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Vertical Acoustic Propagation in the Non-Homogeneous Layered Atmosphere for a Time-Harmonic, Compact Source

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Vertical Acoustic Propagation in the Non-Homogeneous, Layered Atmosphere for a Time-Harmonic, Compact Source

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TABLE OF CONTENTS

LIST OF FIGURES

Figure

ABSTRACT

In this work we study vertical, acoustic propagation in a non-homogeneous media for a spatially-compact, time-harmonic source. An analytical, 2-layer model is developed representing the acoustic pressure disturbance propagating in the atmosphere. The validity of the model spans the distance from the Earth's surface to 30,000 meters. This includes the troposphere (adiabatic), ozone layer (isothermal), and part of the stratosphere (isothermal). The results of the model derivation in the adiabatic region yield pressure solutions as Bessel functions of the First (J) and Second (Y) Kind of order $-\frac{7}{2}$ with an argument of $2\Omega\tau$ (where Ω represents a dimensionless frequency and τ is a dimensionless vertical height in z (vertical coordinate)). For an added second layer (isothermal region), the pressure solution is a decaying sinusoidal, exponential function above the first layer.

In particular, the vertical, acoustic propagation is examined for various configurations. These are divided into 2 basic classes. The first class consists of examining the pressure response function when the source is located on boundary interfaces, while the second class consists of situations where the source is arbitrarily located within a finite layer. In all instances, a time-harmonic, compact source is implicitly understood. However, each class requires a different method of solution. The first class conforms to a general boundary value problem, while the second requires the use of Green's functions method.

In investigating problems of the first class, 3 different scenarios are examined. In the first case, we apply our model to a semi-infinite medium with a time-harmonic source ($e^{-i\omega t}$) located on the ground. In the next 2 cases, a semi-infinite medium is overlain on the previous medium at a height of z=13,000 meters. Thus, there exist two boundaries: the ground and

the layer interface between the 2 media. Sources placed at these interfaces represent the 2nd and 3rd scenarios, respectively. The solutions to all 3 cases are of the form $A\frac{J_{-\frac{7}{2}}(2\Omega\tau)}{\tau^{-\frac{7}{2}}} + B\frac{Y_{-\frac{7}{2}}(2\Omega\tau)}{\tau^{-\frac{7}{2}}}$, where A and B are constants determined by the boundary conditions.

For the 2nd class, we examine the application to a time-harmonic, compact source placed arbitrarily within the 1st layer. The method of Green's functions is used to obtain a particular solution for the model equations. This result is compared with a Fast Field Program (FFP) which was developed to test these solutions. The results show that the response given by the Green's function compares favorably with that of the FFP.

Keywords: Linear Acoustics, Inhomogeneous Medium, Layered Atmosphere, Boundary Value Problem, Green's Function Method

CHAPTER I

Introduction

1.1 Historical Background

The phenomenon of sound and its propagation has been a source of mystery, confusion, conjecture, and curiosity over the history of man. The investigation of sound goes back as far as Aristotle's idea of sound propagation (c. 384-322 B.C.) i.e. air motion was generated by a source. The idea of the wave nature of sound grew from the observation of surface water waves. This idea was picked up by many Greek and Roman philosophers (e.g. Chrysippus (c. 240 B.C.) and Boethius (A.D. 480-524)[12]. Over the centuries, much speculation and experimentation was done. In the 17th century, the idea of sound generation was extended to the experimental result that a body vibrating at a single frequency produced air molecules vibrating at that same frequency. This was addressed particularly by Martin Marsenne (1636, Harmonie universelle) and Galileo (1564-1642)[12]. However, it wasn't until the appearance of Newton's Principia in 1686 that the study of sound was placed on a more firm, mathematical footing. Newton considered sound as "pressure" pulses transmitted through neighboring fluid particles. Greater development was made during the 18th century by the likes of Euler, Lagrange, and d'Alembert through the development of continuum physics[11]. This was followed by Lord Rayleigh's two volume treatise, "Theory of Sound" (1877 and 1878)[2].

Acoustics is the study of the science of sound as applied to various media. Today's theories can be viewed as refinements to the contributions made by Euler and his contemporaries. The use of the "new" continuum physics by Euler as applied to fluids is generally considered the starting point of the development of today's linear acoustic theory. Certainly, two of Euler's most famous equations are used unaltered to this day in the development of linear acoustics. Since the study of acoustics is a very mature field, one may be tempted to believe that there is little work left to be done. However, the numerous papers and journal articles published daily speak otherwise.

In 1755, Euler developed the application of the important equations concerning the Conservation of Mass (or Continuity) Equation and the Momentum Equation. These equations are the first 2 equations shown below. The third equation represents the Equation of State which is derived from the fact that the acoustic pressure, p, is a function of the density, ϱ, and the entropy, η, namely, $p = p(\varrho, \eta)$.

$$(D\varrho/Dt) = \varrho\nabla \cdot u \qquad \text{Mass Conservation} \tag{1.1a}$$

$$\varrho(Du/Dt) = -\nabla p + \varrho b \qquad \text{Momentum Conservation} \tag{1.1b}$$

$$Dp/Dt = c^2(D\varrho/Dt) \qquad \text{Equation of State} \tag{1.1c}$$

$$(D\eta/Dt = 0) \qquad \text{Entropy Equation} \tag{1.1d}$$

From these, one can obtain a wave equation for the propagation of sound through a medium. Here $(D/Dt) \equiv \partial/\partial t + \mathbf{u} \cdot \nabla$ is the Material Derivative and $\mathbf{u}$ is the flow velocity (wind). For our purposes, the medium of concern is the atmosphere (air) which is treated as an ideal gas. Additionally, it should be noted that the pressure, p, is the sum of an ambient (background) quantity plus the pressure perturbation caused by the passage of an acoustic wave. A similar situation exists for the density, ϱ. The term $\mathbf{b}$ represents a body force acting on the medium which will become $\mathbf{g}$, the acceleration of gravity in the negative z-direction.

The difficulty in obtaining and solving an appropriate, acoustic, wave equation for atmospheric conditions is due in large part to the complexity of the medium. Propagation through the atmosphere must include known (experimental) functions for the air density, the wind velocity, ground surface, turbulence, terrain, and any barriers to propagation. To solve these problems, several mathematical models have been developed which are based on the success of models in underwater acoustics, electro-magnetics, and seismology. These models treat the encountered surfaces as a boundary value problem. To this end, there are 3 accurate models developed to treat accoustic propagation. Without going into detail, these are called the Fast Field Program (FFP), the Crank-Nicholson Parabolic Equation (CNPE), and the Green's Function Parabolic Equation (GFPE).

1.2 Historical Relevance

In this study, we examine acoustic wave propagation in non-homogeneous, layered medium. In this respect, the layering of the atmosphere provides a perfect backdrop for such an analysis. As one moves vertically upwards in the atmosphere, the pressure and density change dramatically. Much information and data are known about how these variables change with height and, thus, provide a well-defined model system for investigation.

The origin of these ideas (studying vertical, sound propagation) goes back a century and a half as they were first addressed in the literature by Poisson and Rayleigh around 1890. At the same time (1879), Horace Lamb [10] published his famous work, *Hydrodynamics*, in which he commenced his study of Atmospheric Waves in Section 309, where he stated, "The theory of waves travelling vertically in the atmosphere is of some interest as an example of wave-propagation in a variable medium." Then, he commences to analyze this problem for an atmosphere with a constant temperature and one with a temperature gradient. However, his analysis is performed stating particle displacement, and, later, extends it to acoustic pressure.

He proceeds to study this problem by assuming an adiabatic relation between pressure and density (we do likewise) as a function of 2 atmospheres and solves for the particle displacement for a constant temperature (Sec. 309).

The study we embark upon in this work does not include any horizontal flow (wind), but it is none the less relevant. Such velocity fields are pertinent to such areas as sound interaction with the upper boundary layer as well as remote sensing of the vertical, velocity structure. The problem posed is that for an inhomogeneous atmosphere which was first studied in-depth by Lamb [10]. It is a topic addressed in many texts [14, 17, 7] and articles [9, 15, 6, 13, 5, 8] as well.

In this work, an analytical model is developed in Chapter 2 which addresses the effect on vertical, acoustic propagation from a spatially-compact, time-harmonic source through non-homogeneous, layered media. In particular, the parameters of this study examine acoustic propagation in a layered atmosphere ranging from the Earth's surface to a vertical height of 30,000 meters. This range includes the troposphere, ozone layer, and part of the stratosphere. The density profiles for the layered media used in this work are derived in Appendix B and are of an algebraic and exponential form. The model yields an equation for the pressure perturbation with height. Generally speaking, this is advantageous as pressure quantities offer more physical insight into the various effects than, perhaps, particle displacement or velocity. In all scenarious studied, our source is harmonic in time ($e^{-i\omega t}$) and compact in space ($\delta(z - z_s)$). Here ω is the angular frequency, t represents time, δ is the Dirac delta function, and its arguments are the independent variable z with z_s being the source location.

In the remaining chapters numerous scenarios are examined with our model. These scenarios are divided into 2 basic classes where each class requires a different method of solution. The first class is analyzed in Chapter 3. This consists of examining the pressure response function when the source is located on boundary interfaces. Thus, this class conforms to a general boundary value problem. In investigating problems of this class, 3 different scenarios are examined. In the first case, we investigate the pressure response function in a semi-infinite, adiabatic medium with a time-harmonic source resting on the ground. In the next 2 cases, a semi-infinite medium is overlain on the previous medium at a height of z=13000 meters. This results in a finite layer between the ground and the semi-infinite medium above (adiabatic region). Thus, there now exist two boundaries: the ground and the layer interface between the 2 media. These represent the geometries for the 2nd and 3rd scenarios, respectively.

Chapter 4 examines the second class which consists of geometries where the source is arbitrarily located within the 1st layer. The nature of this problem requires a different method of solution whereby we chose the use of Green's functions. This method is used to obtain a particular solution for the model equations. These are examined individually and in conjunction with the complementary solution. Next, Chapter 5 develops a Fast Field Program for the wave field within the finite layer and is used to compare results from the Green's function problem. The comparisons show that the response given by the Green's function compares favorably with that from the FFP.

Finally, in Chapter 6, conclusions are given resulting from the foregoing analyses and comparisons.

CHAPTER II

Development of an Analytic Pressure Model

In this work, we develop a model representing vertical, acoustic propagation in a layered, non-homogeneous atmosphere. A depiction of this model is shown in Figure 2.1. Here the 2 layers of the model are shown with the first layer terminating at a height of z=13000 meters with the second layer terminating at 30,000 meters. The lower layer is representative of the troposhere (adiabatic region), while the second layer extends into the stratosphere (isothermal region). This work develops an ordinary differential equation (ODE) which represents sound propagation through each of these layers. After developing the ODE, various scenarios are investigated by locating the acoustic source at various locations of Layer 1 (shown in the model diagram). In particular, we study the results for the source located on the 2 boundaries of Layer 1, as well as, the results for arbitrarily placing the source within Layer 1.

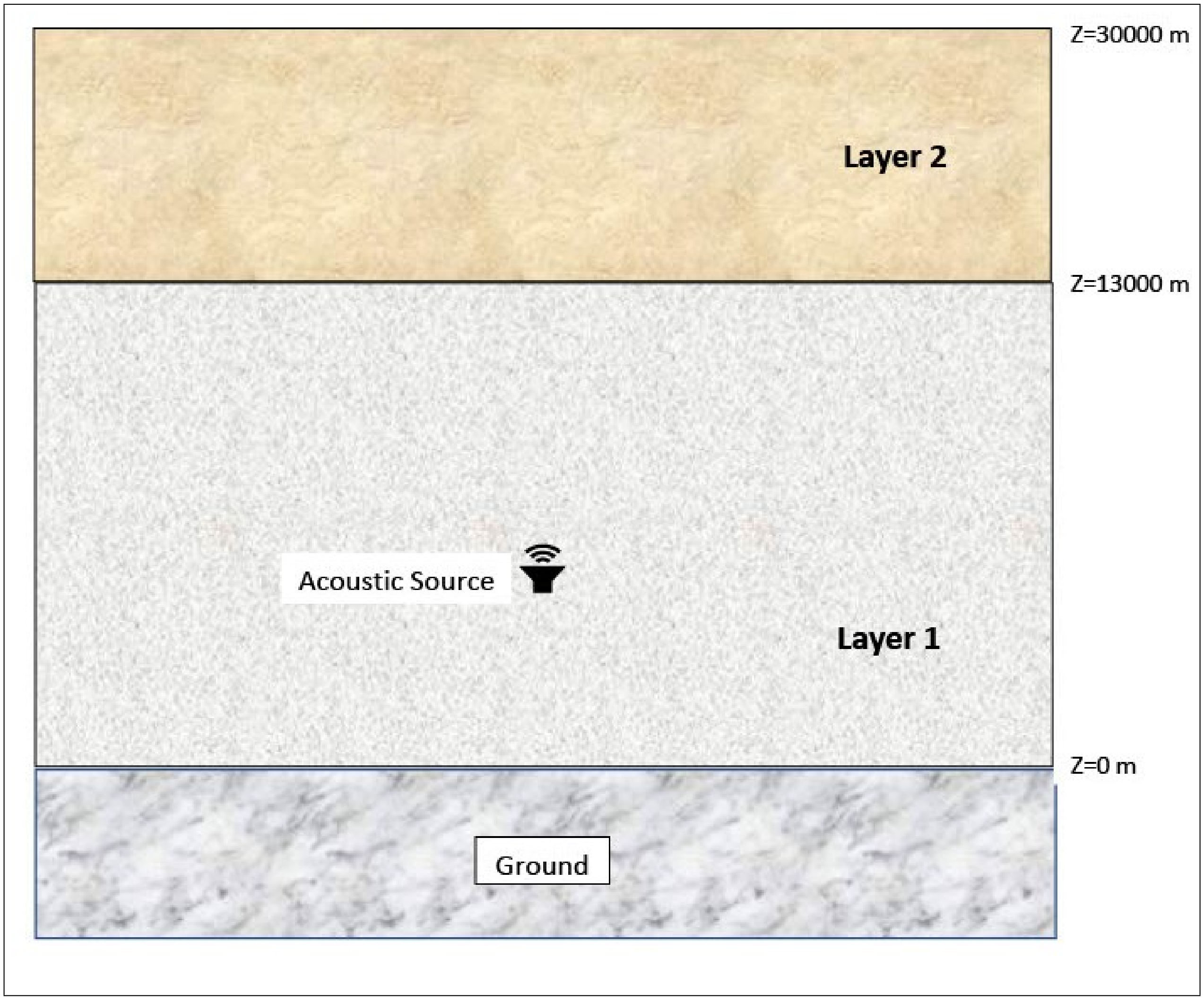

Figure 2.1: 2-Layered, Non-Homogeneous Atmosphere Model

2.1 Development of Pressure ODE for a Layered, Non-Homogeneous Medium

In order to develop our model, one starts with Euler's equation for Mass Conservation and the Continuity Equation as well as an Equation of State.

$$(D\varrho/Dt) = -\varrho\nabla \cdot u \tag{2.1a}$$

$$\varrho(Du/Dt) = -\nabla p + \varrho\mathbf{b} \tag{2.1b}$$

$$Dp/Dt = c^2(D\varrho/Dt), (D\eta/Dt = 0) \tag{2.1c}$$

where c is sound velocity, ϱ is the density function in the vertical direction (z), p is the acoustic pressure variation, u is the acoustic velocity in the vertical (z) direction, $\mathbf{b}$ is the body force, and $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V} \cdot \nabla$ is the Material Derivative. At this point, one assumes that the total quantity measured is the sum of an ambient function plus a propagating disturbance. The ambient function may also be referred to as the average or background function. This is indicated here where the subscript "av" represents the average or ambient function and the unscripted variable is the propagated disturbance:

$$p_a = p_{av}(z) + p \tag{2.2a}$$

$$\varrho_a = \varrho_{av}(z) + \varrho \tag{2.2b}$$

$$\boldsymbol{v}_a = (u_{av}(z) + u, v_{av}(z) + v, w) \tag{2.2c}$$

At this point, we consider the fact that we are working towards a windless model and set the velocities in the x- and y-directions to zero. This leaves only 3 remaining equations which require solving. To this end, we write the remaining variables explicitly in terms of their time-harmonic dependence and make a variable transformaion as shown below:

$$p \rightarrow Pe^{-i\omega t}$$

$$\varrho \rightarrow \Omega_D e^{-i\omega t}$$

$$u \rightarrow We^{-i\omega t}.$$

With this transformation and linearization, equations (2.1a)–(2.1c) and (2.2a)–(2.2c) yield our 3 algebraic expressions below which are to be solved simultaneously:

$$-i\omega\Omega_D + \varrho_{av}^{'} W + \varrho_{av} W^{'} = 0 \tag{2.3a}$$

$$-i\omega W + \varrho_{av}^{-1} P^{'} + g\varrho_{av}^{-1}\Omega_D = 0 \tag{2.3b}$$

$$-i\omega P - \varrho_{av} g W = -i\omega c^2 \Omega_D + c^2 \varrho_{av}^{'} W, \tag{2.3c}$$

where ω is the angular frequency of the continuous source. Although these equations are in algebraic form, they contain derivative terms. Thus, the solution in terms of P will be a differential equation for P. Details of linearization with the above transformations are given in Appendix A.

