]
We once more employ integration by parts and simplify the expression.

\[
\phi^{-1} \left( \frac{u_2^{(e)} - u_1^{(e)}}{x_2^{(e)} - x_1^{(e)}} \right) \left[ -1 \right] = \frac{f_0}{b - a} \frac{1}{2} \Delta x^{(e)} \Gamma_{a,b}(x_0) \left[ 1 \right] + \left[ Q_1^{(e)} \right]_1^{(e)} \left[ Q_2^{(e)} \right]
\]

From which we once again construct the global finite element system. We define the interval immediately following the point \( a \), which defines the beginning of our interval and sits upon a node, as the interval \( \alpha \), and the interval immediately preceding the point \( b \), which defines the end of our interval and is also on a node, as the interval \( \beta \).

\[
\begin{bmatrix}
-\phi^{-1} \left( \frac{u_2^{(1)} - u_1^{(1)}}{\Delta x^{(1)}} \right) \\
\phi^{-1} \left( \frac{u_2^{(1)} - u_1^{(1)}}{\Delta x^{(1)}} \right) - \phi^{-1} \left( \frac{u_2^{(2)} - u_1^{(2)}}{\Delta x^{(2)}} \right) \\
\phi^{-1} \left( \frac{u_2^{(N-2)} - u_1^{(N-2)}}{\Delta x^{(N-2)}} \right) \\
\phi^{-1} \left( \frac{u_2^{(N-1)} - u_1^{(N-1)}}{\Delta x^{(N-1)}} \right)
\end{bmatrix}
\begin{bmatrix}
\psi_1^{(e)} \\
\psi_2^{(e)} \\
\vdots \\
\vdots
\end{bmatrix}
= \frac{f_0}{b - a} \frac{1}{2} \Delta x^{(e)} \Gamma_{a,b}(x_0) \left[ 1 \right] + \left[ Q_1^{(e)} \right]_1^{(e)} \left[ Q_2^{(e)} \right]
\]

with \( u_1^{(1)} \) and \( Q_2^{(N-1)} \) both zero.

### 3.2 Hollomon

We now apply Galerkin to (1.2) under our three loads.

#### 3.2.1 Constant Load

\[
\int_{x_1^{(e)}}^{x_2^{(e)}} \left( K \frac{du}{dx} \right)^{n-1} \frac{du}{dx} \left[ \psi_1^{(e)} \right] \left[ \psi_2^{(e)} \right] dx + \int_{x_1^{(e)}}^{x_2^{(e)}} \frac{f_0}{L} \left[ \psi_1^{(e)} \right] \left[ \psi_2^{(e)} \right] dx = 0
\]

We integrate the left-most term by parts and take the remaining integrals and simplify our expression.

\[
- \int_{x_1^{(e)}}^{x_2^{(e)}} \left( K \frac{du}{dx} \right)^{n-1} \frac{du}{dx} \left[ \frac{\psi_1^{(e)}}{\psi_2^{(e)}} \right] dx + \left( K \frac{du}{dx} \right)^{n-1} \frac{du}{dx} \left[ \frac{\psi_1^{(e)}}{\psi_2^{(e)}} \right] \bigg|_{x_1^{(e)}}^{x_2^{(e)}}
\]

\[
+ \int_{x_1^{(e)}}^{x_2^{(e)}} \frac{f_0}{L} \left[ \frac{\psi_1^{(e)}}{\psi_2^{(e)}} \right] dx = 0
\]

\[
\Rightarrow K \left[ \frac{u_2^{(e)} - u_1^{(e)}}{x_2^{(e)} - x_1^{(e)}} \right]^{n-1} \left[ \frac{u_2^{(e)} - u_1^{(e)}}{x_2^{(e)} - x_1^{(e)}} \right] \left[ -1 \right] = \frac{f_0}{2L} \Delta x^{(e)} \left[ 1 \right] + \left[ Q_1^{(e)} \right]_1^{(e)} \left[ Q_2^{(e)} \right]
\]

From this we construct the global finite element system.
\[ K = \begin{bmatrix} u_2^{(1)} - u_1^{(1)} \Delta x^{(1)} & -u_2^{(1)} - u_1^{(1)} \Delta x^{(1)} & \cdots & -u_2^{(1)} - u_1^{(1)} \Delta x^{(1)} \\ u_2^{(2)} - u_1^{(2)} \Delta x^{(2)} & u_2^{(2)} - u_1^{(2)} \Delta x^{(2)} & \cdots & u_2^{(2)} - u_1^{(2)} \Delta x^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ u_2^{(N-2)} - u_1^{(N-2)} \Delta x^{(N-2)} & u_2^{(N-2)} - u_1^{(N-2)} \Delta x^{(N-2)} & \cdots & u_2^{(N-2)} - u_1^{(N-2)} \Delta x^{(N-2)} \\ \end{bmatrix} \]

\[ K = \begin{bmatrix} \delta \Delta x^{(1)} \\ \delta \Delta x^{(2)} \\ \vdots \\ \delta \Delta x^{(N-1)} \\ \end{bmatrix} + \begin{bmatrix} Q_1^{(1)} \\ Q_1^{(2)} \\ \vdots \\ Q_1^{(N-1)} \\ \end{bmatrix} (3.4) \]

Our boundary conditions imply \( u_1^{(1)} \) and \( Q_2^{(N-1)} \) are both zero.

### 3.2.2 Point Load

\[ \int_{x_1^{(e)}}^{x_2^{(e)}} \frac{d}{dx} \left( K \frac{du}{dx} \right)^{n-1} \left( K \right) dx = f_0 \delta(x-x_0) \left[ \psi_1^{(e)} \right]^{x_2^{(e)}}_{x_1^{(e)}} + f_0 \delta(x-x_0) \left[ \psi_2^{(e)} \right]^{x_2^{(e)}}_{x_1^{(e)}} = 0 \]

We integrate by parts and simplify our expression.

\[ -\int_{x_1^{(e)}}^{x_2^{(e)}} \frac{d}{dx} \left( K \frac{du}{dx} \right)^{n-1} dx + \int_{x_1^{(e)}}^{x_2^{(e)}} f_0 \delta(x-x_0) \left[ \psi_1^{(e)} \right]^{x_2^{(e)}}_{x_1^{(e)}} = 0 \]

\[ K \left[ \frac{u_2^{(e)} - u_1^{(e)}}{x_2^{(e)} - x_1^{(e)}} \right]^{n-1} \left[ \frac{u_2^{(e)} - u_1^{(e)}}{x_2^{(e)} - x_1^{(e)}} \right]^{x_2^{(e)}}_{x_1^{(e)}} = \frac{f_0}{2} x_i(x_0) + \left[ \frac{Q_1^{(e)}}{Q_2^{(e)}} \right] \]

This allows us to construct the global finite element system.
where again \( u_1^{(1)} \) and \( Q_2^{(N-1)} \) are both zero and \( x_0 \) lies upon a node.

### 3.2.3 Stepped Load

\[
\int_{x_1^{(e)}}^{x_2^{(e)}} \left( \frac{d}{dx} \left( K \frac{d}{dx} \left( u^{(1)} - u_1^{(1)} \right) \right) + \frac{f_0}{b-a} (H(x-a) - H(x-b)) \right) \begin{bmatrix} \psi_1^{(e)} \\ \psi_2^{(e)} \end{bmatrix} \, dx = \int_{x_1^{(e)}}^{x_2^{(e)}} \frac{d}{dx} \begin{bmatrix} u_1^{(1)} - u_1^{(1)} \\ u_2^{(e)} - u_2^{(e)} \end{bmatrix} \, dx + \int_{x_1^{(e)}}^{x_2^{(e)}} \frac{f_0}{b-a} (H(x-a) - H(x-b)) \begin{bmatrix} \psi_1^{(e)} \\ \psi_2^{(e)} \end{bmatrix} \, dx = 0
\]

Again, we integrate by parts and will then take the remaining integrals and simplify the expression.

\[
K \begin{bmatrix} \frac{u_1^{(e)} - u_1^{(1)}}{x_2^{(e)} - x_1^{(e)}} \\ \frac{u_2^{(e)} - u_2^{(e)}}{x_2^{(e)} - x_1^{(e)}} \end{bmatrix} \left[ \begin{array}{c} -1 \\ 1 \end{array} \right] = \frac{f_0}{b-a} \begin{bmatrix} \Delta x^{(e)} f_{1,b}(x_0) \\ 0 \end{bmatrix} + \begin{bmatrix} Q_1^{(e)} \\ Q_2^{(e)} \end{bmatrix}
\]

We use this result to construct the global finite element system as follows.

\[
\begin{bmatrix} 0 \\ \vdots \\ f_0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} Q_1^{(1)} \\ 0 \\ \vdots \\ 0 \\ Q_2^{(N-1)} \end{bmatrix}
\]

(3.5)

Both \( u_1^{(1)} \) and \( Q_2^{(N-1)} \) are zero and, as in the Ramberg-Osgood case, the interval immediately following the point \( a \) is the interval \( \alpha \) and the interval immediately preceding the point \( b \) is the interval \( \beta \).
Chapter 4

Point-wise Convergence and Error Estimation

For the finite element system of a constant load above in both the Ramberg-Osgood and Hollomon models with a very small number of nodes, around four, the values calculated for each node point are grossly inaccurate, but once the number of nodes is increased to around ten we see the solutions produced for each node are very close to the analytic solution for those same points within several significant digits. So, do our finite element model solutions actually approach the analytic solution if we were to increase the number of nodes to infinity?