First, solving (2.3c) for Ω and substituting into (2.3b), one arrives at

$$\begin{aligned} &- i\omega W + \varrho_{av}^{-1} P^{'} \\ &+ g\frac{\varrho_{av}^{-1}}{c^2}(P + \frac{1}{i\omega}(\varrho_{av} g + c^2 \varrho_{av}^{'}) W) = 0. \end{aligned} \tag{2.4}$$

This yields the following equation for W

$$W = \frac{\varrho_{av}^{-1} P^{'} + \frac{g\varrho_{av}^{-1}}{c^2} P}{(i\omega - g\frac{\varrho_{av}^{-1}}{c^2}\frac{1}{i\omega}(\varrho_{av} g + c^2 \varrho_{av}^{'}))}, \tag{2.5}$$

from which an equation for the derivative, $W^{'}$, can be found as

$$\begin{aligned} W^{'} = &\frac{P^{''} + \frac{g}{c^2} P^{'}}{(i\omega\varrho_{av} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{'} + \varrho_{av} g))} \\ &- \frac{(P^{'} + \frac{g}{c^2}P)(i\omega\varrho_{av}^{'} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{''} + \varrho_{av}^{'} g))}{(i\omega\varrho_{av} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{'} + \varrho_{av} g))^2}. \end{aligned} \tag{2.6}$$

Using (2.5) and (2.6) in (2.3a), a clearer expression for the pressure develops as shown here

$$
\begin{aligned}
&P\frac{\omega^2}{c^2}\varrho_{av} + \varrho_{av}^{'}\frac{g}{c^2}P + \varrho_{av}P^{''} + \\
&\frac{-\varrho_{av}(P^{'} + \frac{g}{c^2}P)(i\omega\varrho_{av}^{'} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{''} + \varrho_{av}^{'}g))}{(i\omega\varrho_{av} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{'} + \varrho_{av}g))} = 0.
\end{aligned} \tag{2.7}
$$

After collecting terms common to $P^{''}$, $P^{'}$, and P, (2.7) can be rewritten as

$$
\begin{aligned}
&P^{''} - P^{'}\left[\frac{(i\omega\varrho_{av}^{'} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{''} + \varrho_{av}^{'}g))}{(i\omega\varrho_{av} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{'} + \varrho_{av}g))}\right] + \\
&P\left[\frac{\omega^2}{c^2} + \frac{\varrho_{av}^{'}}{\varrho_{av}}\frac{g}{c^2} + \frac{g}{c^2}\frac{(i\omega\varrho_{av}^{'} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{''} + \varrho_{av}^{'}g))}{(i\omega\varrho_{av} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{'} + \varrho_{av}g))}\right] = 0,
\end{aligned} \tag{2.8}
$$

which can be manipulated into the following form

$$
\begin{aligned}
&P^{''} - P^{'}\left\{\frac{d}{dz}ln(\varrho_{av}) + \frac{d}{dz}ln\left[\omega^2\right]\right\} \\
&+ P\left[\frac{\omega^2}{c^2} + \frac{\varrho_{av}^{'}}{\varrho_{av}}\frac{g}{c^2} - \frac{g}{c^2}\left\{\frac{d}{dz}ln(\varrho_{av}) + \frac{d}{dz}ln\left[\omega^2 c^2\right]\right\}\right] = 0.
\end{aligned} \tag{2.9}
$$

Finally, arguments can be made regarding the magnitude and significance of various terms above. These are given in Appendix A prior to equation (A.35). With these reductions, one arrives at the desired equation:

$$
P^{''} - \left(\frac{\varrho_{av}^{'}}{\varrho_{av}}\right)P^{'} + \left(\frac{\omega^2}{c^2}\right)P = 0. \tag{2.10}
$$

2.2 Pressure Equation with Linear Density Variation (1st Layer)

Next, we study (2.10) in an adiabatic layer. The density, pressure, temperature, and sound speed profiles are given by (see Appendix B) :

$$\varrho_{av}(z) = \varrho_0 \left[1 - \frac{z}{l}\right]^{\chi} \quad \text{Density Equation} \tag{2.11a}$$

$$p = p_0(1 - \frac{z}{l})^{\chi+1} \quad \text{Pressure Equation} \tag{2.11b}$$

$$T = T_0(1 - \frac{z}{l}) \quad \text{Temperature Equation} \tag{2.11c}$$

$$c(z) = c_0 \left[1 - \frac{z}{l}\right]^{\frac{1}{2}} \quad \text{Sound Speed Equation} \tag{2.11d}$$

In the above formulas, all of the variables with "0" subscript represent the corresponding quantity at the ground level i.e. $z = 0$. Here, z, of course, is the vertical coordinate and is positive upwards. Consequently, z varies from the surface height $z = 0$ to a height l, where we have $l \coloneqq (\chi + 1)H$ and $\chi(\geq 0)$ is a (known) constant such that the temperature T is always decreasing as we move upwards in the atmosphere (z increasing). Or, in other words, as stated by Reference[9] (as taken from Reference[10]), T', the gradient of the temperature, is always negative and constant-valued. Additionally, the constant H, as referenced in [10], is the height of the 'homogeneous atmosphere' defined as $H \coloneqq \frac{c_0^2}{(\gamma g)}$. Here γ is equal to the ratio of specific heats at constant pressure and volume. That is, $\gamma \coloneqq \frac{c_p}{c_v}$. This ratio can assume the values of $\frac{5}{3}$ or $\frac{7}{5}$ for an ideal monoatomic or polyatomic gas, respectively. Our work will use the value for $\gamma = \frac{7}{5}$.

Proceding with the derivation, it should be noted that all primes (') on variables represent a derivative with respect to z. Using Equation (2.11a), its derivative may be written as

$$\varrho_{av}'(z) = \varrho_0 \chi \left[1 - \frac{z}{l}\right]^{\chi-1} (-\frac{1}{l}) \tag{2.12}$$

Hence, the ratio $\frac{\varrho_{av}'}{\varrho_{av}}$ required in Equation(2.10) is:

$$\frac{\varrho_{av}'}{\varrho_{av}} = -\frac{1}{l}\frac{\chi}{(1 - \frac{z}{l})} \tag{2.13}$$

Substituting the newly found expressions for $\frac{\varrho_{av}'}{\varrho_{av}}$ and $c(z)$ into (2.10), one obtains

$$P^{''} + \left(\frac{1}{l}\frac{\chi}{(1-\frac{z}{l})}\right)P^{'} + \left(\frac{\omega^2}{c_0^2\left[1-\frac{z}{l}\right]}\right)P = 0 \quad (2.14)$$

(Please note that a power of 2 term is not included on the term $\left[1-\frac{z}{l}\right]$ with the definition for c^2 in Equation (2.10).)

We can now derive a dimensionless equation for Equaton (2.14) by substituting $Z = \frac{z}{l}$. This substitution yields

$$\frac{1}{l^2}\frac{d^2P}{dZ^2} + \frac{1}{l^2}\frac{\chi}{1-Z}\frac{dP}{dZ} + \frac{\omega^2}{c_0^2(1-Z)}P = 0 \quad (2.15)$$

Finally, defining $\Omega = \frac{\omega l}{c_0}$, the end result is

$$(1-Z)\frac{d^2P}{dZ^2} + \chi\frac{dP}{dZ} + \Omega^2 P = 0 \quad (2.16)$$

Continuing our developement to arrive at a Bessel's equation, we make the substitution of $\tau = \sqrt{1-Z}$. One can now take the various derivatives with respect to τ using the chain rulle as follows:

$$\frac{dP}{dZ} = \frac{dP}{d\tau}\frac{d\tau}{dZ}$$

where

$$\frac{d\tau}{dZ} = -\frac{1}{2}\frac{1}{\sqrt{1-Z}} = -\frac{1}{2\tau} \quad (2.17)$$

Therefore,

$$\frac{dP}{dZ} = -\frac{1}{2\tau}\frac{dP}{d\tau} \quad (2.18)$$

Solving for the 2nd derivative term, we have

$$
\begin{aligned}
\frac{d^2P}{dZ^2} = \frac{d}{dZ}\frac{dP}{dZ} &= -\frac{1}{2\tau}\frac{d}{d\tau}\left[-\frac{1}{2\tau}\frac{dP}{d\tau}\right] \\
&= -\frac{1}{2\tau}\left\{\frac{1}{2\tau^2}\frac{dP}{d\tau} - \frac{1}{2\tau}\frac{d^2P}{d\tau^2}\right\} \\
&= \left\{-\frac{1}{4\tau^3}\frac{dP}{d\tau} + \frac{1}{4\tau^2}\frac{d^2P}{d\tau^2}\right\}
\end{aligned}
\tag{2.19}
$$

Putting all of the parts of Equation (2.16) together, one gets

$$
\tau^2\left\{-\frac{1}{4\tau^3}\frac{dP}{d\tau} + \frac{1}{4\tau^2}\frac{d^2P}{d\tau^2}\right\} + -\frac{\chi}{2\tau}\frac{dP}{d\tau} + \Omega^2 P = 0 \tag{2.20}
$$

This finally becomes

$$
\frac{d^2P}{d\tau^2} - \frac{(2\chi+1)}{\tau}\frac{dP}{d\tau} + 4\Omega^2 P = 0 \tag{2.21}
$$

We now redefine χ by the substitution

$$
-(2\chi+1) = (2\chi^{'}+1). \tag{2.22}
$$

where the prime (') is not indicating a derivative, but it merely marks the change of variables.

As explained above, one interesting choice for χ is $\chi = \frac{1}{\gamma-1}$. Here we choose $\gamma = \frac{7}{5}$ since air is a diatomic molecule. This yields a value $\chi = \frac{5}{2}$. And a corresponding value of $\chi^{'} = -\frac{7}{2}$. Hence, Equation (2.21) now becomes

$$
\frac{d^2P}{d\tau^2} + \frac{(2\chi^{'}+1)}{\tau}\frac{dP}{d\tau} + 4\Omega^2 P = 0 \tag{2.23}
$$

Equation (2.23) is an Euler-type equation and, thus, one way of transforming it is to assume a solution of the form $P = \frac{X}{\tau^{\chi}}$. Here we are looking for a final equation for X and we have also changed $\chi^{'}$ to χ for convenience. With this in mind, we commence finding the individual derivatives:

$$\frac{dP}{d\tau} = \frac{d(\frac{X}{\tau^\chi})}{d\tau} = X\frac{d(\frac{1}{\tau^\chi})}{d\tau} + \frac{1}{\tau^\chi}\frac{dX}{d\tau} = -\frac{\chi}{\tau^{\chi+1}}X + \frac{1}{\tau^\chi}\frac{dX}{d\tau} \tag{2.24}$$

Continuing, we have for the 2nd order derivative:

$$\frac{d^2P}{d\tau^2} = \frac{d}{d\tau}\left[-\frac{\chi}{\tau^{\chi+1}}X + \frac{1}{\tau^\chi}\frac{dX}{d\tau}\right] \tag{2.25}$$

Or,

$$\frac{d^2P}{d\tau^2} = -\frac{\chi}{\tau^{\chi+1}}\frac{dX}{d\tau} + \frac{\chi(\chi+1)}{\tau^{\chi+2}}X - \frac{\chi}{\tau^{\chi+1}}\frac{dX}{d\tau} + \frac{1}{\tau^\chi}\frac{d^2X}{d\tau^2} \tag{2.26}$$

Combining terms common to the 1st order derivative, we have

$$\frac{d^2P}{d\tau^2} = \frac{1}{\tau^\chi}\frac{d^2X}{d\tau^2} - \frac{2\chi}{\tau^{\chi+1}}\frac{dX}{d\tau} + \frac{\chi(\chi+1)}{\tau^{\chi+2}}X \tag{2.27}$$

Now, combining all of the terms of Equations (2.24) and (2.26) with Equation (2.23), we have

$$\left[\frac{1}{\tau^\chi}\frac{d^2X}{d\tau^2} - \frac{2\chi}{\tau^{\chi+1}}\frac{dX}{d\tau} + \frac{\chi(\chi+1)}{\tau^{\chi+2}}X\right] + \frac{(2\chi+1)}{\tau}\left[-\frac{\chi}{\tau^{\chi+1}}X + \frac{1}{\tau^\chi}\frac{dX}{d\tau}\right] + 4\Omega^2\frac{X}{\tau^\chi} = 0 \tag{2.28}$$

This may distributed as below:

$$\left[\frac{1}{\tau^\chi}\frac{d^2X}{d\tau^2} - \frac{2\chi}{\tau^{\chi+1}}\frac{dX}{d\tau} + \frac{\chi(\chi+1)}{\tau^{\chi+2}}X\right] + \left[-\frac{\chi(2\chi+1)}{\tau^{\chi+2}}X + \frac{(2\chi+1)}{\tau^{\chi+1}}\frac{dX}{d\tau}\right] + 4\Omega^2\frac{X}{\tau^\chi} = 0 \tag{2.29}$$

Adding terms with like differential orders, we have

$$\frac{1}{\tau^\chi}\frac{d^2X}{d\tau^2} + \frac{1}{\tau^{\chi+1}}\frac{dX}{d\tau} - \frac{\chi^2}{\tau^{\chi+2}}X + 4\Omega^2\frac{X}{\tau^\chi} = 0 \tag{2.30}$$

This finally yields our Bessel function equation for the lower layer:

$$\tau^2 \frac{d^2 X}{d\tau^2} + \tau \frac{dX}{d\tau} + (4\Omega^2\tau^2 - \chi^2)X = 0 \tag{2.31}$$

where X satisfies Equation (2.31). Additionally, (2.31) is a Bessel equation of order $\chi = -\frac{7}{2}$ and of argument $2\Omega\tau$. The solution for X admits solutions of Bessel's equation of the First ($J_\chi(2\Omega\tau)$) and Second Kind ($Y_\chi(2\Omega\tau)$) . Thus a general solution for the pressure,P, may be written as

$$P = A\frac{J_\chi(2\Omega\tau)}{\tau^\chi} + B\frac{Y_\chi(2\Omega\tau)}{\tau^\chi}, \tag{2.32}$$

where A and B are constants to be determined from boundary conditions. As a reminder, $\tau = \sqrt{1 - \frac{z}{l}} = \sqrt{1 - Z}$, where $l = 30,000$ meters. Both τ and Z are dimensionless and are used interchangeably when needed for purposes of clarifying various results.

Hence, (2.32) is the governing Bessel's equation of order χ and argument $2\Omega\tau$ which will be used for finding solutions in the lower layer(s) of all the models herein.

2.3 Pressure Equation with Exponential Density (2nd Layer)

We now turn our attention to developing the equation which represents the top semi-infinite layer, overlaying a finite layer. This result will then be coupled with that for the bottom layer (via the boundary conditions at the common interface) in order to form the complete model. Hence, we start with the differential equation for the pressure (2.10). But, instead of using the dimensionless variable Z, we use the coordinate variable z. Then, after working with the density function $\varrho_{av}(z)$, we will renormalize the equation appropriately.

Next, we start with a common model for $\varrho_{av}(z)$ which may be found in [11] , that is, the exponential form

$$\varrho_{av} = \varrho_0 e^{(-\frac{z}{H})} \Rightarrow \varrho_{av}^{'} = -\frac{1}{H}\varrho_{av}, \tag{2.33}$$

where H corresponds to the height of the 'homogeneous atmosphere' as quoted by Lamb.

For such an atmosphere, one may write the sound speed in the medium as

$$c^2 = \gamma g H \Rightarrow H = \frac{c^2}{\gamma g}, \tag{2.34}$$

where γ is the ratio of specific heats.

Writing the pressure equation in terms of the Cartesian variable z, we have

$$\frac{d^2 P}{dz^2} - \left(\frac{\varrho'_{av}}{\varrho_{av}}\right)\frac{dP}{dz} + \left(\frac{\omega^2}{c^2}\right)P = 0. \tag{2.35}$$

Expressing this again in terms of Z and multiplying through by l^2, then, (with $\Omega^2 = \frac{\omega^2 l^2}{c^2}$), we obtain

$$\frac{d^2 P}{dZ^2} + \left(l\frac{\gamma g}{c^2}\right)\frac{dP}{dZ} + \Omega^2 P = 0. \tag{2.36}$$

Next, we designate the term $\left(l\frac{\gamma g}{c^2}\right)$ by α and notice that all of the coefficients of Equation (2.36) are constants. Given that a differential equation whose coefficients are constants has solutions which are of the form e^{mz}, one can easily solve the characteristic equation and get the pressure, P.

The characteristic equation resulting from (2.36) is

$$m^2 + \alpha m + \Omega^2 = 0, \tag{2.37}$$

and its solution is given by

$$m = -\alpha\frac{1 \mp i\sqrt{\frac{4\Omega^2}{\alpha^2} - 1}}{2}. \tag{2.38}$$

Substituting the result for the positive root in the exponential, the pressure P may be written with respect to the bottom of the 2nd layer (Z_2) as

$$P = Ce^{\frac{-\alpha[Z-Z_2]}{2}} cos(\alpha\frac{\sqrt{\frac{4\Omega^2}{\alpha^2} - 1}}{2}(Z - Z_2)), \tag{2.39}$$

where C is the boundary value constant which must be solved in order to match the solution in the lower layer. This will be done below for each specific model configuration.

Due to our previous choice of m from (2.38), one notices that propagation in (2.39) only occurs when the term $\frac{4\Omega^2}{\alpha^2} - 1 \geq 0$. Thus, one can define the minimum value for Ω for which this occurs and label it Ω_c where the c means "critical value." This value is then determined to be

$$\Omega_c = \frac{\alpha}{2}, \tag{2.40}$$

where α is defined for a specific sound speed c_1 as

$$\alpha = \frac{\gamma g l}{c_1^2}, \tag{2.41}$$

and c_1 is the sound speed occurring at $z = z_2$ (which is the minimum z at which (2.39) applies).

Since two of the models below have this critical propagation criterion, frequencies for all of the models will be couched in terms of multiples of Ω_c. The frequencies will be labeled as fc with $\Omega = fc * \Omega_c$.

Furthermore, Ω can be redefined from earlier as $\Omega = \frac{\omega l}{c_1}$, where c_1 is the sound speed at the base of layer 2.. Combining the definition of Ω, Ω_c, and α, one may calculate the lowest propagating angular frequency ω by equating $\Omega = \frac{\alpha}{2}$. Doing so reveals a cut-off frequency for ω_c of 26.5 mHz!

CHAPTER III

The 3 Models

3.1 Model 1: Simple, Semi-Infinite Half-Space

In order to understand the solutions to the adiabatic atmosphere more clearly, it is helpful to examine plots of the relevant Bessel functions. Figure 3.1 shows the Bessel functions of the First (J) and Second (Y) Kind of order $-\frac{7}{2}$. The behavior of these functions is quite different from their counterparts of non-negative, integer order. For these orders, it is common for the Y solution to be infinite at the origin ($z = 0$). However, here both functions are finite at the origin while the J function becomes infinite at infinity (modeled by the top boundary). This will become important for solving the boundary problems below.