The average length of elements in our finite element system is the length of the rod divided by the total number of elements. We define a function, $c(i)$, to be the percentage of the average value of elements which defines a given element, $i$.

$$\Delta x^{(i)} = c(i) \frac{L}{N-1}$$

We note that this would satisfy the property

$$\sum_{i=1}^{N-1} c(i) = N - 1$$

We can choose any value initially for each $c(i)$ so long as it meets the above conditions. So, we can define a system with elements of any size and distribution in this manner. We want to ensure that as the number of nodes in our system increases to infinity that each element goes to zero at the same rate. Each time we add a new node we create a new element, and this new element must define some interval. This interval which the new element defines must be produced from the existing domain, so it must come from points within other currently existing elements. That is, the elements and nodes must be shifted around in some manner so as to accommodate the addition of a new element. So, we further define $c(i)$ such that as a node is added to our system the size of the elements shifts such that bigger elements would shrink and smaller elements would grow distributing the weight more equitably across the whole finite element system. To that end, we define $c(i)$ such that it satisfies the following limit.

$$\lim_{N \to \infty} c(i) = 1, \quad \forall i$$

We define $q(N - 1) \in \mathbb{N}$ such that for some point $x$ where there exists a node we may define $x$ as follows.

$$x = \sum_{i=1}^{q(N-1)} c(i) \frac{L}{N-1}$$

If we take the limit of the right hand side as the number of nodes approaches infinity so as to encompass all possible points $x$ within the domain, then we may define $q$ as follows.

$$q := \frac{x}{L}$$
We can then express \( u \) recursively. We will employ the convention \( u = \),

We see that the finite element representation of the Ramberg-Osgood equation (3.1) can be solved for each node recursively. We will employ the convention \( u^{(1)} = u_{1}, u^{(2)} = u_{2}, \ldots, u^{(N-2)} = u_{N-1}, u^{(N-1)} = u_{N} \).

We can then express \( u_{i} \) as follows.

\[
\frac{u_{i}}{d} = \frac{1}{N-1} \left( \sum_{i=1}^{q(N-1)} f_{0} \left( c_{i} + 2 \frac{N-1}{j_{i+1}} \right) \right) + u_{i-1}
\]

This expression has a base term added to which is the value of the previous node. We can see that the value of any particular node is just the sum of all of these base terms up to and including said node. Whereas \( q(N-1) \) is the element index to which we are summing we define \( \tau = q(N-1) + 1 \) to be the index of the node immediately following this element.

\[
\frac{u_{\tau}}{d} = \frac{1}{N-1} \left( \sum_{i=1}^{q(N-1)} f_{0} \left( c_{i} + 2 \frac{N-1}{j_{i+1}} \right) \right)
\]

We can see then that \( u_{\tau} \) is bounded above and below by replacing each \( c_{i} \) with, respectively, the supremum, \( d \), and the infimum, \( p \).

\[
\frac{p}{N-1} \sum_{i=1}^{q(N-1)} f_{0} \left( p + 2 \frac{N-1}{j_{i+1}} \right) \leq \frac{u_{\tau}}{d} = \frac{1}{N-1} \left( \sum_{i=1}^{q(N-1)} f_{0} \left( c_{i} + 2 \frac{N-1}{j_{i+1}} \right) \right) \\
\leq \frac{d}{N-1} \sum_{i=1}^{q(N-1)} f_{0} \left( d + 2 \frac{N-1}{j_{i+1}} \right)
\]

We begin by expanding the latter expression with the supremum, \( d \).

\[
\frac{d}{N-1} \sum_{i=1}^{q(N-1)} A \left( f_{0} \left( d + 2 \frac{N-1}{j_{i+1}} \right) \right) \\
+ \frac{d}{N-1} \sum_{i=1}^{q(N-1)} B \left( f_{0} \left( d + 2 \frac{N-1}{j_{i+1}} \right) \right) \\
= d \frac{1}{2} a \left( f_{0} \left( (L^2 - (L - Lq)^2) + d^{m+1} \frac{1}{N-1} BL^{m+1} \right) \right) \\
= \frac{f_{0}}{L} \sum_{i=1}^{m-1} \left( 1 - \frac{2i-1}{2(N-1)} \right)^{m}
\]