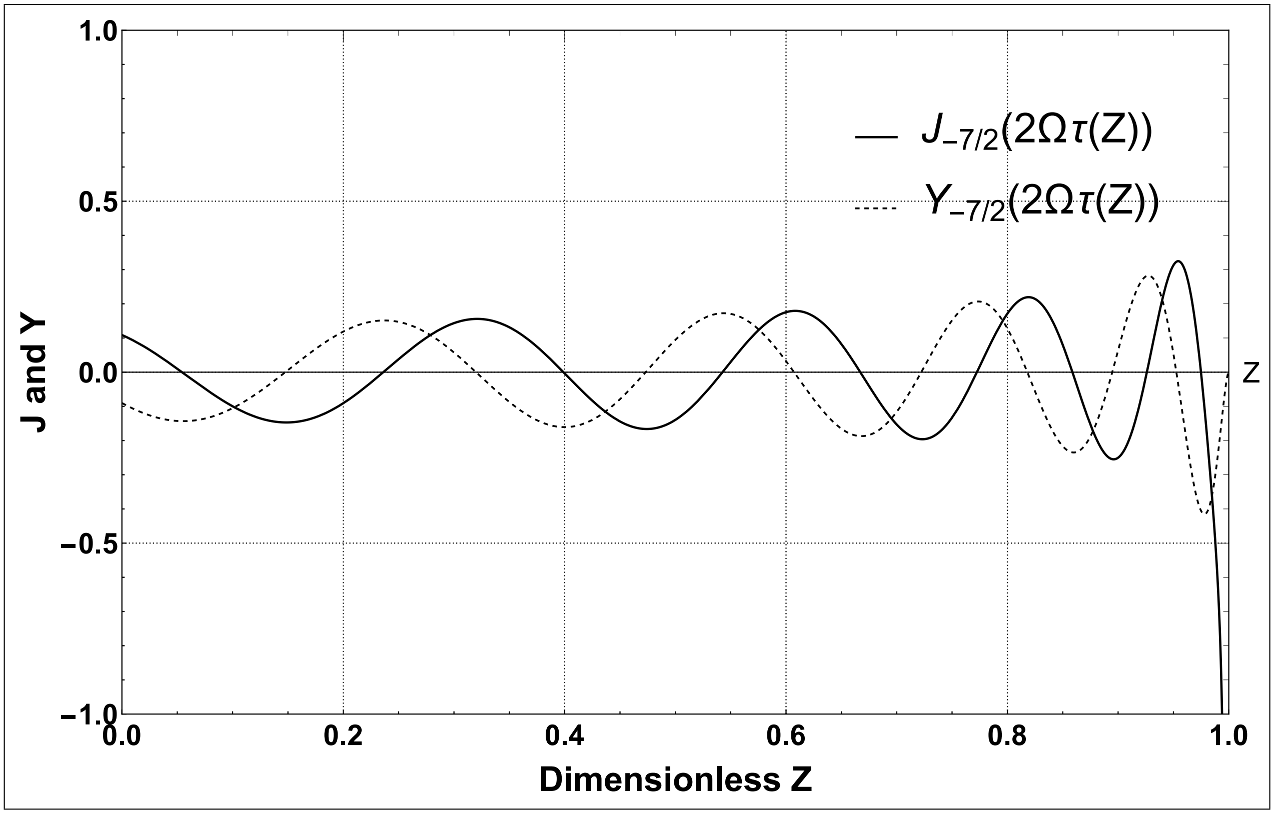

Figure 3.1: Bessel Functions of First and Second Kind for Order $-\frac{7}{2}$ in z

The adiabatic density model is the non-homogeneous, layered function chosen for the semi-infinite, half-space of the 1st model. As previously found, the relevant Bessel's equation is given by (2.31) while the total solution is expressed in (2.32). Only the solution of the 2nd Kind (Y) is allowed since the J solution goes to infinity as $l \to \infty$.

Since there is only one boundary in this model, there exists only one boundary condition. However, the solution contains 2 unknown, boundary values. We write the only boundary value condition as

$$P_1|_{Z=0} = P_0, \tag{3.1}$$

where the general solution is given by

$$P = A\frac{J_\chi(2\Omega\tau)}{\tau^\chi} + B\frac{Y_\chi(2\Omega\tau)}{\tau^\chi}. \tag{3.2}$$

as already found in Chapter II.

For resolving the 2nd boundary condition, one considers the behavior as the function approaches infinity ($\tau = 1$ or $Z = l$). This behavior is seen in figure 3.1. One notices that the J Bessel function blows-up as $Z \to l$, therefore, the coefficient A must be 0.

Thus, the new pressure equation to be solved for this model becomes:

$$P = B\frac{Y_\chi(2\Omega\tau)}{\tau^\chi}, \tag{3.3}$$

where B is the unknown boundary value constant. Here, again, $\chi = -\frac{7}{2}$ and $\tau = \sqrt{1 - \frac{z}{l}} = \sqrt{1 - Z}$.

The only boundary condition for this model is that the pressure $P = P_0$ at the base of the half-space($\tau = 0$). Applying this condition to (3.3), we find that

$$B = \frac{P_0}{Y_{-\frac{7}{2}}(2\Omega)}, \tag{3.4}$$

with the total pressure solution becoming

$$P = \frac{P_0}{Y_{-\frac{7}{2}}(2\Omega)}\frac{Y_{-\frac{7}{2}}(2\Omega\tau)}{\tau^{-\frac{7}{2}}}. \tag{3.5}$$

$$P = \frac{P_0}{Y_\chi(2\Omega)}\frac{Y_\chi(2\Omega\tau)}{\tau^\chi}. \tag{3.6}$$

3.2 Model 1 Results

Model 1 consists only of a semi-infinite, half-space with an adiabatic density function. Hence, it is a single layer for which the Bessel functions are the only solutions. Since, the medium is semi-infinite, there is no cut-off frequency (as in Models 2 and 3) which would need to be addressed when considering the possible range of propagating frequencies. However, since a cut-off does arise for the other 2 models, it is useful to consider the propagating frequencies for all of the models in terms of multiples (f_c) of the cut-off frequency (Ω_c).

Since Model 1 is the simplest of the 3 models, it is very useful in identifying some interesting behavior common to all of the models. In the course of this research, the author noticed large, amplitude fluctuations as a function of the frequency multiplier (f_c) among the solutions for each model. In particular, there appeared certain frequencies in which the functions grew considerably larger than at other frequencies. This is illustrated with figures 3.2 and 3.3.

The first plot, Figure 3.2 below, shows the raw pressure functions without the boundary value constant (BVC), B, multiplying the solution. Here, the frequency multiplier, f_c, takes on the 3 values of 2.0, 2.2, and 2.213. There are a couple of results to observe in this plot. First, one can notice that there is not a large amplitude range among the 3 functions. Additionally, one should note that all of these solutions are initially negative.

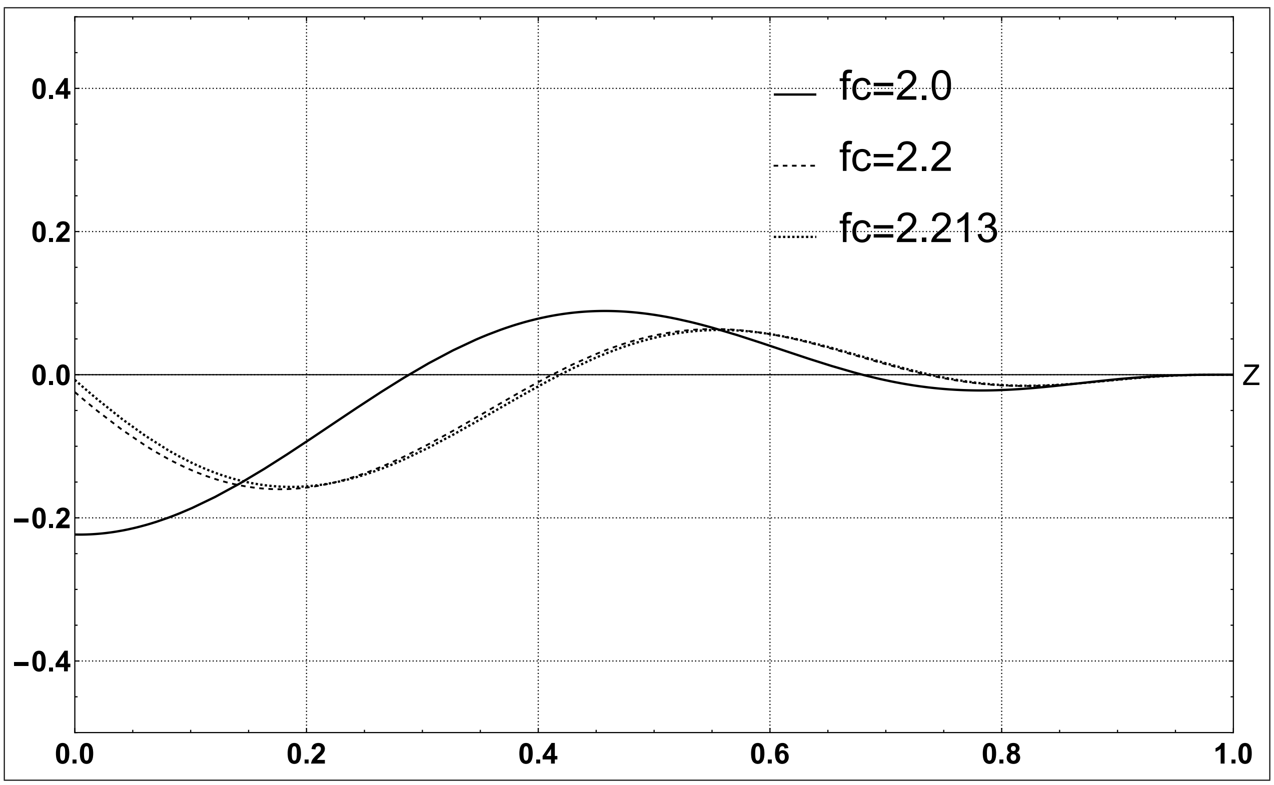

Figure 3.2: Semi-Infinite Half-Space Pressure Functions with $f_c = 2.0$, 2.2, and 2.213 with Ω_c in terms of τ.

However, upon multiplying by the boundary value constant B, one can observe the important effect of this constant on the solution in Figure 3.3. First, one notices the large spread in amplitudes in contrast to those in Figure 3.2. (Note the scale change between the

2 figures.) Second, the amplitudes now have opposite signs compared to that of the previous figure. And, third, as required, the boundary value constant, B, fixes the pressure to $P_0 = 1$ at $Z = 0$. Additionally, the source location in all pressure plots is indicated by a circle. On this plot, one will notice the circle at (0,1) on the plot.

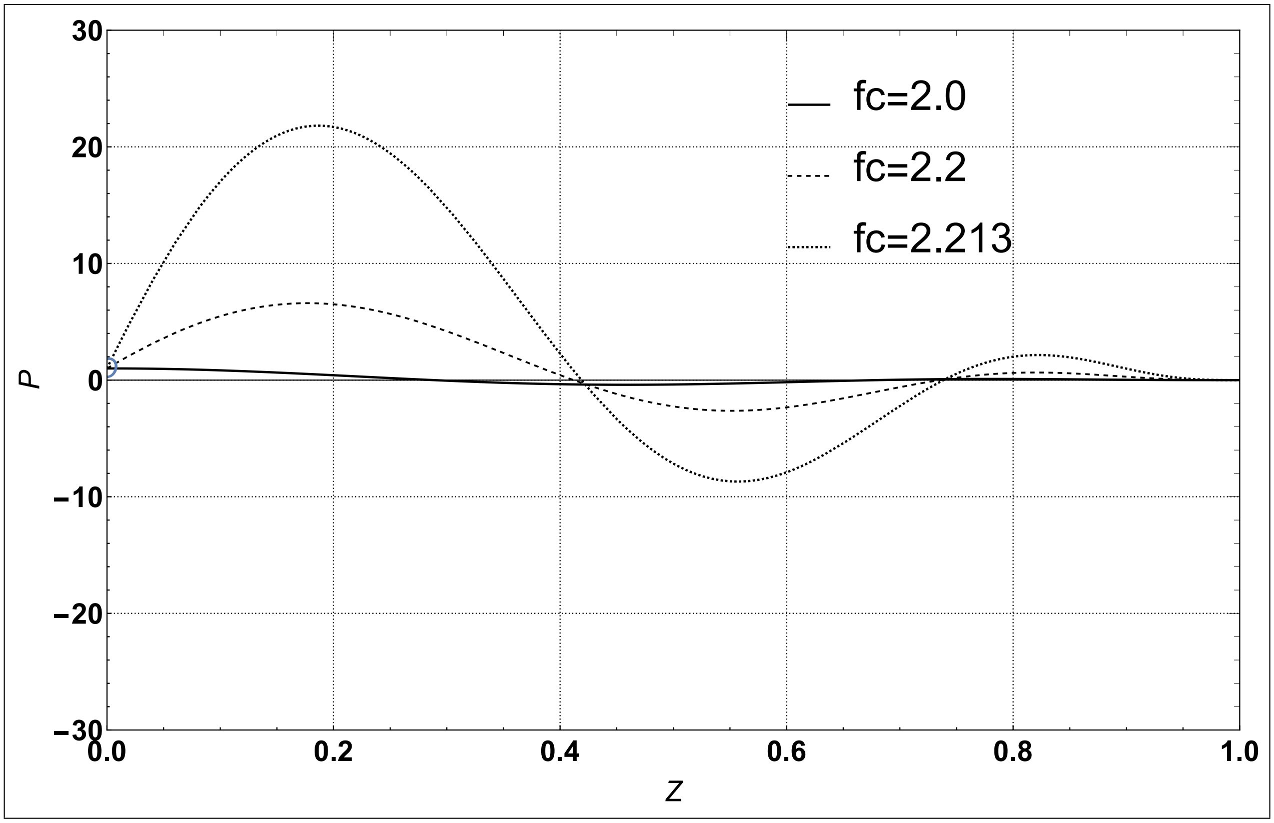

Figure 3.3: Semi-Infinite Half-Space Bessel Function with $f_c = 2.0$, 2.2, and 2.213 with Ω_c in terms of $\tau(Z)$.

Since the difference between the plots in Figures 3.2 and 3.3 is the presence of the boundary value, it appears that amplitude sensitivity is attributable to the boundary value constant, B, as a function of frequency. This can be shown more quantitatively and conclusively in the next figure.

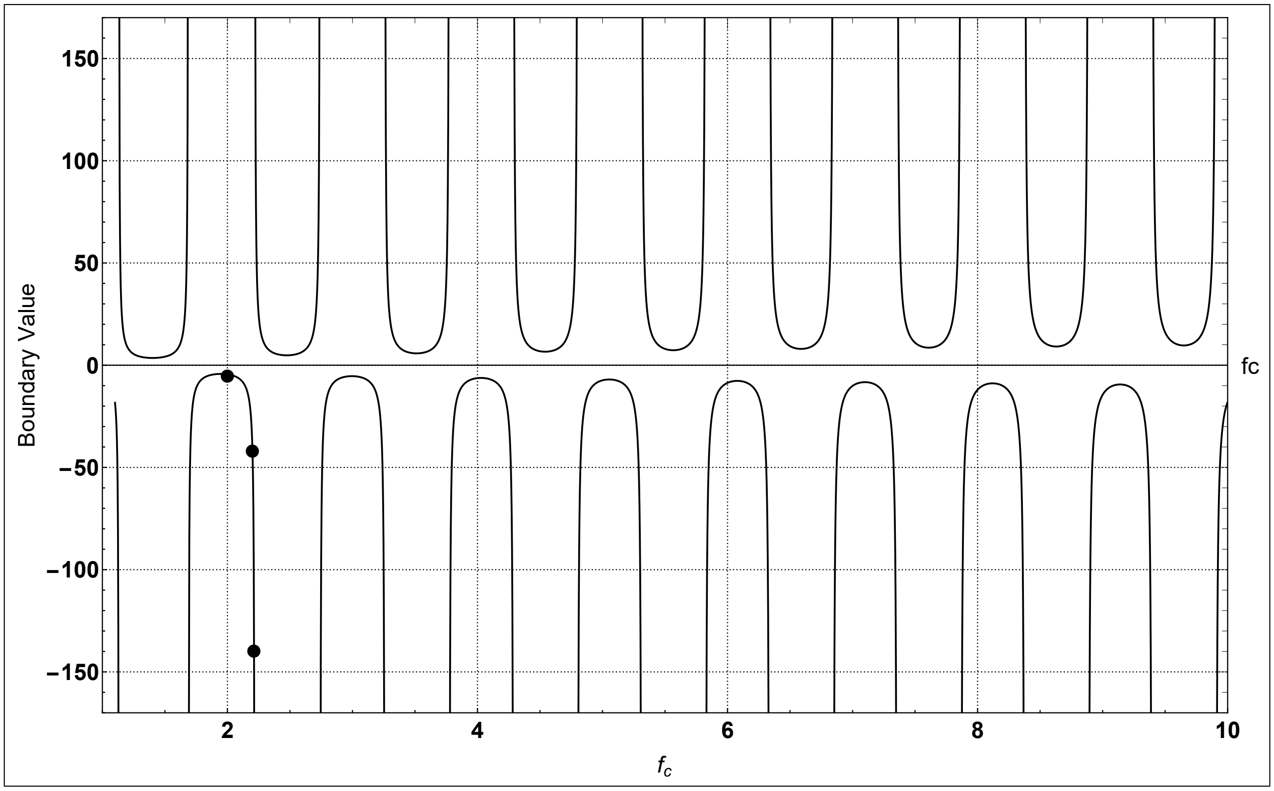

Figure 3.4: Boundary Value Constant as a function of f_c. Dots identify values for $f_c = 2.0$, 2.2, and 2.213 (left to right).

Figure 3.4 shows the values of the boundary value constant B plotted as a function of the frequency, f_c, on the boundary at $z = 0$. One will notice that the 3 "dots" on this plot represent the 3 relatively close frequencies $f_c = 2.0$, 2.2, and 2.213. One may also observe that these points occur on the negative (bottom) half of the plot. This substantiates that the minus B values are responsible for the inversion of the original functions. Hence, from looking at this plot, it is relatively easy to choose frequencies whose resulting solutions may be either large or small, or positive or negative with respect to the original functions. Usually, one solves these boundary value problems for a single frequency, possibly, unaware of what may be occurring at other frequencies. Since the boundary constant is given by $\frac{1}{J(2\Omega)}$, infinities of this function as a function of frequency correspond to the zeros of $J(2\Omega)$.

A similar example is shown in the following figures for Model 1.

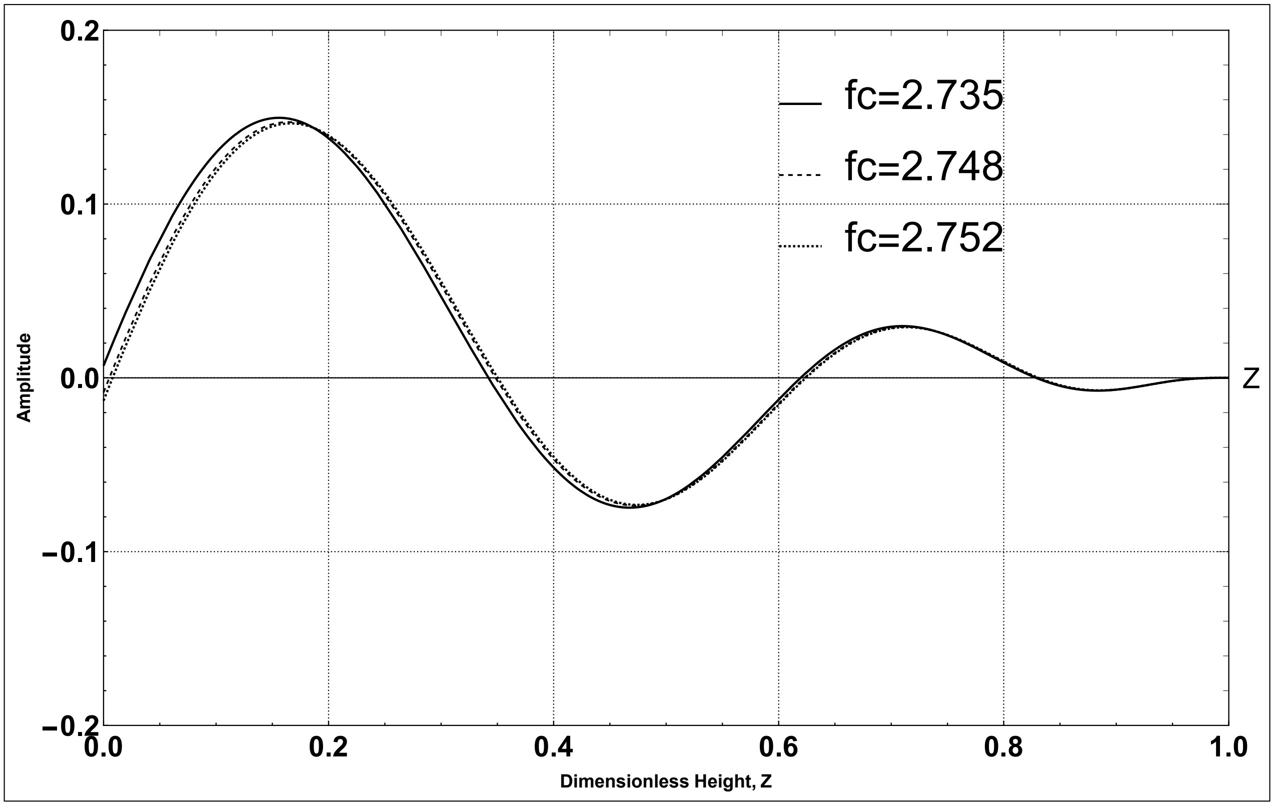

Figure 3.5: Semi-Infinite, Half-Space Bessel Function for f_c = 2.735, 2.748, and 2.752 with Ω_c in terms of $\tau(Z)$.