We will focus our attention now solely on the right-most portion of the previous expression and then come back to the full expression in a moment. First, we notice that we may expand the exponentiated term within the sum into a sum itself using the binomial theorem.

\[
d^{m+1} \frac{1}{N-1} BL^{m+1} \left( \sum_{i=1}^{m-1} \binom{m}{j} (-1)^{j} \left( \frac{2i-1}{2(N-1)} \right)^{j} \right)
\]
We note that the absolute value of the fractional term within the sum is less than one so the infinite sum is absolutely convergent and the preceding sum is finite so we may swap their order.

\[ d^{m+1} \frac{1}{N-1} BL^{m+1} \left| \frac{f_0}{L} \right|^{m-1} \frac{f_0}{L} \sum_{j=0}^{\infty} \binom{m}{j} \left( \frac{-1}{2(N-1)} \right)^j \sum_{i=1}^{\infty} (2i-1)^j \]

We apply the binomial theorem once more to the right-most term in the expression.

\[ d^{m+1} \frac{1}{N-1} BL^{m+1} \left| \frac{f_0}{L} \right|^{m-1} \frac{f_0}{L} \sum_{j=0}^{\infty} \binom{m}{j} \left( \frac{-1}{2(N-1)} \right)^j \sum_{i=1}^{\infty} \sum_{k=0}^{i} \binom{i}{k} (-1)^k (2i-1)^j \]

This new sum is finite, so we may swap its order with the sum that precedes it.

\[ d^{m+1} \frac{1}{N-1} BL^{m+1} \left| \frac{f_0}{L} \right|^{m-1} \frac{f_0}{L} \sum_{j=0}^{\infty} \binom{m}{j} \sum_{k=0}^{j} \binom{j}{k} \frac{(-1)^k j^i}{2^k(N-1)^j} \sum_{i=1}^{\infty} (i)^{j-k} \]

We now use Bernoulli’s formula to express the right-most sum on the end.

\[ d^{m+1} \frac{1}{N-1} BL^{m+1} \left| \frac{f_0}{L} \right|^{m-1} \frac{f_0}{L} \sum_{j=0}^{\infty} \binom{m}{j} \sum_{k=0}^{j} \binom{j}{k} \frac{(-1)^k j^i}{2^k(N-1)^j} \sum_{k=0}^{j-k+1} \frac{(j-k+1)!}{h} B_h(q(N-1))^{j-k-h+1} \]

We may now factor out from this expression \((q(N-1))^{j+1}\) which results in the following.

\[ d^{m+1} BL^{m+1} \left| \frac{f_0}{L} \right|^{m-1} \frac{f_0}{L} \sum_{j=0}^{\infty} \binom{m}{j} \sum_{k=0}^{j} \binom{j}{k} \frac{(-1)^k j^i}{2^k(N-1)^j} \sum_{h=0}^{j-k+1} \binom{j-k+1}{h} B_h(q(N-1))^{j-k-h} \]

Finally, we are ready to take the limit of our expression. We note that the exponent in the right most term is \(-k-h\) where \(k\) and \(h\) are both indices always greater than or equal to zero. When both \(k\) and \(h\) are zero this term is one. When either of \(k\) or \(h\) are not zero when the limit is taken this term will go to zero. So, after taking the limit our entire expression is zero except for when the indices \(k\) and \(h\) are both zero. We also note that when the number of nodes increases to infinity the supremum will go towards one and we will also have \(q = x/L\). We have, then, the following.

\[ BL^{m+1} \left| \frac{f_0}{L} \right|^{m-1} \frac{f_0}{L} \sum_{j=0}^{\infty} \binom{m}{j} \frac{1}{j+1} \left( \frac{x}{L} \right)^{j+1} \]

We shift the index \(j\) up one and adjust our sum accordingly.

\[ \frac{1}{m+1} BL^{m+1} \left| \frac{f_0}{L} \right|^{m-1} \frac{f_0}{L} \sum_{j=0}^{\infty} \binom{m+1}{j} \frac{1}{j+1} \left( \frac{x}{L} \right)^{j+1} \]

If the index were zero in this new expression we would have a value of minus one. We add and subtract one incorporating the minus one into the sum and factor out a minus one.

\[ \frac{1}{m+1} BL^{m+1} \left| \frac{f_0}{L} \right|^{m-1} \frac{f_0}{L} \left[ 1 - \sum_{j=0}^{\infty} \binom{m+1}{j} \left( \frac{x}{L} \right)^{j} \right] \]