Figure 3.5 shows the Bessel pressure solutions for 3 relatively close frequencies of f_c = 2.735, 2.748, and 2.752. One can observe that these plots virtually overlay each other. However, when the boundary value constant B is applied, one can notice its effect. Two of the curve's initial amplitudes become negative, while that of the third is positive. This is the same behavior as seen in the previous example due to the effect of the BVC, B. Nevertheless, this can again be explained by the plot for the values of B versus f_c in Figure 3.7. Here one observes that the point with the highest value for B is the lowest frequency curve (f_c=2.735) and is positive. Hence, it has the greatest amplitude of all three curves. The next two curves of higher frequencies are seen to have smaller values for B which are negative. This resuts not only in a decreased amplitude with respect to the first curve (f_c=2.735), but also an inversion from their initial form.

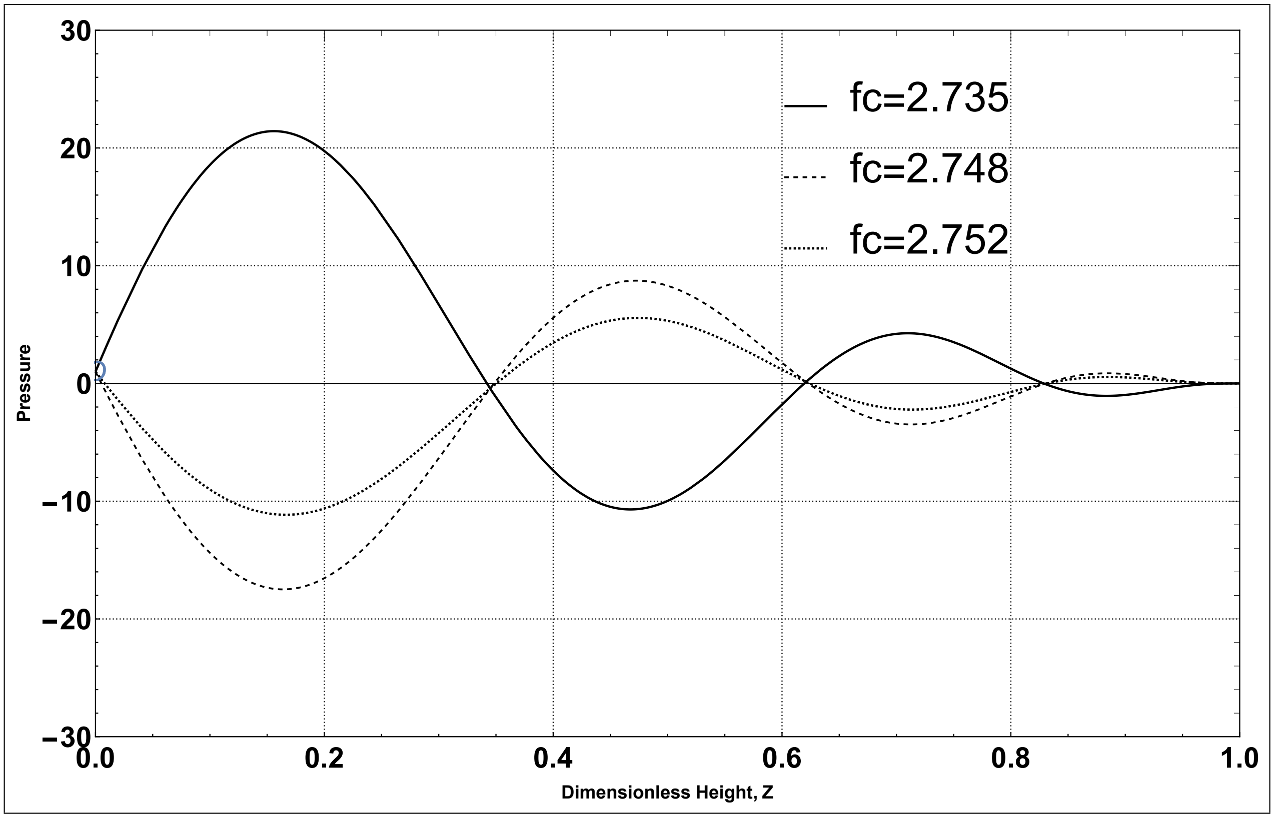

Figure 3.6: Semi-Infinite Half-Space Bessel Function for f_c = 2.735, 2.748, and 2.752 with Ω_c in terms of $\tau(Z)$.

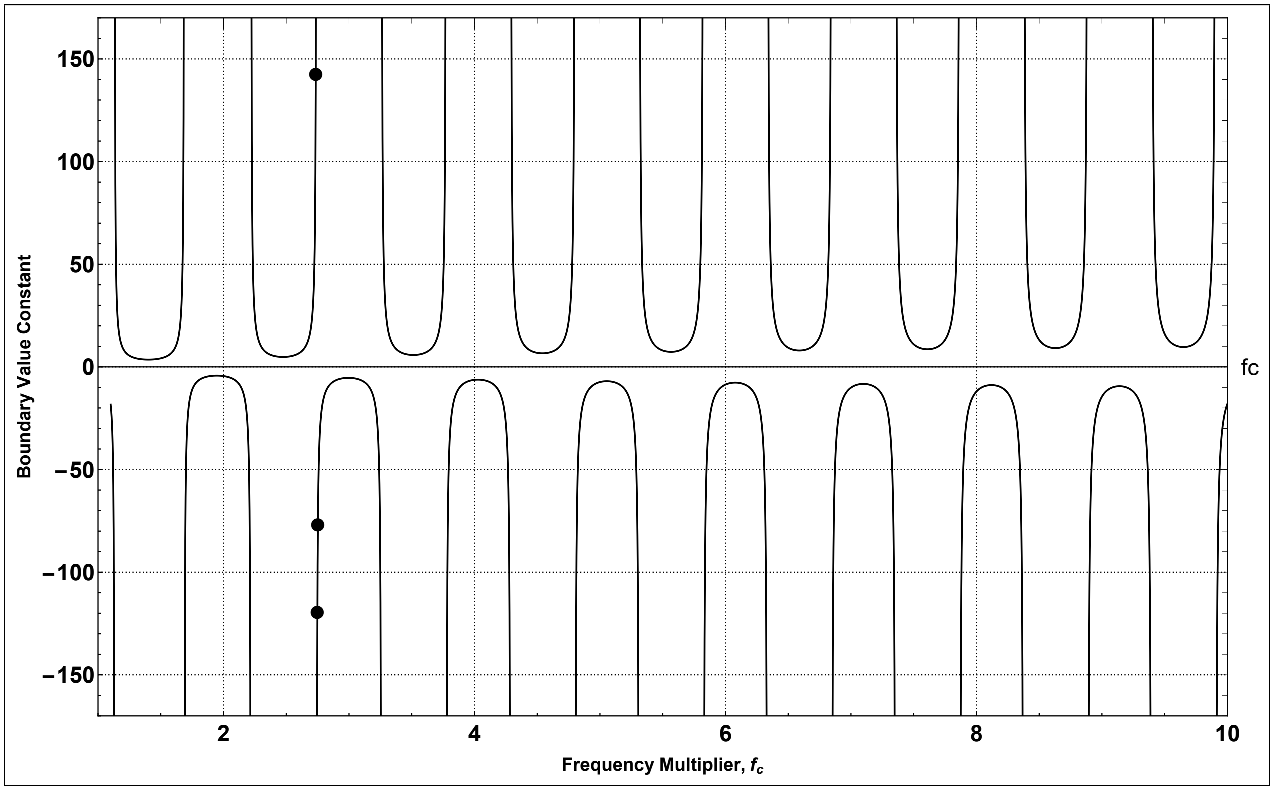

Figure 3.7: Boundary Value Constant as a function of f_c. Dots identify values for $f_c = 2.735$, 2.748, and 2.752 (left to right).

3.3 Prelude: Models 2 and 3

Models 2 and 3 consist of 2 layers in which both scenarios having a finite-layered medium at the bottom overlain with a semi-infinite one on top. Lamb's density function is used to represent the atmosphere in the lower layer and an exponential model is used in the upper layer. We take the height of the 1st layer to be $z_2 = 13{,}000$ meters and perform calculations in the 2nd medium only to a height of l=30,000 meters. The solution in the lower medium is termed P_1 and that of the upper region is termed P_2.

The general pressure solution in layer 1 (as found previously) is

$$P_1 = A\frac{J_\chi(2\Omega\tau)}{\tau^\chi} + B\frac{Y_\chi(2\Omega\tau)}{\tau^\chi}, \tag{3.7}$$

and that for the upper region is

$$P_2 = Ce^{\frac{-\alpha[Z-Z_2]}{2}} cos(\alpha \frac{\sqrt{\frac{4\Omega^2}{\alpha^2} - 1}}{2}(Z - Z_2)), \tag{3.8}$$

where the value for α is set using the value of the sound speed at $Z = Z_2$, namely, c_1. We will still call this α as shown here:

$$\alpha = \frac{\gamma g l}{c_1^2} \tag{3.9a}$$

$$\Omega_c^2 = \frac{\omega^2 l^2}{2c_1^2}. \tag{3.9b}$$

3.4 Model 2: Bottom Source, Layered, Semi-Infinite Half-Space

In this model, the source is located at the lower boundary of the 1st medium. In order to find the solution for a time-harmonic signal, one must apply the appropriate boundary conditions to (3.7) at the bottom of the 1st layer ($Z = 0$ or $\tau = 1$) and (3.8) at the top of the 1st layer (bottom of the 2nd layer) ($Z = Z_2$ or $\tau = \tau_2$) in order to determine the constants A, B, and C.

These 3 boundary conditions can be illustrated as shown below:

$$P_1|_{Z=0} = P_0 \tag{3.10a}$$

$$P_1|_{Z=Z_2} = P_2|_{Z=Z_2} \tag{3.10b}$$

$$P_1'\Big|_{Z=Z_2} = P_2'\Big|_{Z=Z_2}, \tag{3.10c}$$

where P_0 is the source pressure. Applying these boundary conditions to equations (3.7) and (3.8), one obtains the following set of equations:

$$P_0 = P|_{\tau=1} = A\left[\frac{J_\chi(2\Omega\tau)}{\tau^\chi}\right]\Bigg|_{Z=0} + B\left[\frac{J_\chi(2\Omega\tau)}{\tau^\chi}\right]\Bigg|_{Z=0} \tag{3.11}$$

$$A\left[\frac{J_\chi(2\Omega\tau)}{\tau^\chi}\right]\bigg|_{\tau=\tau_2}+B\left[\frac{J_\chi(2\Omega\tau)}{\tau^\chi}\right]\bigg|_{\tau=\tau_2}=C\left[e^{\frac{-\alpha[Z-Z_2]}{2}}cos(\alpha\frac{\sqrt{\frac{4\Omega^2}{\alpha^2}-1}}{2}(Z-Z_2))\right]\Bigg|_{Z=Z_2} \tag{3.12}$$

$$A\left[\frac{J_\chi(2\Omega\tau)}{\tau^\chi}\right]'\bigg|_{\tau=\tau_2}+B\left[\frac{J_\chi(2\Omega\tau)}{\tau^\chi}\right]'\bigg|_{\tau=\tau_2}=C\left[e^{\frac{-\alpha[Z-Z_2]}{2}}cos(\alpha\frac{\sqrt{\frac{4\Omega^2}{\alpha^2}-1}}{2}(Z-Z_2))\right]'\Bigg|_{Z=Z_2}. \tag{3.13}$$

In the above equations, the single quote mark ($'$) represents the derivative of the quantity in brackets ([ ]) with respect to z. The system of Equations (3.11)–(3.13) can be solved analytically for the 3 constants A, B, and C. This effort is undertaken in Appendix C. It should be noted that the free variable in these equations is Ω. For purposes of later analysis, this variable is cast in terms of the cut-off frequency in the top layer. That is, we frame our solutions in terms of multiples of the cut-off frequency. This is given by $\Omega_c = \frac{\alpha}{2}$ and $\Omega = f_c * \Omega_c$. Additionally, one must be aware that the boundary value constants need to be recalculated each time the frequency multiple,f_c, is changed.

3.5 Results for Model 2

Below, we show some pertinent results for Model 2 with a time-harmonic source of various frequency multiples (f_c) of the cut-off frequency (Ω_c). Here, we have chosen 3 frequencies which are reasonably near the cut-off frequency which is defined as f_c=1. The values chosen in Figure 3.8 are f_c = 2.0, 2.2, and 2.58 times the cut-off frequency. There are a couple of observations to be noted about these plots. First, the plots span both the 1st and 2nd layer and extend to the top of the 2nd layer. Second, there is no discontinuity at the top of the 1st layer (z = 13,000 m or Z = 0.43). This is due to the effect of the 3rd boundary condition (Equation (3.13)) which matches the slopes at the boundary between the 2 layers. And, third, the envelope of the pressure function is decreasing with increasing height.

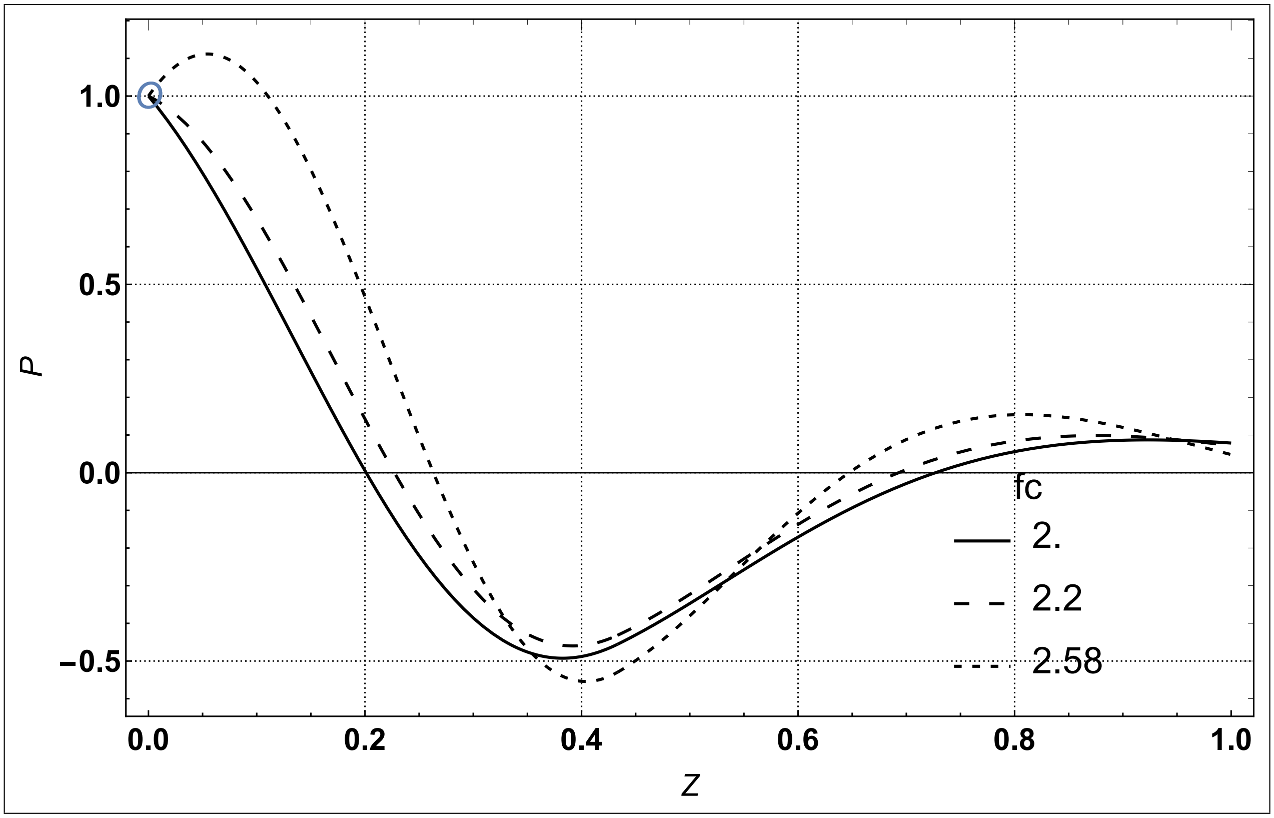

Figure 3.8: Pressure (P) vs. Height (Z) for $f_c = 2.0$, 2.2, and 2.58 (left to right).

As was previously done for Model 1 in Figure 3.4, a similar frequency dependence of the boundary value constants is derived and plotted. However, whereas in Model 1, there was only one constant (B), in Model 2 (and 3), there are two constants A and B. In order to show the effect of these constants, as a function of frequency, we choose to plot the ratio $\frac{B}{A}$. The equation showing the relationship between these two constants is given in (C.17) in Appendix C. The plot showing the ratio $\frac{B}{A}$ is shown below in Figure 3.9.

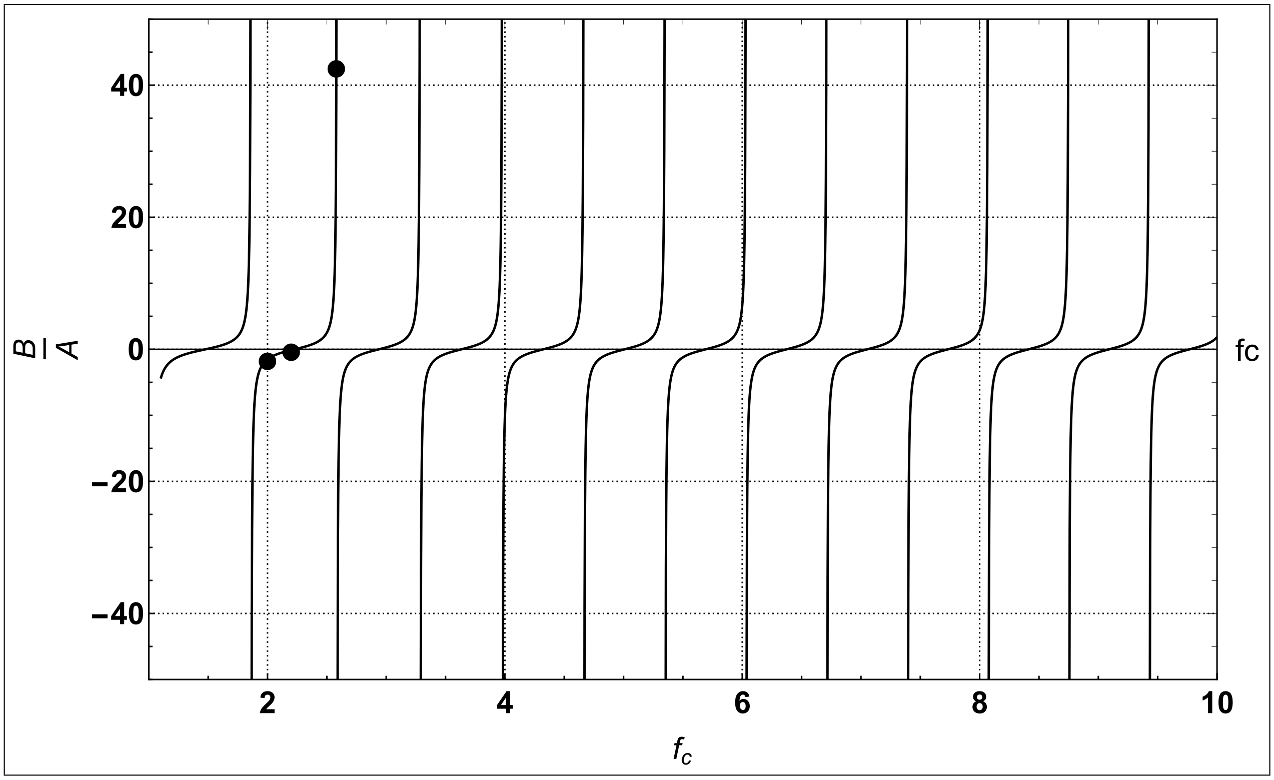

Figure 3.9: Boundary Value Constant ratio, $\frac{B}{A}$, where dots identify values for f_c=2.0, 2.2, and 2.58 (left to right).

As done previously, we have plotted "dots" on the graph for the corresponding frequencies plotted in figure 3.8. One should note that these all lie on the same curve. Additionally, for frequencies $f_c = 2.0$ and 2.2 (which have negative amplitudes in figure 3.9), their plots start downward as shown in 3.8. However, for $f_c = 2.58$, which has a positive pressure amplitude, its graph starts upward as seen in 3.8.

3.6 Model 3: Top Source, Layered, Semi-Infinite Half-Space

In the 3rd model, one has the same two-layered medium as in the 2nd case except that the source has been moved to the top of the first layer. That is, on the boundary between the 1st and 2nd layers. Here, once again, one is faced with solving a boundary value problem for the 3 constants A,B, and C. However, the boundary conditions are different than previously found and are enumerated here:

$$P_1'\Big|_{Z=0} = 0 \tag{3.14a}$$

$$P_2|_{Z=Z_2} = P_0 \tag{3.14b}$$

$$P_1|_{Z=Z_2} = P_2|_{Z=Z_2}\,. \tag{3.14c}$$

Again, these conditions are applied to Equations (3.7) and (3.8). A complete derivation for the constants A, B, and C is developed in Appendix E. A plot showing the pressure curves for the 3 previous frequencies is shown in Figure 3.10. Here the source is located on the boundary between the 2 layers. Hence, the source radiates into both layers simultaneously. From the plot, one can observe the expected effect of the decline in pressure with distance from the source.

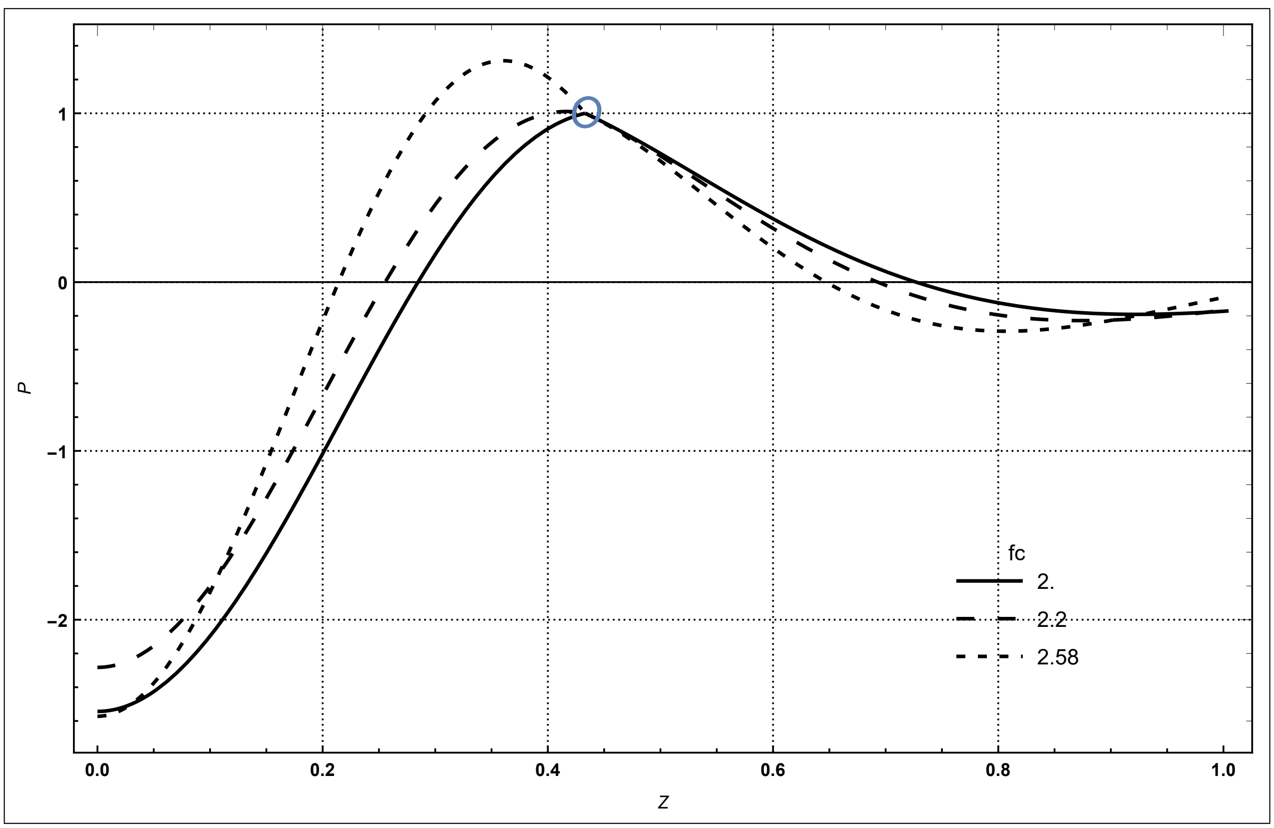

Figure 3.10: Model 3 Pressure Curves for f_c=2.0, 2.2, and 2.58 (left to right).

3.7 Phase Velocity

For Models 2 and 3, we find that standing wave solutions for vertical, atmospheric, wave propagation can be represented by equation (3.7). Their general form is composed of Bessel functions of the First and Second Kind. However, it is well known that these standing waves are comprised of traveling waves (Hankel functions) moving in opposite directions with phase velocity, v_p. Thus, one may ask to what familiar phenomena could these functions be related and what attributes could be ascribed to our results.

Physically speaking, if one thinks of our problem as a large cylinder whose axis passes through the source and is perpendicular to the Earth's radius at the surface, then traveling waves in the z-direction become radial waves. Hence, our analogy allows us to look for the phase velocities of these waves.

With this in mind, we define the radial waves as Hankel function [16], [3] of order $-\frac{7}{2}$

$$H^{(1)}_{-\frac{7}{2}} = \left\{ J_{-\frac{7}{2}}(2\Omega\tau) + iY_{-\frac{7}{2}}(2\Omega\tau) \right\}. \tag{3.15}$$

The phase function is then defined as

$$\phi = \tan^{-1}\left\{ \frac{Y_{-\frac{7}{2}}(2\Omega\tau)}{J_{-\frac{7}{2}}(2\Omega\tau)} \right\} - \omega t, \tag{3.16}$$

where the phase velocity, v_p, is determined by

$$v_p = -\frac{\partial\phi}{\partial t} / \frac{\partial\phi}{\partial z}. \tag{3.17}$$

Explicitly writing $\tau(Z)$ is a helpful reminder that $\tau(Z)$ is defined as

$$\tau(Z) = (1 - Z)^{\frac{1}{2}} = (1 - \frac{z}{l})^{\frac{1}{2}}, \tag{3.18}$$

where Z is the variable in the vertical direction (z) normalized by highest altitude (l) of the model.

Finding $-\frac{\partial \phi}{\partial t}$, one has

$$-\frac{\partial \phi}{\partial t} = -\frac{\partial}{\partial t}\left\{ \tan^{-1}\left\{ \frac{(Y_{-\frac{7}{2}}(2\Omega\tau(Z))}{(J_{-\frac{7}{2}}(2\Omega\tau(Z))} \right\} - \omega t \right\} \\ = \omega. \tag{3.19}$$

Next, it is necessary to find $\frac{\partial \phi}{\partial z}$. Writing the spatial part of ϕ only, we have

$$\frac{\partial \phi}{\partial z} = \frac{\partial}{\partial z}\left\{ \tan^{-1}\left\{ \frac{(Y_{-\frac{7}{2}}(2\Omega(1-\frac{z}{l})^{\frac{1}{2}})}{(J_{-\frac{7}{2}}(2\Omega(1-\frac{z}{l})^{\frac{1}{2}})} \right\} \right\}. \tag{3.20}$$

For clarity of the remaining derivation, the order and argument of the Bessel functions are not explicitly indicated, but still remain $-\frac{7}{2}$ and $2\Omega\tau$, respectively. Thus, continuing with the fraction $\frac{Y(\cdot)}{J(\cdot)} = \nu$, one may write

$$\frac{\partial \phi}{\partial z} = \frac{\partial tan^{-1}(\nu)}{\partial \nu}\frac{\partial \nu}{\partial z}. \tag{3.21}$$

Performing the first partial derivative on the right-hand side, we have

$$\frac{\partial \phi}{\partial z} = \frac{1}{1+\nu^2}\frac{\partial \nu}{\partial z}. \tag{3.22}$$

Then, the other derivative on the right-hand side yields

$$\frac{\partial \phi}{\partial z} = \frac{1}{1+\nu^2}\left\{ \frac{\partial \frac{Y(u)}{J(u)}}{\partial u}\frac{\partial u}{\partial z}, \right\} \tag{3.23}$$

where $u = (2\Omega(1-\frac{z}{l})^{\frac{1}{2}})$. Continuing, we have

$$\frac{\partial \phi}{\partial z} = \frac{1}{1+\nu^2}\left\{ \frac{JY' - YJ'}{J^2}\frac{\partial u}{\partial z} \right\}. \tag{3.24}$$

Substituting for ν in (3.24), we obtain

$$\frac{1}{1+\nu^2} = \frac{J^2}{J^2+Y^2}, \tag{3.25}$$

resulting in

$$\frac{\partial \phi}{\partial z} = \left\{ \frac{JY' - YJ'}{J^2 + Y^2} \right\} \frac{\partial u}{\partial z}. \tag{3.26}$$

Here we should note that the expression $JY' - YJ'$ is the negative Wronskian, W, of $J(u)$ and $Y(u)$. For all orders of Bessel functions, it can be shown that

$$JY' - YJ' = -W[J(u), Y(u)] = -\frac{2}{\pi u}, \tag{3.27}$$

which for our value of u becomes

$$-W[J(u), Y(u)] = \frac{-2}{\pi 2\Omega(1 - \frac{z}{l})^{\frac{1}{2}}}. \tag{3.28}$$

And, for the remaining partial derivative, one has

$$\begin{aligned} \frac{\partial u}{\partial z} &= \frac{\partial(2\Omega(1 - \frac{z}{l})^{\frac{1}{2}})}{\partial z} \\ &= 2\Omega \frac{1}{2(1 - \frac{z}{l})^{\frac{1}{2}}} \frac{-1}{l} \\ &= \frac{-\Omega}{l(1 - \frac{z}{l})^{\frac{1}{2}}}. \end{aligned} \tag{3.29}$$

Finally, combining (3.19), (3.26), (3.27), (3.29) in (3.17), we obtain

$$v_p = \frac{\omega(J^2 + Y^2)}{\frac{-1}{\pi\Omega(1 - \frac{z}{l})^{\frac{1}{2}}}} \frac{-l(1 - \frac{z}{l})^{\frac{1}{2}}}{\Omega}. \tag{3.30}$$

In reducing (3.30), it should be remembered that one is addressing only those frequencies in the layer which will propagate into the top layer. Hence, we use Ω such that

$$\Omega = f_c * \Omega_c \tag{3.31a}$$

$$\text{where } \Omega_c = \frac{\omega l}{c_1} \tag{3.31b}$$

and $c_1 = c_0\sqrt{1 - \frac{z_2}{l}}$. Thus,

$$v_p = \pi \frac{\Omega c_0 (1 - \frac{z_2}{l})^{\frac{1}{2}}}{l} (J^2 + Y^2)(l(1 - \frac{z}{l})) \quad (3.32)$$

or

$$\frac{v_p}{c_0} = \pi \Omega \sqrt{1 - Z_2}(1 - Z) \left[J^2 + Y^2 \right], \quad (3.33)$$

where Z has been reconstituted.

Equation (3.33) represents the phase velocity for all of the models presented in this work. And, as seen, all of the phase velocities are functions of the frequency f_c through the Bessel equations and Ω. Consequently, we show an example of these velocities for several different frequencies in the figure below:

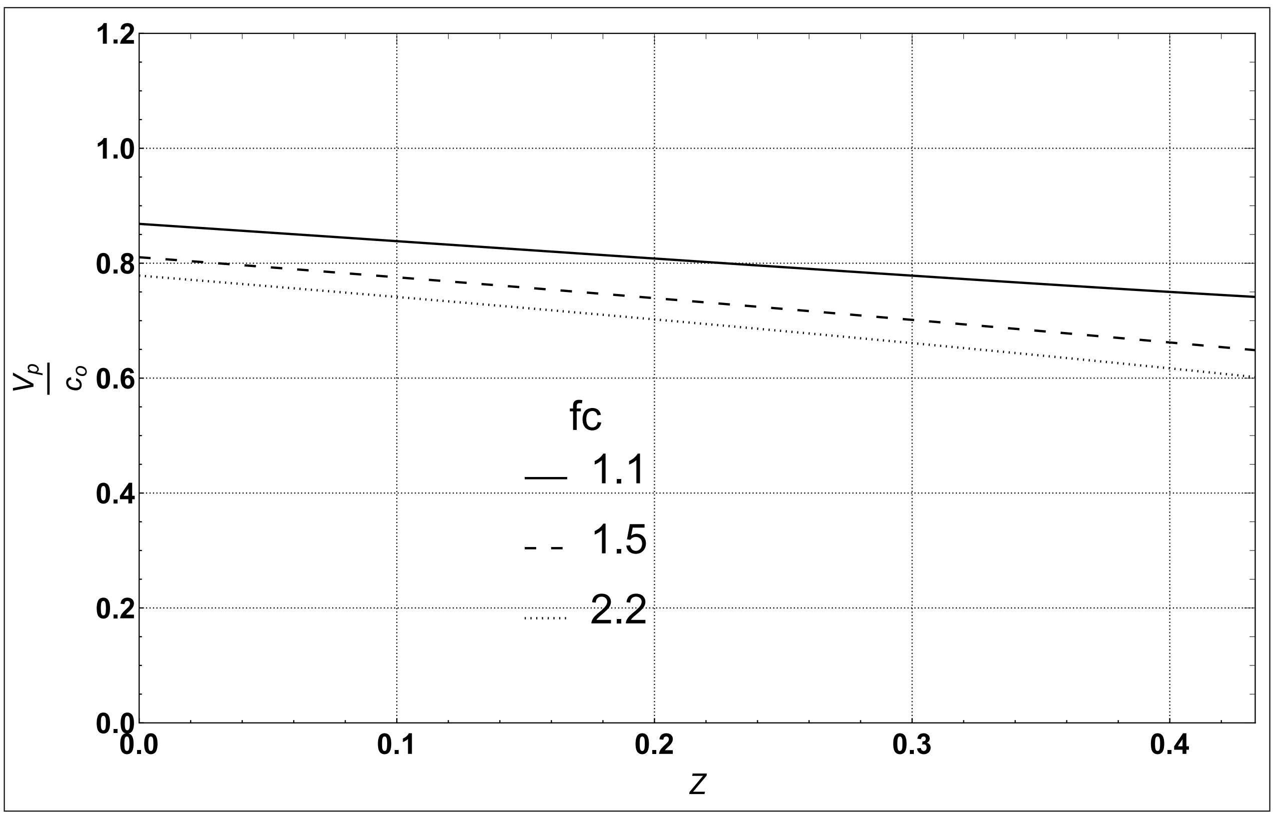

Figure 3.11: Phase Velocity for $f_c = 1.1$, 1.5, and 2.2.

Figure 3.11 shows phase velocities calculated for 3 different frequencies as a ratio to the ground sound speed c_0. They are shown as a function of the normalized coordinate Z for the entire length of the 1st layer.

3.8 Discussion

In this work, we have obtained analytical solutions to the propagation of acoustic waves in the z-direction for a spatially-compact, time-harmonic source, assuming no wind conditions. The analytical solutions are found to be Bessel equations of the First and Second Kind of order $-\frac{7}{2}$ expressed in terms of a variable $\tau = \sqrt{1-Z}$ where Z is a dimensionless height. To investigate the realistic nature of this model, we devised 3 scenarios to test the model's behavior.

The first scenario was termed Model 1 and consisted of a harmonic source placed at the bottom of a semi-infinite,adiabatic half-space. The resulting pressure function was very sensitive to the boundary value (B) at $Z = 0$ and, which, in turn, was critically dependent on the frequency. The boundary value constant, B, was plotted as a function of the frequency parameter, f_c, and found that relative amplitudes could be predicted based on their positions on the same curve of this plot (see Figure 3.4). Additionally, it could be seen that the greater the magnitude of B, then, the steeper the increase in pressure curves from the source P_0 (see Figure 3.3).