The sum is now the binomial theorem expression of \((1-x/L)^{m+1}\). We rewrite it as such and distribute the \(L^{m+1}\) term.
We add to this the term which we had left behind previously noting that we have taken the limit, so \( d \) will have gone to one and \( q \) will have gone to \( \sqrt[4]{L} \).

\[
\frac{1}{m+1}B \left[ \frac{f_0}{L} \right]^{m-1} \frac{f_0}{L} \left( L^{m+1} - (L-x)^{m+1} \right)
\]

We have then the full expression for the upper bound is exactly the previously found analytic solution for the Ramberg-Osgood relationship. We note that the lower bound differed from the upper bound only in the usage of the infimum, \( p \). When the limit is taken here \( p \) will go to one as did the supremum, \( d \). So, the lower bound converges to the same thing as the upper bound.

In the point load case the finite element system (3.2) provides the exact analytic solution to the problem for as many nodes as necessary for there to be a node at the point where the load is applied. So, for as few as two elements the exact analytic solution is provided and the system converges pointwise to the analytic solution. The proof of this is obvious.

The stepped load finite element system (3.3) can be proved by handling all elements to the left of the point \( a \), the left most bound of the stepped load, as a point load and all elements between \( a \) and \( b \) on the step as a constant load. This provides convergence for the exact analytic solution as well.

### 4.1.2 Point-wise Error

To find the amount of error in (3.1) at some node for a system of a particular size we can take the difference between the known analytic solution and the recursively defined finite element solution at some node. We will substitute \( x \) in the analytic solution for its comparable representation \( qL \) to correspond with the point of the node in the finite element solution. Then we will take advantage of the relationship, \( \tau = q(N - 1) + 1 \) to express the error as dependent upon as few terms as possible.

\[
E_N(\tau) = \frac{Af_0L}{2} \left[ 1 - \left( \frac{N - \tau}{N - 1} \right)^2 \right] - \frac{c(\tau)}{N - 1} \sum_{i=1}^{\tau-1} \left( \frac{c(i)}{N - 1} + \frac{\sum_{j=i+1}^{N-1} c(j)}{2(N - 1)} \right) + B \left[ \frac{f_0}{L} \right]^{m-1} f_0L^m
\]

\[
\left[ \frac{1}{m+1} \left( 1 - \left( \frac{N - \tau}{N - 1} \right)^{m+1} \right) - \frac{c(\tau)}{N - 1} \sum_{i=1}^{\tau-1} \left( \frac{c(i)}{N - 1} + \frac{\sum_{j=i+1}^{N-1} c(j)}{2(N - 1)} \right) \right]^{m}
\]

### 4.2 Hollomon

#### 4.2.1 Point-wise Convergence

We will approach the constant load case of the Hollomon finite element system (3.4) just as we did above with Ramberg-Osgood. We can solve this system recursively reusing in the following expression.

\[
u_i = c(i) \frac{L}{N - 1} \left| f_0 \right|^{\frac{L}{K}} \left[ \frac{1}{2} \left( \frac{c(i)}{N - 1} + \frac{\sum_{j=i+1}^{N-1} c(j)}{N - 1} \right) \right]^{\frac{1}{2}} + u_{i-1}
\]

We can then express the value at any particular node as the sum of the base terms as follows.

\[
u_\tau = c(\tau) \frac{L}{N - 1} \left| f_0 \right|^{\frac{L}{K}} \left[ \frac{1}{2} \left( \frac{c(i)}{N - 1} + \frac{\sum_{j=i+1}^{N-1} c(j)}{N - 1} \right) \right]^{\frac{1}{2}} + u_{i-1}
\]

As with Ramberg-Osgood, we bound \( u_\tau \) above and below replacing each \( c(i) \) with the supremum, \( d \), and the infimum, \( p \), respectively.
\[
\frac{P}{N-1} \left| \frac{f_0}{KL} \right|^{1/n} \sum_{i=1}^{N-1} \left[ \frac{1}{2} \left( \frac{P}{N-1} + 2 \frac{\sum_{j=i+1}^{N-1} P}{N-1} \right) \right]^{\frac{1}{n}}
\]

\[
\leq c(\tau) \frac{L}{N-1} \left| \frac{f_0}{KL} \right|^{1/n} \sum_{i=1}^{N-1} \left[ \frac{1}{2} \left( c(i) \frac{L}{N-1} + 2 \frac{\sum_{j=i+1}^{N-1} c(j)}{N-1} \right) \right]^{\frac{1}{n}} \leq
\]

\[
d\frac{L}{N-1} \left| \frac{f_0}{KL} \right|^{1/n} \sum_{i=1}^{N-1} \left[ \frac{1}{2} \left( d \frac{L}{N-1} + 2 \frac{\sum_{j=i+1}^{N-1} d}{N-1} \right) \right]^{\frac{1}{n}}
\]