The next 2 models were considered two layered media. The first layer was a finite, adiabatic medium and extended from the ground ($z = 0$) to a height of 13,000 meters. On top of this layer was a second, isothermal layer of semi-infinite extent. Although its extent was infinite, for computational purposes, the point at infinity was set to a height of l=30,000 meters. Model 2 considered a harmonic source at the bottom of the first layer. Model 3 considers a source placed at this interface of two layers.

The results for Model 2 are plotted for frequencies of f_c = 2.0, 2.2, and 2.58. These plots showed similar behavior to Model 1 due to the close nature of these frequencies (see Figure 3.8). The dependence on the ratio of the boundary value constants $\frac{B}{A}$ were plotted in Figure 3.9. This result showed that the 3 pressure curves remained on the same $\frac{B}{A}$ curve (see Figure 3.9). Furthermore, it was seen that, whether pressure function proceeding from the source was below P_0 or above it, showed dependence on the sign of $\frac{B}{A}$.

In Model 3, the source was placed on the boundary between the 2 layers. The frequencies for the pressure curves were the same used for Model 2. Consequently, this showed a close grouping of the 3 curves. With the source placed on the layer boundaries, the source radiated simultaneously into the lower and upper layers. The resulting pressure showed less oscillation than that for Models 1 and 2, and, additionally, showed a continuous pressure decline as one moved further from the source, in either direction.

CHAPTER IV

Green's Functions (GF) and Fast Field Program (FFP)

As seen in the previous chapters, an analytic model for the vertical propagation of time-harmonic, acoustic waves has been developed. Three model scenarios were studied: 1) Source at the bottom of a semi-infinite medium, 2) and 3) a finite, adiabatic medium with the source on the bottom/top overlain by a semi-infinite, isothermal medium. Here, we consider time-harmonic, point sources located within a finite, non-homogeneous medium. The lower (1st) layer is represented by a Bessel differential equation of order $\chi = -7/2$ and argument $2\Omega\tau$. The top (2nd) layer is represented by damped harmonic oscillator ODE. All solutions are couched in terms of multiples (f_c) of the cut-off frequency (Ω_c) in the 2nd layer. For a source within a layer, the solution is derived via the Green's function method. To this end, the model equations are solved using a Green's function technique with validation performed using a Fast Field Program (FFP) with a source function.

4.1 Model Review

The present model consists of a finite-layered atmosphere at the Earth's surface topped by a semi-infinite medium. Propagation equations were derived for both media.

We derived our model from the linearization of Euler's equations for compressible flow in an non-homogeneous fluid in Chapter II for a single-frequency source [14]. It was found that the lower layer solution produced a Bessel's ODE of the form shown here

$$\tau^2 \frac{d^2X}{d\tau^2} + \tau \frac{dX}{d\tau} + (4\Omega^2\tau^2 - \chi^2)X = 0. \tag{4.1}$$

The solution for the pressure is given by

$$P = \frac{X}{\tau^\chi}, \tag{4.2}$$

where X satisfies Equation (4.1). Additionally, (4.1) is a Bessel equation of order $\chi = -\frac{7}{2}$ of argument $2\Omega\tau$. The solution for X admits solutions of Bessel's equation of the First ($J_\chi(2\Omega\tau)$) and Second Kind ($Y_\chi(2\Omega\tau)$) in the argument $2\Omega\tau$. Thus a general solution for the pressure,P, in the lower layer may be written as:

$$P = A\frac{J_\chi(2\Omega\tau)}{\tau^\chi} + B\frac{Y_\chi(2\Omega\tau)}{\tau^\chi}, \tag{4.3}$$

where A and B are constants to be determined from boundary conditions. For clarification, $\tau = \sqrt{1 - \frac{z}{l}} = \sqrt{1 - Z}$, where $l = 30,000$ meters and z is the height above the Earth's surface with Z the corresponding dimensionless height with respect to l. Consequently, the finite layer extends to a height of $z = z_2$, where z_2 represents the top of the troposhpere at 13,000 meters and l terminates in the stratosphere.

The solution in the top layer was developed for an atmosphere which had an exponentially decreasing density with height (z). It was found to be

$$P = Ce^{\frac{-\alpha[Z-Z_2]}{2}} cos(\alpha\frac{\sqrt{\frac{4\Omega^2}{\alpha^2} - 1}}{2}(Z - Z_2)), \tag{4.4}$$

with

$$\alpha = \frac{\gamma g l}{c^2} \tag{4.5a}$$

$$\Omega = \frac{\omega l}{c}. \tag{4.5b}$$

In (4.4), the pressure in the upper layer, P, does not exist until $Z = Z_2$. At this point, α and Ω are determined by the sound speed c at Z_2.

4.2 Green's Function Introduction

Here we begin the derivation and study of a point-source model. In this part, we concentrate solely on the bottom layer which has a point source (in spatial dimension) located within the layer at an arbitrary location $Z = Z_1$. When solving any ordinary differential equation (ODE), there are always 2 general solutions: the particular solution and the complementary solution. If the ODE is homogeneous, then there is no particular solution to be found. However, if there is a source term, it is not homogeneous, and then there will be a particular solution, if it can be found. One method of finding the particular solution involves using Green's functions. This method involves finding the particular solution where the source term is a Dirac *delta* function [1] and which is defined here:

$$\delta(x) = 0, x \neq 0 \tag{4.6a}$$

$$f(0) = \int_{-\infty}^{+\infty} f(x)\delta(x)\,\mathrm{d}x. \tag{4.6b}$$

The method usually entails assuming homogeneous boundary conditions at the bottom and top of layer "1". The homogeneous boundary conditions may be Dirichilet, Neumann, Cauchy, or Robin. The type of boundary condition is not critical, but it is helpful if they are homogeneous. If there are boundary conditions which are not zero, then these may be addressed by inclusion of the complementary solution with the appropriate boundary conditions. One will have the complete solution, when both the particular (Green's function) and the complementary solutions are combined.

Proceeding with finding the particular solution, the Green's function method was a natural choice due to the inclusion of the point source in the lower layer. The 4 "boundary" conditions which must be solved are readily found in most mathematical physics texts which cover Green's functions[1]. These are listed here for easy reference:

$$G_1'(Z|Z_1)\Big|_{Z=0} = 0 \tag{4.7a}$$

$$G_1(Z|Z_1)|_{Z=Z_1} = G_2(Z|Z_1)|_{Z=Z_1} \tag{4.7b}$$

$$G_2'(Z|Z_1)\Big|_{Z=Z_1} - G_1'(Z|Z_1)\Big|_{Z=Z_1} = \frac{1}{2\tau_1^3} \tag{4.7c}$$

$$G_2(Z|Z_1)|_{Z=Z_2} = 0. \tag{4.7d}$$

The above steps involve separating the layer at the source point ($Z = Z_1$) into a "top" (2) and "bottom" (1) layer where the numbers in parentheses correspond to the subscripts in the above equations (these now change their meaning as used previously for our 2-layer problem).

After the Green's function is found, it must be convolved with a source function in order to produce the correct solution for the given source distribution. For example, if one has the source distribution of $f(\zeta)$, then the correct solution would be

$$u(Z) = \int_a^b G(Z|\zeta)f(\zeta)\,d\zeta \tag{4.8}$$

where the interval $[a,b]$ includes the source distribution.

However, in the problem posed here, we are using a point source which can be represented by $P_0\delta(Z_1 - \zeta)$ where P_0 would be the pressure amplitude. Substituting this source in (4.8), we get

$$u(Z) = \int_a^b G(Z|\zeta)P_0\delta(Z_1 - \zeta)\,d\zeta = P_0 G(Z|Z_1) \tag{4.9}$$

It should be noted that the solution to the Bessel's equation is not the total solution for the pressure. To find the pressure, we must form the pressure equation as $P = \frac{g(Z|Z_1)}{\tau^\chi}$. However, the utility of the Green's function is that once it is found, then the solution for any distributive source function may be easily found by convolving the Green's function with the source term.

The details of applying the steps in Equations (4.7a)–(4.7d) to the Bessel's equation are carried out in the next section. It should be noted that, as previously, the boundary value constants depend on the variable Ω. Yet, furthermore, the results also depend on the source location $Z = Z_1$ as one would expect. Additionally, we emphasize that the solution found in the next section is the Green's function solely for the Bessel ODE. The complete solution still needs to be determined and that will involve the complementary solution as well.

4.3 Particular Solution with Green's Function and Determining Boundary Values

As in the previous case for a harmonic source, we now must solve another boundary value problem for an point source. However, here we must follow the boundary conditions set forth in finding the Green's functions. Consequently, we are only concerned with the lower layer, as the complete solution is found in the main text. Again, we are concerned with Bessel's equation as previously derived and restated below:

$$\tau^2 \frac{d^2 G}{d\tau^2} + \tau \frac{dG}{d\tau} + (4\Omega^2\tau^2 - \chi^2)G = \delta(\tau(Z) - \tau(Z_1)) \tag{4.10}$$

where $\tau_1 = \tau Z_1$ indicates the source location. Again, has, as our starting point, the general solution:

$$G_h = AJ_\chi\left(2\Omega\sqrt{1-Z}\right) + BY_\chi\left(2\Omega\sqrt{1-Z}\right) \tag{4.11}$$

where A and B are constants to be determined by boundary conditions and the subscript h refers to the homogeneous solution of (4.10). The reader is reminded that $\tau = \sqrt{1 - \frac{z}{l}}$ and $Z = \frac{z}{l}$.

Now, as in the nature of the Green's function method, we must solve the above equation (with different coefficients) in the region below and above the source location. Hence, the lower layer is split by the location of the source into 2 layers in which Equation (4.11) is solved in each layer. Thus, we will state here the general boundary conditions which must be solved to determine constants A, B, C, and D. It should also be recognized that the boundary conditions at the top and bottom of the 1st layer are chosen to be homogeneous. In order to solve for the constants A, B, C, and D, 4 boundary conditions are necessary. These are now restated below in detail with reference to the parameters of the problem such that the subscript "1" refers to the layer below the source location and "2" refers to the layer above the source location. We'll refer to the source location as $Z = Z_1$.

Since we will be needing the derivative of G, we will find this expression first. Using the chain rule on (4.11), we find the derivatives to be:

$$\frac{d}{dZ}\left[J_\chi\left(2\Omega\tau\right)\right] = -\frac{\Omega}{\tau}J'_\chi\left(2\Omega\tau\right) \tag{4.12a}$$

$$\frac{d}{dZ}\left[Y_\chi\left(2\Omega\tau\right)\right] = -\frac{\Omega}{\tau}Y'_\chi\left(2\Omega\tau\right) \tag{4.12b}$$

Using the expressions for the Bessel function and its derivatives in Equations (4.7a)–(4.7d), we have the following substitutions for the Green's functions:

$$1.\ \left\{ A(-\frac{\Omega}{\tau}) J'_{\chi}(2\Omega\tau) + B(-\frac{\Omega}{\tau}) Y'_{\chi}(2\Omega\tau) \right\}\Bigg|_{Z=0} = 0 \tag{4.13a}$$

$$\begin{aligned} &2.\ \{AJ_{\chi}(2\Omega\tau) + BY_{\chi}(2\Omega\tau)\}|_{Z=Z_1} \\ &= \{CJ_{\chi}(2\Omega\tau) + DY_{\chi}(2\Omega\tau)\}|_{Z=Z_1} \end{aligned} \tag{4.13b}$$

$$\begin{aligned} &3.\ \left\{ C(-\frac{\Omega}{\tau}) J'_{\chi}(2\Omega\tau) + D(-\frac{\Omega}{\tau}) Y'_{\chi}(2\Omega\tau) \right\}\Bigg|_{Z=Z_1} \\ &- \left\{ A(-\frac{\Omega}{\tau}) J'_{\chi}(2\Omega\tau) + B(-\frac{\Omega}{\tau}) Y'_{\chi}(2\Omega\tau) \right\}\Bigg|_{Z=Z_1} \\ &= \frac{1}{\tau^2}\Bigg|_{Z=Z_1} \end{aligned} \tag{4.13c}$$

$$4.\ \{CJ_{\chi}(2\Omega\tau) + DY_{\chi}(2\Omega\tau)\}|_{Z=Z_2} = 0 \tag{4.13d}$$

Solving (1) above in Equation (4.13a), we find

$$B = -A\frac{J'_{\chi}(2\Omega)}{Y'_{\chi}(2\Omega)} = -A \cdot g \tag{4.14}$$

where we are using lower case letters to represent the ratio of various quantities. This will be quite useful later on in effecting a solution as these expressions become more and more unweilding.

Next we rearrange the terms in (2) in Equation (4.13b) to get

$$[A - C] = [D - B]\, \frac{Y_{\chi}(Z_1)}{J_{\chi}(Z_1)} \tag{4.15}$$

Rewriting Equation (4.13c) in (3) above, we have

$$-[A - C]\, J'_{\chi}(Z_1) + [D - B]\, Y'_{\chi}(Z_1) = \frac{-\tau}{\Omega\tau^2(Z_1)} = \frac{-1}{\Omega\tau(Z_1)} \tag{4.16}$$

Substituting Equation (4.15) into Equation (4.16), one has

$$-\left[D-B\right]\frac{Y_\chi(Z_1)}{J_\chi(Z_1)}J'_\chi(Z_1)+\left[D-B\right]Y'_\chi(Z_1)=\frac{-1}{\Omega\tau(Z_1)} \tag{4.17}$$

Factoring and multiplying by $J_\chi(Z_1)$, one has

$$\begin{aligned}&\left[D-B\right]\Big\{-Y_\chi(Z_1)J'_\chi(Z_1)+Y'_\chi(Z_1)J_\chi(Z_1)\Big\}\\&=\frac{-J_\chi(Z_1)}{\Omega\tau(Z_1)}\end{aligned} \tag{4.18}$$

The term in braces is the Wronskian $[W]$ of $J_\chi(Z_1)$ and $Y_\chi(Z_1)$. That is,

$$\begin{aligned}&W[J_\chi(Z_1),\,Y_\chi(Z_1)]\\&=\Big\{-Y_\chi(Z_1)J'_\chi(Z_1)+Y'_\chi(Z_1)J_\chi(Z_1)\Big\}\end{aligned} \tag{4.19}$$

Thus, Equation (4.18) may be rewritten as

$$\left[D-B\right]=\frac{-J_\chi(Z_1)}{W[J_\chi(Z_1),\,Y_\chi(Z_1)]\,\Omega\tau(Z_1)}=e \tag{4.20}$$

A similar process may be used with Equations (4.15) and (4.16) to solve for $[A-C$ resulting in:

$$\left[A-C\right]=\frac{-Y_\chi(Z_1)}{W[J_\chi(Z_1),\,Y_\chi(Z_1)]\,\Omega\tau(Z_1)}=f \tag{4.21}$$

Finally, the last boundary condition in (4) of Equation (4.13d) becomes

$$D=-C\frac{J_\chi(Z_2)}{Y_\chi(Z_2)}=-C\cdot h \tag{4.22}$$

Equations (4.14), (4.20), (4.21), and (4.22) now form a reduced system of 4 equations for the 4 unknowns A, B, C, and D. These can be solved in terms of the lower case variables and the results are:

$$A = \left[\frac{e - fh}{g - h}\right] \tag{4.23a}$$

$$B = \left[\frac{e - fh}{h - g}\right] g \tag{4.23b}$$

$$C = \left[\frac{e - fg}{g - h}\right] \tag{4.23c}$$

$$D = \left[\frac{e - fg}{h - g}\right] h \tag{4.23d}$$

Below we expand the expressions in Equations (4.23a)–(4.23d) in terms of their representative functions. Thus, we have

$$e = \frac{-J_\chi(Z_1)}{W[J_\chi(Z_1),\, Y_\chi(Z_1)]\, \Omega\tau(Z_1)} \tag{4.24a}$$

$$f = \frac{-Y_\chi(Z_1)}{W[J_\chi(Z_1),\, Y_\chi(Z_1)]\, \Omega\tau(Z_1)} \tag{4.24b}$$

$$g = \frac{J'_\chi(2\Omega)}{Y'_\chi(2\Omega)} \tag{4.24c}$$

$$h = \frac{J_\chi(Z_2)}{Y_\chi(Z_2)} \tag{4.24d}$$

Hence, the constants are found to be:

$$A = \frac{\frac{-J_\chi(Z_1)}{W[J_\chi(Z_1),Y_\chi(Z_1)]\Omega\tau(Z_1)} + \left\{\frac{Y_\chi(Z_1)}{W[J_\chi(Z_1),Y_\chi(Z_1)]\Omega\tau(Z_1)}\right\}\left\{\frac{J_\chi(Z_2)}{Y_\chi(Z_2)}\right\}}{\left[\frac{J'_\chi(2\Omega)}{Y'_\chi(2\Omega)}\right] - \left[\frac{J_\chi(Z_2)}{Y_\chi(Z_2)}\right]} \tag{4.25a}$$

$$B = \left\{\frac{\frac{-J_\chi(Z_1)}{W[J_\chi(Z_1),Y_\chi(Z_1)]\Omega\tau(Z_1)} + \left\{\frac{Y_\chi(Z_1)}{W[J_\chi(Z_1),Y_\chi(Z_1)]\Omega\tau(Z_1)}\right\}\left\{\frac{J_\chi(Z_2)}{Y_\chi(Z_2)}\right\}}{\left[\frac{J_\chi(Z_2)}{Y_\chi(Z_2)}\right] - \left[\frac{J'_\chi(2\Omega)}{Y'_\chi(2\Omega)}\right]}\right\}\left[\frac{J'_\chi(2\Omega)}{Y'_\chi(2\Omega)}\right] \tag{4.25b}$$

$$C = \frac{\frac{-J_\chi(Z_1)}{W[J_\chi(Z_1),Y_\chi(Z_1)]\Omega\tau(Z_1)} + \left\{\frac{Y_\chi(Z_1)}{W[J_\chi(Z_1)Y_\chi(Z_1)]\Omega\tau(Z_1)}\right\}\left\{\frac{J'_\chi(2\Omega)}{Y'_\chi(2\Omega)}\right\}}{\left[\frac{J'_\chi(2\Omega)}{Y'_\chi(2\Omega)}\right] - \left[\frac{J_\chi(Z_2)}{Y_\chi(Z_2)}\right]} \tag{4.25c}$$

$$D = \left\{\frac{\frac{-J_\chi(Z_1)}{W[J_\chi(Z_1)Y_\chi(Z_1)]\Omega\tau(Z_1)} + \left\{\frac{Y_\chi(Z_1)}{W[J_\chi(Z_1),Y_\chi(Z_1)]\Omega\tau(Z_1)}\right\}\left\{\frac{J'_\chi(2\Omega)}{Y'_\chi(2\Omega)}\right\}}{\left[\frac{J_\chi(Z_2)}{Y_\chi(Z_2)}\right] - \left[\frac{J'_\chi(2\Omega)}{Y'_\chi(2\Omega)}\right]}\right\}\left[\frac{J_\chi(Z_2)}{Y_\chi(Z_2)}\right] \tag{4.25d}$$

Therefore, our Green's function solution can now be written as

$$g(Z|Z_1) = \begin{cases} AJ_\chi + BY_\chi, 0 \le Z \le Z_1 \\ CJ_\chi + DY_\chi, Z_1 \le Z \le Z_2 \end{cases}$$

Or, more compactly written:

$$g(Z|Z_1) = \begin{pmatrix} g_1(Z|Z_1) \\ g_2(Z|Z_1) \end{pmatrix} = \left[\begin{pmatrix} A \\ C \end{pmatrix} J_\chi(2\Omega\tau) + \begin{pmatrix} B \\ D \end{pmatrix} Y_\chi(2\Omega\tau)\right], \begin{pmatrix} Z \le Z_1 \\ Z \ge Z_1 \end{pmatrix} \tag{4.26}$$

4.4 Green's Function Examples

Since we have now found the Green's function solution to our layered medium, it would prove useful to show what these functions look like. In the following diagrams, we show the Green's functions for various cut-off frequency multipliers, f_c. The layer, within which the Green's function was found, is overlain by a semi-infinite medium represented by an exponentially decaying density. The solution for this layer has a natural cut-off frequency which we term Ω_c. Our operating frequency, Ω, is couched in terms of f_c through the relationship $\Omega = f_c\Omega_c$. In the plots below, we show the Green's function results for f_c =2.0, 2.2, and 2.25.

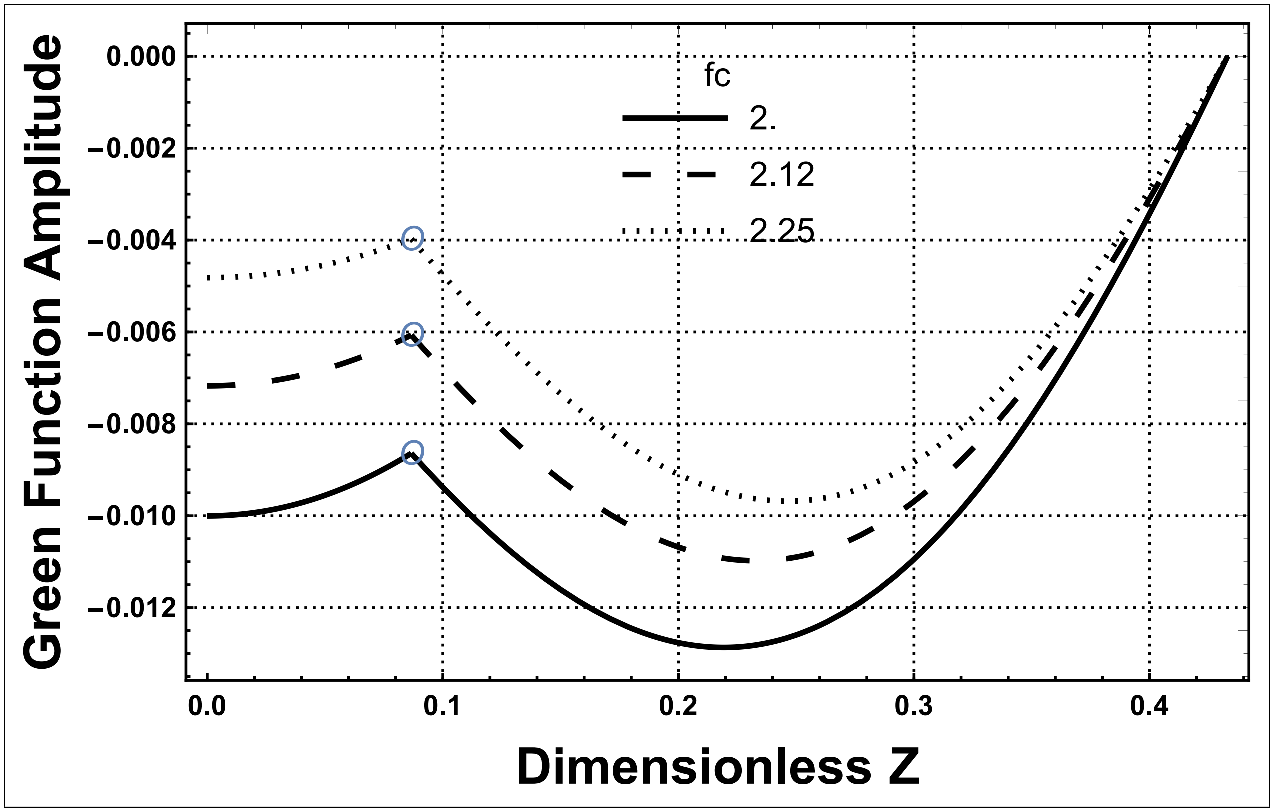

Figure 4.1: Green's Function Solution for f_c = 2.0, 2.2, 2.25

The plots shown above in Figure 4.5 shows the Green's function for f_c = 2.0. The location of the source is circled by an oval with a red line through the center.

Comparably Figure 4.2 shows a similar plot for the Green's function for a value of f_c = 2.2. Please note the scale differences between Figure 4.1 and Figure 4.2. Also note that these functions only extend over the dimensions of the bottom layer. The entire solution for the 2-layer problem will be found by connecting the corresponding pressure functions to the complementary functions.

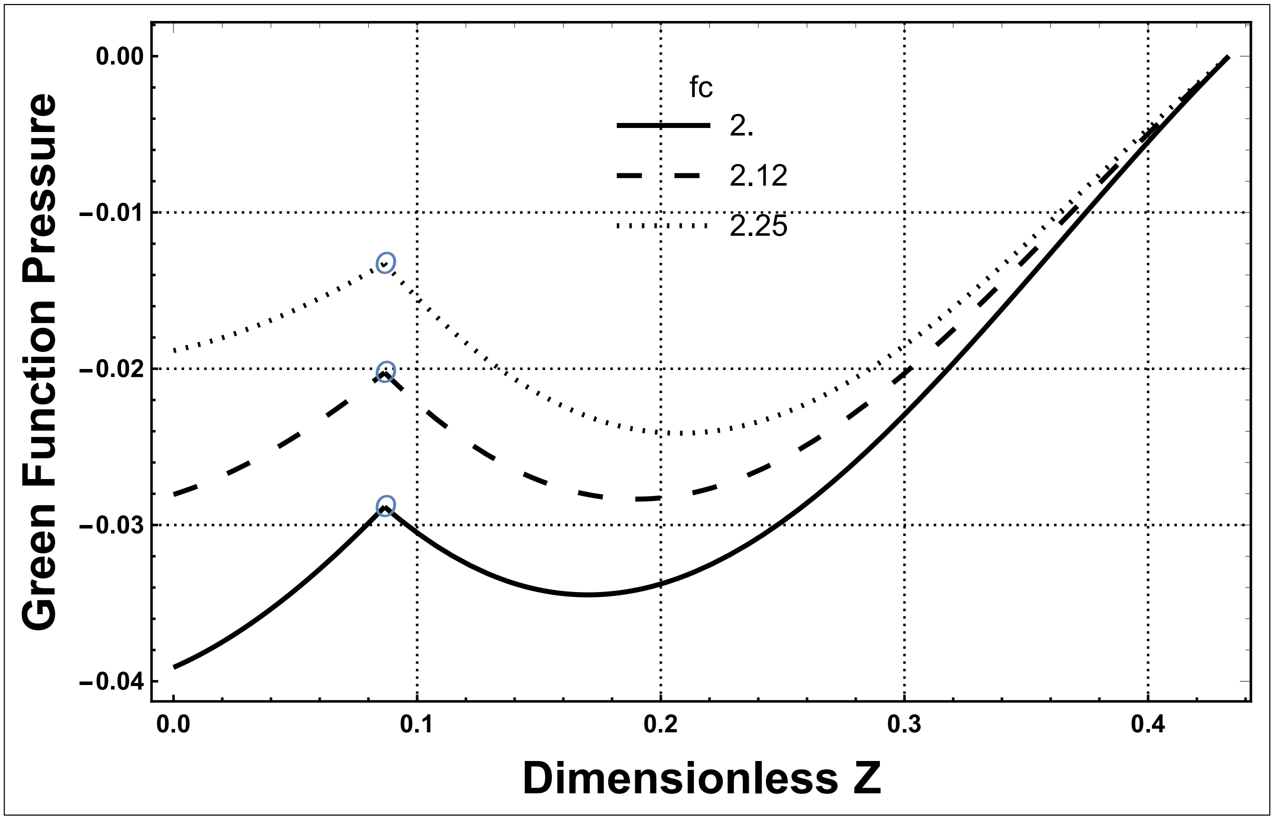

Figure 4.2: Pressure Solution for f_c = 2.0, 2.2, 2.25

4.5 Complementary Solution

As with all differential equations, there is usually a particular solution and a complementary solution. The complementary solution is usually the most frequently encountered equation as it is homogeneous with regards to a source term. Here, we solve for the complementary solution in the finite, bottom layer. This will require again solving a boundary value problem in which 3 constants are to be determined. These we name E, F, and H. Here E and F are the coefficients for the Bessel function of the First and Second Kind which constitute our Bessel solution and H is the constant for the exponentially damped function in the top layer. The 3 conditions which will determine the boundary value constants are:

1. Derivative of the Total Pressure at Z=0 is zero.

2. Continuity of Total Pressure at $Z = Z_2$.

3. Continuity of the Derivative of the Total Pressure at $Z = Z_2$.

As found previously, solutions in the lower medium require Bessel functions of the form:

$$X = EJ_\chi\,(2\Omega\tau) + F\,Y_\chi\,(2\Omega\tau) \tag{4.27}$$

where E and F are the constants we seek. However, while (4.27) is the Bessel function solution, it does not represent the full pressure equation. For that we must divide Equation (4.27) by τ^χ . So the general pressure solution looks like

$$P_C = E\frac{J_\chi(2\Omega\tau)}{\tau^\chi} + F\frac{Y_\chi(2\Omega\tau)}{\tau^\chi} \tag{4.28}$$

where P_C represents the pressure corresponding to the complementary solution.

Following from Equation (4.28), we have

$$\frac{\partial P_C}{\partial Z} = \frac{\partial\tau^{-\chi}}{\partial Z}\left[EJ_\chi\,(2\Omega\tau) + F\,Y_\chi\,(2\Omega\tau)\right] + \tau^\chi\frac{\partial}{\partial Z}\left[EJ_\chi\,(2\Omega\tau) + F\,Y_\chi\,(2\Omega\tau)\right] \tag{4.29}$$

In the interest of being clear in the derivation of these constants, we will use the following abbreviations for the pressure function and its derivatives:

$$J_\tau \equiv \frac{J_\chi}{\tau^\chi} \tag{4.30a}$$

$$dJ_\tau \equiv \frac{\partial}{dZ}\left(\frac{J_\chi}{\tau^\chi}\right) = J_\chi\frac{\partial}{dZ}\left(\frac{1}{\tau^\chi}\right) + \frac{1}{\tau^\chi}\frac{\partial J_\chi}{dZ} \tag{4.30b}$$

$$Y_\tau \equiv \frac{Y_\chi}{\tau^\chi} \tag{4.30c}$$

$$dY_\tau \equiv \frac{\partial}{dZ}\left(\frac{Y_\chi}{\tau^\chi}\right) = Y_\chi\frac{\partial}{dZ}\left(\frac{1}{\tau^\chi}\right) + \frac{1}{\tau^\chi}\frac{\partial Y_\chi}{dZ} \tag{4.30d}$$

Using this terminology and writing Equations (4.28) and (4.29) in terms of Equations (4.30a)–(4.30d), we have:

$$P_C = EJ_\tau + F\,Y_\tau \tag{4.31}$$

and

$$\frac{\partial}{\partial Z}(P_C) = EdJ_\tau + FdY_\tau \tag{4.32}$$

where for convenient reference, the reader is reminded of the definitions below:

$$\frac{d}{dZ}(\frac{1}{\tau^\chi}) = \frac{\chi}{2}\frac{1}{\tau^{\chi+2}} \tag{4.33a}$$

$$\frac{d}{dZ}[J_\chi(2\Omega\tau)] = -\frac{\Omega}{\tau}J'_\chi(2\Omega\tau) \tag{4.33b}$$

$$\frac{d}{dZ}[Y_\chi(2\Omega\tau)] = -\frac{\Omega}{\tau}Y'_\chi(2\Omega\tau) \tag{4.33c}$$

Consequently, (4.30b) and (4.30d) can be written using the above equations as

$$dJ_\tau = \frac{\chi}{2}\frac{1}{\tau^{\chi+2}}J_\chi(2\Omega\tau) - \frac{\Omega}{\tau^{\chi+1}}J'_\chi(2\Omega\tau) \tag{4.34a}$$

$$dY_\tau = \frac{\chi}{2}\frac{1}{\tau^{\chi+2}}Y_\chi(2\Omega\tau) - \frac{\Omega}{\tau^{\chi+1}}Y'_\chi(2\Omega\tau) \tag{4.34b}$$

It should be noted that the complementary solution cannot be found on its own merit from the conditions listed previously. In particular, the 2nd condition is not complete. That is, one does not have a value for the pressure at Z_2 for the complementary equation. To remedy this shortcoming, one must consider the total pressure when considering conditions 2 and 3 above. The total pressure consists of the particular solution plus the complementary function. In our case, the particular solution is the Green's function found previously. The Green's functions, or more appropriately the pressure function formed from the Green's functions, completes the conditions that are needed to find the complementary solution. To write this more clearly, we write the total pressure as

$$P_T = P_G + P_C \tag{4.35}$$

where P_t is the total pressure, P_G is the pressure derived from the Green's Function, and P_c

is the complementary pressure function as shown in (4.31).

We have already found the Green's function G, however, we will need the pressure Green's function, P_G, and the derivative of the pressure Green's function, $\frac{\partial}{\partial Z}(P_G)$. These can be derived as follows:

$$P_G = \frac{G}{\tau^{\chi}} \tag{4.36}$$

Applying the first boundary condition above to (4.32), one can write

$$\begin{aligned} \left\{\frac{\partial}{\partial Z}(P_t)\right\}\Big|_{Z=0} &= \left\{\frac{\partial}{\partial Z}(P_G) + \frac{\partial}{\partial Z}(P_c)\right\}\Big|_{Z=0} \\ &= \left\{\frac{\partial}{\partial Z}(\frac{g}{\tau^{\chi}})\right\}\Big|_{Z=0} + \{E dJ_\tau(2\Omega\tau) + F dY_\tau(2\Omega\tau)\}|_{Z=0} = 0 \end{aligned} \tag{4.37}$$

In order to proceed, we need to find $\frac{\partial}{\partial Z}(\frac{G}{\tau^{\chi}})$. Thus,

$$\frac{\partial}{\partial Z}(\frac{G}{\tau^{\chi}}) = \tau^{-\chi}\frac{\partial G}{\partial Z} + G\frac{\partial}{\partial Z}(\tau^{-\chi}). \tag{4.38}$$

For future use, we need to evaluate (4.38) at $Z = 0$ and for $Z = Z_2$. For $Z = 0$, we have

$$\left\{\frac{\partial}{\partial Z}(\frac{G}{\tau^{\chi}})\right\}\Big|_{Z=0} = \left\{\tau^{-\chi}\cancelto{0}{\frac{\partial G}{\partial Z}}\right\}\Bigg|_{Z=0} + \left\{G\frac{\partial}{\partial Z}(\tau^{-\chi})\right\}\Big|_{Z=0} \tag{4.39}$$

or, with the aid of (4.33a) and (4.26), we have

$$\left\{\frac{\partial}{\partial Z}(\frac{G_1}{\tau^{\chi}})\right\}\Big|_{Z=0} = \left\{\frac{\chi}{2}\frac{G_1}{\tau^{\chi+2}}\right\}\Big|_{Z=0} \tag{4.40}$$

Similarly, for $Z = Z_2$ and again using (4.26), we have

$$\left\{\frac{\partial}{\partial Z}(\frac{G_2}{\tau^{\chi}})\right\}\Big|_{Z=Z_2} = \left\{\tau^{-\chi}\frac{\partial G_2}{\partial Z}\right\}\Big|_{Z=Z_2} + \left\{\frac{\partial}{\partial Z}(\tau^{-\chi})\cancelto{0}{G_2}\right\}\Big|_{Z=Z_2}. \tag{4.41}$$

Thus, we now have for $Z = Z_2$

$$\frac{\partial}{\partial Z}(\frac{G}{\tau^{\chi}}) = \left\{\tau^{-\chi}\frac{\partial G_2}{\partial Z}\right\}\bigg|_{Z=Z_2}. \tag{4.42}$$

Using (4.26), (4.30b), (4.30d), (4.34a),(4.34b) , (4.42) may be written as

$$\left\{\tau^{-\chi}\frac{\partial G_2}{\partial Z}\right\}\bigg|_{Z=Z_2} = C\left\{-\frac{\Omega}{\tau^{\chi+1}}J'_{\chi}(2\Omega\tau)\right\}\bigg|_{Z=Z_2} + D\left\{-\frac{\Omega}{\tau^{\chi+1}}Y'_{\chi}(2\Omega\tau)\right\}\bigg|_{Z=Z_2}. \tag{4.43}$$

Now that we have obtained expressions for the pressure Green's functions at the boundaries, we can form the boundary value equations in order to find the unknown constants E, F, and H. Starting with the lower boundary condition at Z=0 and using (4.40) in (4.37), we have

$$\left\{\frac{\partial}{\partial Z}(P_T)\right\}\bigg|_{Z=0} = \left\{\frac{\chi}{2}\frac{G}{\tau^{\chi+2}}\right\}\bigg|_{Z=0} + \{E\,dJ_{\tau}(2\Omega\tau) + F\,dY_{\tau}(2\Omega\tau)\}|_{Z=0} = 0. \tag{4.44}$$

Next we substitute G_1 from (4.26) for the Green's function G in (4.44) and obtain

$$\left\{\frac{\partial}{\partial Z}(P_T)\right\}\bigg|_{Z=0} = \left\{\frac{\chi}{2}\frac{AJ_{\chi}(2\Omega\tau) + BY_{\chi}(2\Omega\tau)}{\tau^{\chi+2}}\right\}\bigg|_{Z=0} + \{E\,dJ_{\tau} + F\,dY_{\tau}\}|_{Z=0} = 0. \tag{4.45}$$

Expanding the other terms using (4.34), we have

$$\begin{aligned}\left\{\frac{\partial}{\partial Z}(P_T)\right\}\bigg|_{Z=0} &= \left\{\frac{\chi}{2}\frac{AJ_{\chi}(2\Omega\tau) + BY_{\chi}(2\Omega\tau)}{\tau^{\chi+2}}\right\}\bigg|_{Z=0} \\ &\quad + E\left\{\frac{\chi}{2}\frac{1}{\tau^{\chi+2}}J_{\chi}(2\Omega\tau) - \frac{\Omega}{\tau^{\chi+1}}J'_{\chi}(2\Omega\tau)\right\}\bigg|_{Z=0} \\ &\quad + F\left\{\frac{\chi}{2}\frac{1}{\tau^{\chi+2}}Y_{\chi}(2\Omega\tau) - \frac{\Omega}{\tau^{\chi+1}}Y'_{\chi}(2\Omega\tau)\right\}\bigg|_{Z=0} = 0.\end{aligned} \tag{4.46}$$

At $Z=0$, $\tau=1$. Therefore, (4.57) reduces to

$$\left\{\frac{\partial}{\partial Z}(P_T)\right\}\Bigg|_{Z=0} = \frac{\chi}{2}\left\{AJ_\chi(2\Omega)+BY_\chi(2\Omega)\right\} + E\left\{\frac{\chi}{2}J_\chi(2\Omega)-\Omega J'_\chi(2\Omega)\right\} + F\left\{\frac{\chi}{2}Y_\chi(2\Omega)-\Omega Y'_\chi(2\Omega)\right\}=0. \tag{4.47}$$

For easier manipulation, we can rename the various terms above as

$$a_1 = \frac{\chi}{2}\left\{AJ_\chi(2\Omega)+BY_\chi(2\Omega)\right\} \tag{4.48a}$$

$$e_1 = \left\{\frac{\chi}{2}J_\chi(2\Omega)-\Omega J'_\chi(2\Omega)\right\} \tag{4.48b}$$

$$f_1 = \left\{\frac{\chi}{2}Y_\chi(2\Omega)-\Omega Y'_\chi(2\Omega)\right\} \tag{4.48c}$$

and solve for E as

$$E = -\frac{\{f_1F+a_1\}}{e_1} \tag{4.49}$$

which in longer form is

$$E = -\frac{F\left\{\frac{\chi}{2}Y_\chi(2\Omega)-\Omega Y'_\chi(2\Omega)\right\}+\frac{\chi}{2}\left\{AJ_\chi(2\Omega)+BY_\chi(2\Omega)\right\}}{\left\{\frac{\chi}{2}J_\chi(2\Omega)-\Omega J'_\chi(2\Omega)\right\}}. \tag{4.50}$$

This equation relates the coefficients E and F.