Focusing on the upper bound, we may rewrite this expression as follows.

\[
d^{\frac{m+1}{n}} L^{\frac{n+1}{n}} \frac{1}{N-1} \left| \frac{f_0}{KL} \right|^{1/n} \sum_{i=1}^{N-1} \sum_{j=0}^{\infty} \left( \frac{1}{n} \right)^j (1 - \frac{2i - 1}{2(N-1)})^j
\]

We can see that the sum expressions are the same as we had when we began with Ramberg-Osgood above, the two differing only in the exponent, \( \frac{1}{n} \) here instead of \( m \) previously. So, we will arrive at the same answer as we did before but exchanging \( m + 1 \) with \( \frac{1}{n} + 1 \). After distributing the \( L^{\frac{n+1}{n}} \) term we will then find the exact analytic solution.

\[
\frac{n}{n+1} \left| \frac{f_0}{KL} \right|^{1/n} \sum_{i=1}^{N-1} \sum_{j=0}^{\infty} \left( \frac{1}{n} \right)^j (1 - \frac{2i - 1}{2(N-1)})^j
\]

The point load (3.5) and stepped load (3.6) will be handled just as it was previously.

### 4.2.2 Point-wise Error

To find the amount of error in (3.4) at some node for a system of a particular size we take the difference between the known analytic solution and the finite element solution at some node in the same manner as which we handled Ramberg-Osgood above.

\[
E_N(\tau) = L^{\frac{n+1}{n}} \left| \frac{f_0}{KL} \right|^{1/n} \sum_{i=1}^{N-1} \left[ \frac{1}{n+1} \left( 1 - \left( \frac{N - \tau}{N-1} \right)^{\frac{n+1}{n}} \right) - \frac{c(\tau)}{N-1} \sum_{i=1}^{N-1} \frac{c(i)}{2(N-1)} + \frac{\sum_{j=i+1}^{N-1} c(j)}{N-1} \right]^{\frac{1}{n+1}}
\]
5.1 Linear Model Comparison

We will perform some examples below to demonstrate visually where and why error would appear when using a linear model as opposed to one of the more precise power-law models. Generally, in modern finite element software it would be standard to input between five and ten linear segments. This of course can provide some fairly accurate approximations. The linear models can even be concentrated in a more complex area such as the knee bend in the Ramberg-Osgood relationship to provide better results. Depending on the specific needs of a project this approach can be sufficient. However, using the power laws which model much more precisely the behavior of the material leads to almost no error, at least in the case of the model itself. This can provide much more exacting results where such results are necessary as well as a higher degree of confidence in the project design.

There are many ways in which linear models are used to construct results in finite element software. Below we will focus on two simple and common methods. We will use a series of three tangent moduli, these are three lines tangent to points along the curve, and we will use a series of three secant moduli, which are three lines between a set of points on the curve. These models, using so few linear segments will tend to exaggerate the appearance of error especially in the Hollomon case because of the nature of the relationship. It is almost assured that in modern engineering no engineer would use such a model to model material behavior. However, these simple models besides being easy to understand do serve to highlight the nature of the error between the different approaches as well as the possibility for mitigating said error.

5.1.1 Ramberg-Osgood

For the trilinear tangent modulus model we take line segments tangent to the points along the curve at the origin, where we assume Young’s modulus as the slope, at the point (0.0475,36), and at ultimate. The points of transition where the linear tangent moduli intersect are first at 31.057 kip and at 41.464 kip. In the secant modulus model we take four points between which we put secant linear segments. We start at the origin then (0.0353,30), (0.0534,38), and to ultimate. The points of transition in the secant modulus model are at their respective stress values.

We use material constants derived from material tests of a Titanium alloy done in the Naval Architecture and Marine Engineering Department at the University of New Orleans with $A = 1.11111 \times 10^{-3}$, $B = 3.49873 \times 10^{-14}$, and $m = 7.28249$. Units are in kip.
Figure 5.1: LEFT: The linear Tangent moduli model as compared to the Ramberg-Osgood relationship. RIGHT: The linear secant moduli model as compared to the Ramberg-Osgood relationship.

Constant Load

Figure 5.2: LEFT: A comparison between the trilinear secant modulus and tangent modulus models and the Ramberg-Osgood relationship used in a Finite Element model to find displacement with a rod of 10 inches in length with one end fixed and the other free using 15 nodes and elements of equal size with a constant load distributed over the length of the rod of 20 kip. RIGHT: The error from the Ramberg-Osgood model as the baseline of each of the secant and tangent modulus trilinear models.
Figure 5.3: LEFT: A comparison between the trilinear secant modulus and tangent modulus models and the Ramberg-Osgood relationship used in a Finite Element model to find displacement with a rod of 10 inches in length with one end fixed and the other free using 15 nodes and elements of equal size with a constant load distributed over the length of the rod of 33 kip. RIGHT: The error from the Ramberg-Osgood model as the baseline of each of the secant and tangent modulus trilinear models.

Figure 5.4: LEFT: A comparison between the trilinear secant modulus and tangent modulus models and the Ramberg-Osgood relationship used in a Finite Element model to find displacement with a rod of 10 inches in length with one end fixed and the other free using 15 nodes and elements of equal size with a constant load distributed over the length of the rod of 44 kip. RIGHT: The error from the Ramberg-Osgood model as the baseline of each of the secant and tangent modulus trilinear models.
Point Load

Figure 5.5: LEFT: A comparison between the trilinear secant modulus and tangent modulus models and the Ramberg-Osgood relationship used in a Finite Element model to find displacement with a rod of 10 inches in length with one end fixed and the other free using 15 nodes and elements of equal size with a point load of 20 kip placed at the center of the rod. RIGHT: The error from the Ramberg-Osgood model as the baseline of each of the secant and tangent modulus trilinear models.

Figure 5.6: LEFT: A comparison between the trilinear secant modulus and tangent modulus models and the Ramberg-Osgood relationship used in a Finite Element model to find displacement with a rod of 10 inches in length with one end fixed and the other free using 15 nodes and elements of equal size with a point load of 38 kip placed at the center of the rod. RIGHT: The error from the Ramberg-Osgood model as the baseline of each of the secant and tangent modulus trilinear models.
Figure 5.7: LEFT: A comparison between the trilinear secant modulus and tangent modulus models and the Ramberg-Osgood relationship used in a Finite Element model to find displacement with a rod of 10 inches in length with one end fixed and the other free using 15 nodes and elements of equal size with a point load of 42 kip placed at the center of the rod. RIGHT: The error from the Ramberg-Osgood model as the baseline of each of the secant and tangent modulus trilinear models.

Stepped Load

Figure 5.8: LEFT: A comparison between the trilinear secant modulus and tangent modulus models and the Ramberg-Osgood relationship used in a Finite Element model to find displacement with a rod of 10 inches in length with one end fixed and the other free using 15 nodes and elements of equal size with a stepped load of 20 kip with the beginning of the step interval placed at 10/3 and the end of the interval placed at 20/3. RIGHT: The error from the Ramberg-Osgood model as the baseline of each of the secant and tangent modulus trilinear models.
5.1.2 Hollomon

For the trilinear tangent modulus model we take line segments tangent to the points along the curve at the origin, where we assume Young's modulus as the slope, at the point (0.0272,42), and at ultimate. The points of transition where the linear tangent moduli intersect are first at 24.104 kip and at 66.811 kip. In the secant modulus model we take four points between which we put secant linear segments. We start at the origin then (0.0099,27), (0.0567,58), and to ultimate. The points of transition in the secant modulus model are at their respective stress values.

We use material constants for 304 Stainless Steel Annealed with $K = 205$ and $n = 0.44 \ [3]$. Units are in kip.
Figure 5.11: LEFT: The linear Tangent moduli model as compared to the Hollomon relationship. RIGHT: The linear secant moduli model as compared to the Hollomon relationship.

**Constant Load**

Figure 5.12: LEFT: A comparison between the trilinear secant modulus and tangent modulus models and the Hollomon relationship used in a Finite Element model to find displacement with a rod of 10 inches in length with one end fixed and the other free using 15 nodes and elements of equal size with a constant load distributed over the length of the rod of 45 kip. RIGHT: The error from the Hollomon model as the baseline of each of the secant and tangent modulus trilinear models.
Figure 5.13: LEFT: A comparison between the trilinear secant modulus and tangent modulus models and the Hollomon relationship used in a Finite Element model to find displacement with a rod of 10 inches in length with one end fixed and the other free using 15 nodes and elements of equal size with a constant load distributed over the length of the rod of 60 kip. RIGHT: The error from the Hollomon model as the baseline of each of the secant and tangent modulus trilinear models.

Figure 5.14: LEFT: A comparison between the trilinear secant modulus and tangent modulus models and the Hollomon relationship used in a Finite Element model to find displacement with a rod of 10 inches in length with one end fixed and the other free using 15 nodes and elements of equal size with a constant load distributed over the length of the rod of 80 kip. RIGHT: The error from the Hollomon model as the baseline of each of the secant and tangent modulus trilinear models.
Point Load

Figure 5.15: LEFT: A comparison between the trilinear secant modulus and tangent modulus models and the Hollomon relationship used in a Finite Element model to find displacement with a rod of 10 inches in length with one end fixed and the other free using 15 nodes and elements of equal size with a point load of 30 kip placed at the center of the rod. RIGHT: The error from the Hollomon model as the baseline of each of the secant and tangent modulus trilinear models.

Figure 5.16: LEFT: A comparison between the trilinear secant modulus and tangent modulus models and the Hollomon relationship used in a Finite Element model to find displacement with a rod of 10 inches in length with one end fixed and the other free using 15 nodes and elements of equal size with a point load of 45 kip placed at the center of the rod. RIGHT: The error from the Hollomon model as the baseline of each of the secant and tangent modulus trilinear models.
Figure 5.17: LEFT: A comparison between the trilinear secant modulus and tangent modulus models and the Hollomon relationship used in a Finite Element model to find displacement with a rod of 10 inches in length with one end fixed and the other free using 15 nodes and elements of equal size with a point load of 80 kip placed at the center of the rod. RIGHT: The error from the Hollomon model as the baseline of each of the secant and tangent modulus trilinear models.

Stepped Load

Figure 5.18: LEFT: A comparison between the trilinear secant modulus and tangent modulus models and the Hollomon relationship used in a Finite Element model to find displacement with a rod of 10 inches in length with one end fixed and the other free using 15 nodes and elements of equal size with a stepped load of 30 kip with the beginning of the step interval placed at 10/3 and the end of the interval placed at 20/3. RIGHT: The error from the Hollomon model as the baseline of each of the secant and tangent modulus trilinear models.
Figure 5.19: LEFT: A comparison between the trilinear secant modulus and tangent modulus models and the Hollomon relationship used in a Finite Element model to find displacement with a rod of 10 inches in length with one end fixed and the other free using 15 nodes and elements of equal size with a stepped load of 45 kip with the beginning of the step interval placed at 10/3 and the end of the interval placed at 20/3. RIGHT: The error from the Hollomon model as the baseline of each of the secant and tangent modulus trilinear models.

Figure 5.20: LEFT: A comparison between the trilinear secant modulus and tangent modulus models and the Hollomon relationship used in a Finite Element model to find displacement with a rod of 10 inches in length with one end fixed and the other free using 15 nodes and elements of equal size with a stepped load of 80 kip with the beginning of the step interval placed at 10/3 and the end of the interval placed at 20/3. RIGHT: The error from the Hollomon model as the baseline of each of the secant and tangent modulus trilinear models.
Chapter 6

Conclusions

In this paper we have presented what we believe to be new results for the one dimensional Ramberg-Osgood axial rod under various loads as well as shown the viability of finite element models for both Ramberg-Osgood and Hollomon rods. We have touched upon some shortcomings as to the dependence on the number of nodes in the system to the accuracy of the solutions as well as to the limitations of some linear based models compared to this approach.

It is the case that engineering, by it’s nature, is generally a physical results driven discipline. For linear based models that approximate nonlinear material behavior, such as those used in most modern finite element software packages[4][5], the output calculations are "good enough” for many applications; especially in disciplines and on projects that never exit the very low elastic range, such as most civil projects. Once we leave the comfort of the Hooke’s relationship, however, and move into the higher regions of elasticity and the plastic range of these nonlinear materials errors begin to appear dependent upon the choice of segmentation. Even for a large number of linear segments to reduce the error, some small amount of model dependent error is still present. As we begin to venture into the design of increasingly smaller and more precise machinations and use more exotic specifically crafted alloys and materials a more accurate modelling approach will be necessary to perform a proper design analysis of the limitations and abilities of the given project. Modeling of physical materials is imprecise by nature. It can be dependent upon the purity of the raw material or the environment it operates within. It should not be the case that whatever error exists in our modelling of the behavior of our designs be the product of inexact models of already well understood material behaviors.
Bibliography


Vita

The author was born in New Orleans, Louisiana. He obtained his Bachelors of Science degree in Mathematics from the University of New Orleans in 2010. He joined the University of New Orleans graduate program to pursue a PhD in Engineering and Applied Sciences with a concentration in Mathematics, and became involved in research with Dr. Dongming Wei in 2012.