Next, we apply the 2nd condition above for the total pressure at $Z=Z_2$. This can be expressed generically as

$$P_T\big|_{Z=Z_2} = \cancelto{0}{P_G}\Big|_{Z=Z_2} + P_C\big|_{Z=Z_2} \tag{4.51}$$

Here we see that the pressure term from the Green's function is 0 as this was the other homogeneous condition that was used in finding the Green's function coefficients. Hence, we are left with the following equation for the continuity of pressure at Z_2:

$$EJ_\tau|_{Z=Z_2} + F\ dY_\tau|_{Z=Z_2} = H \tag{4.52}$$

where the equation in the top layer at $Z = Z_2$ (Appendix D, (C.4)) evaluates to H. Expanding (4.52), we find

$$E\frac{J_\chi(2\Omega\tau_2)}{\tau_2^\chi} + F\frac{Y_\chi(2\Omega\tau_2)}{\tau_2^\chi} = H \tag{4.53}$$

Again, renaming the terms in (4.53) as

$$e_2 = \frac{J_\chi(2\Omega\tau_2)}{\tau_2^\chi} \tag{4.54a}$$

$$f_2 = \frac{Y_\chi(2\Omega\tau_2)}{\tau_2^\chi} \tag{4.54b}$$

This allows us to express (4.53) in simple form as

$$e_2E + f_2F = H \tag{4.55}$$

Finally, from the 3rd condition above (which expresses the continuity of the derivative of the total pressure), we have

$$\left.\frac{\partial P_T}{\partial Z}\right|_{Z=Z_2} = \left.\frac{\partial P_G}{\partial Z}\right|_{Z=Z_2} + \left.\frac{\partial P_C}{\partial Z}\right|_{Z=Z_2} = -\frac{\alpha}{2}H \tag{4.56}$$

where again we evaluated the expression for the top layer at $Z = Z_2$.

Substituting for the various derivatives in (4.56), we have

$$
\begin{aligned}
\left\{\frac{\partial}{\partial Z}(P_T)\right\}\bigg|_{Z=Z_2} &= C\left\{-\frac{\Omega}{\tau^{\chi+1}} J'_{\chi}(2\Omega\tau)\right\}\bigg|_{Z=Z_2} + D\left\{-\frac{\Omega}{\tau^{\chi+1}} Y'_{\chi}(2\Omega\tau)\right\}\bigg|_{Z=Z_2} \\
&+ E\left\{\frac{\chi}{2}\frac{1}{\tau^{\chi+2}} J_{\chi}(2\Omega\tau) - \frac{\Omega}{\tau^{\chi+1}} J'_{\chi}(2\Omega\tau)\right\}\bigg|_{Z=Z_2} \\
&+ F\left\{\frac{\chi}{2}\frac{1}{\tau^{\chi+2}} Y_{\chi}(2\Omega\tau) - \frac{\Omega}{\tau^{\chi+1}} Y'_{\chi}(2\Omega\tau)\right\}\bigg|_{Z=Z_2} = -\frac{\alpha}{2}H.
\end{aligned} \tag{4.57}
$$

or, evaluating the expression at $Z = Z_2$, one gets

$$
\begin{aligned}
\left\{\frac{\partial}{\partial Z}(P_T)\right\}\bigg|_{Z=Z_2} &= C\left\{-\frac{\Omega}{\tau_2^{\chi+1}} J'_{\chi}(2\Omega\tau_2)\right\} + D\left\{-\frac{\Omega}{\tau_2^{\chi+1}} Y'_{\chi}(2\Omega\tau_2)\right\} \\
&+ E\left\{\frac{\chi}{2}\frac{1}{\tau_2^{\chi+2}} J_{\chi}(2\Omega\tau_2) - \frac{\Omega}{\tau_2^{\chi+1}} J'_{\chi}(2\Omega\tau_2)\right\} \\
&+ F\left\{\frac{\chi}{2}\frac{1}{\tau_2^{\chi+2}} Y_{\chi}(2\Omega\tau_2) - \frac{\Omega}{\tau_2^{\chi+1}} Y'_{\chi}(2\Omega\tau_2)\right\} = -\frac{\alpha}{2}H.
\end{aligned} \tag{4.58}
$$

Again, renaming the terms in (4.58), we have

$$
a_3 = C\left\{-\frac{\Omega}{\tau_2^{\chi+1}} J'_{\chi}(2\Omega\tau_2)\right\} + D\left\{-\frac{\Omega}{\tau_2^{\chi+1}} Y'_{\chi}(2\Omega\tau_2)\right\} \tag{4.59a}
$$

$$
e_3 = \left\{\frac{\chi}{2}\frac{1}{\tau_2^{\chi+2}} J_{\chi}(2\Omega\tau_2) - \frac{\Omega}{\tau_2^{\chi+1}} J'_{\chi}(2\Omega\tau_2)\right\} \tag{4.59b}
$$

$$
f_3 = \left\{\frac{\chi}{2}\frac{1}{\tau_2^{\chi+2}} Y_{\chi}(2\Omega\tau_2) - \frac{\Omega}{\tau_2^{\chi+1}} Y'_{\chi}(2\Omega\tau_2)\right\} \tag{4.59c}
$$

allowing one to rewrite (4.58) as

$$
a_3 + e_3E + f_3F = -\frac{\alpha}{2}H \tag{4.60}
$$

Multiplying (4.55) by $\frac{\alpha}{2}$ and adding it to (4.60), we have

$$
\frac{\alpha}{2}\left\{e_2E + f_2F\right\} + a_3 + e_3E + f_3F = 0 \tag{4.61}
$$

Collecting common terms of E and F, one can write

$$E\left\{e_2\frac{\alpha}{2}+e_3\right\}+F\left\{f_2\frac{\alpha}{2}+f_3\right\}+a_3=0 \tag{4.62}$$

Using E from (4.49) in (4.62), we have

$$-\frac{\{f_1F+a_1\}}{e_1}\left\{e_2\frac{\alpha}{2}+e_3\right\}+F\left\{f_2\frac{\alpha}{2}+f_3\right\}+a_3=0 \tag{4.63}$$

One can now find F as

$$F=\frac{\frac{a_1}{e_1}\left\{e_2\frac{\alpha}{2}+e_3\right\}-a_3}{\left\{-\frac{f_1}{e_1}\left\{e_2\frac{\alpha}{2}+e_3\right\}+\left\{f_2\frac{\alpha}{2}+f_3\right\}\right\}} \tag{4.64}$$

Additionally, F may be substituted in (4.49) to find E and, then, use (4.55) to find H. Thus, all constants are found.

Now that E and F are found, H is easily found from (4.52) which is repeated here for convenience:

$$H=EJ_\tau|_{Z=Z_2}+F\ dY_\tau|_{Z=Z_2}. \tag{4.65}$$

All of the constants are now known for P_G and P_c, hence, the total pressure function, P_t can be calculated and studied. Below in Figure 4.3 is shown an example of the complementary functions for $f_c=2.0$, 2.2, and 2.25.

4.6 Development of General Fast Field Program

In order to validate the outcome of our computations for a Green's function model, it is necessary to establish some form of verification. To this end, we use the model output from the results of a general Fast Field Program (FFP) calculation for comparison. The FFP is a well established model [4] and is particularly accurate for vertical propagation (but not wide-angle types of propagation). An interesting outline of this model can be found

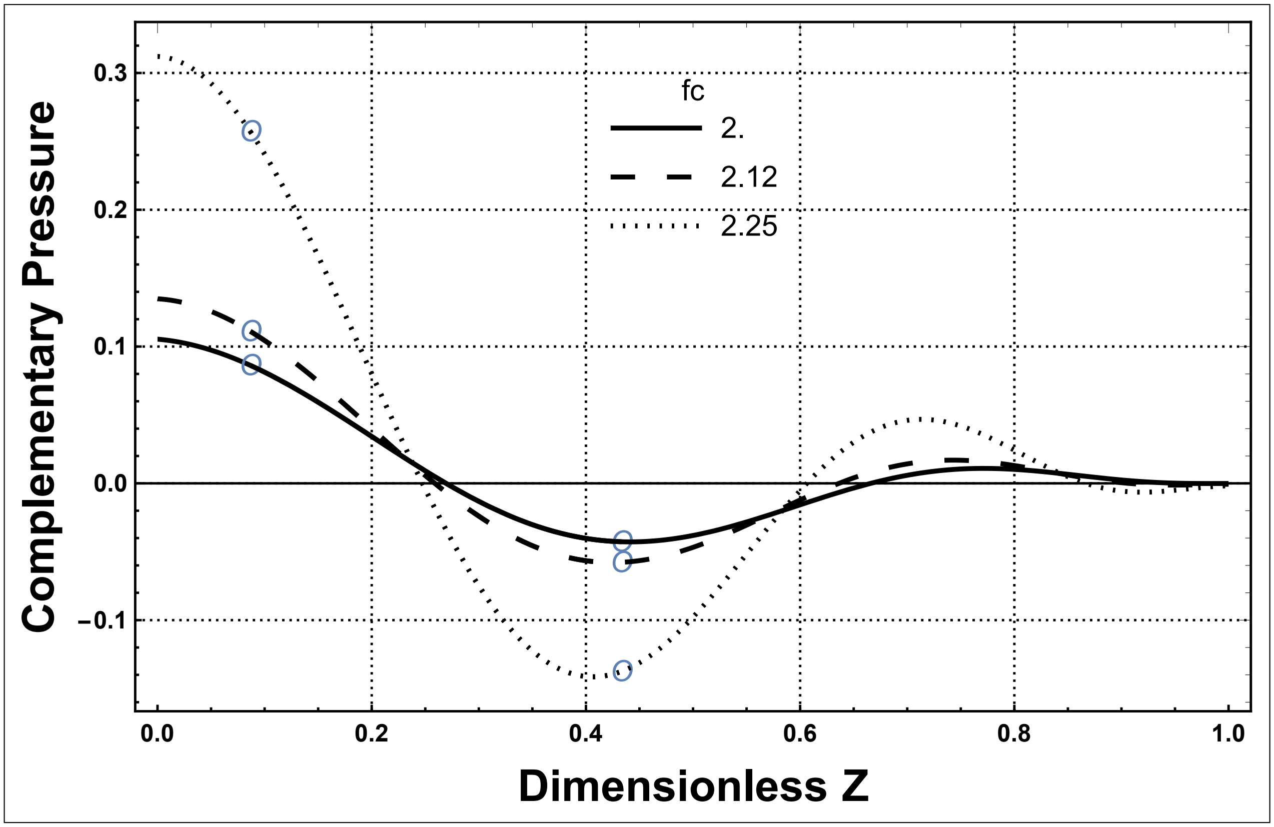

Figure 4.3: Complementary Solution for f_c=2.0, 2.2, 2.25

in *Computational Atmospheric Acoustics*[14]. Since it is well established, it can offer some verification of the results. A brief derivation of the model is given below.

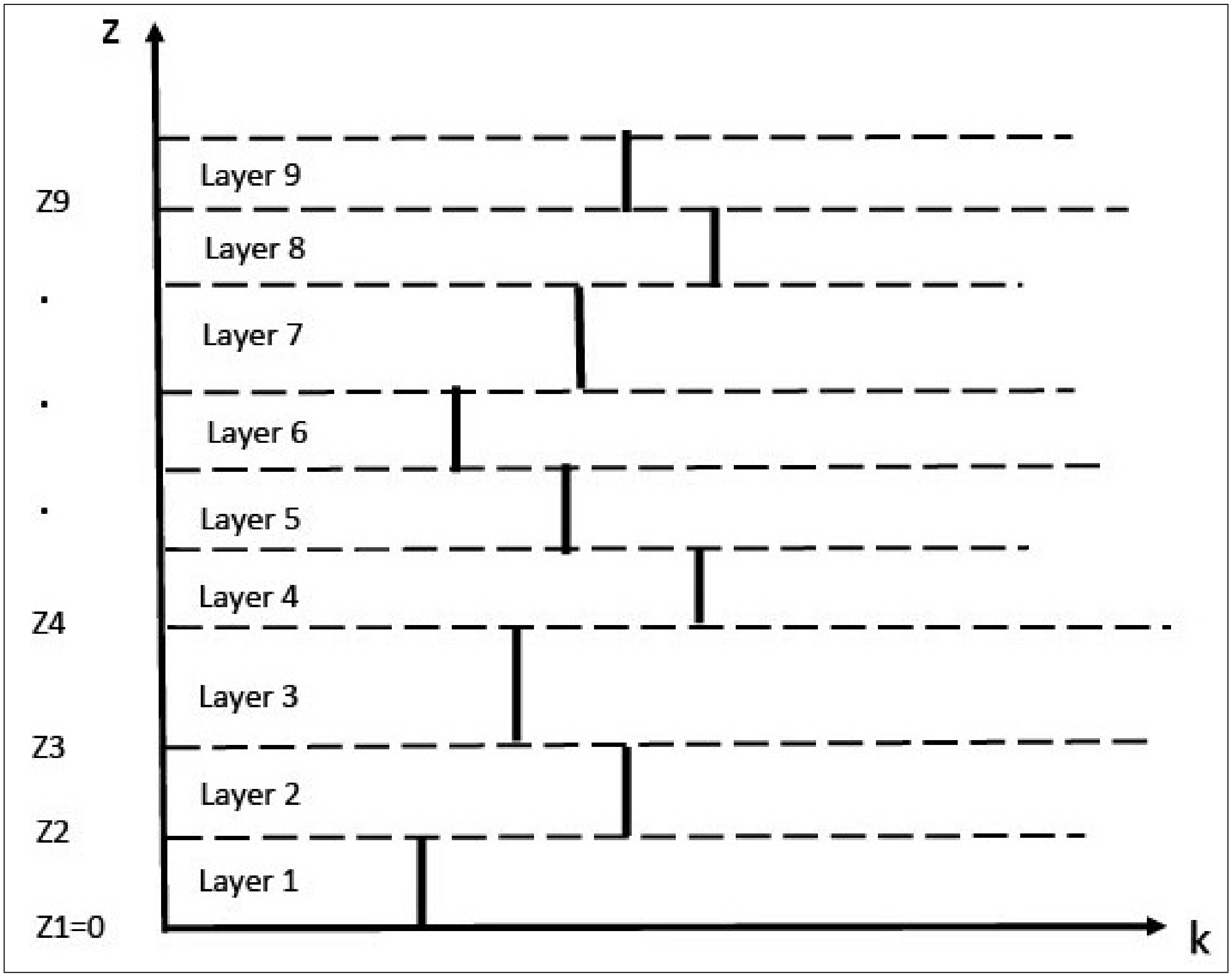

Figure 4.4: Layered Medium for FFP

Figure 4.4 depicts how a medium is divided into various layers for the FFP. In each of the layers shown, the layers are small enough to assume that the wave number, k, within the layer is a constant. For a layer with constant k, one may use the Helmhotz equation to represent a plane wave travelling within the medium. Hence, the idea of the FFP is to approximate a traveling wave within a medium by plane waves propagating within finite layers such that each layer has a constant sound speed. This reduces the problem to solving a 1-D wave equation within a layer for a fixed, angular frequency ω and a source represented by $-S_\delta \delta(z - z_s)$. In other words, one solves the wave equation

$$\nabla^2 P_j(z_j, t) - \frac{1}{c_j^2} \frac{\partial^2}{\partial t^2} P_j(z_j, t) = -S_\delta \delta(z - z_s) e^{-i\omega t} \tag{4.66}$$

where j is the index of the j^{th} interface/layer. Additionally, z_s is the source location. For the homogeneous wave equation part of (4.66), one assumes a solution

$$P_j(z,t) = P_j(z)e^{-i\omega t} \tag{4.67}$$

which leads to the Helmholtz equation (after canceling the time-harmonic part)

$$P_j(z_j) = \nabla^2 P_j(z_j) + k_j^2 P_j(z_j) = 0 \tag{4.68}$$

where $k_j = \frac{\omega}{c_j}$

This admits of 2 solutions: an upward-going wave and a downward-going wave (AB and B coefficients, respectively below (4.69)):

$$P_j(z_j) = A_j e^{ik_j z_j} + B_j e^{-ik_j z_j}. \tag{4.69}$$

Note we have assumed that interface/layer j is not located at $z = z_s$, the source layer. The issue of how to handle the source location, z_s, will be discussed later in this work.

Given an initial set of coefficients at the bottom and top of the layered medium, one could then solve for the boundary conditions (A_j and B_j) at the j^{th} interface as one progresses from the bottom to the source and from the top to the source. This would require matching the boundary conditions at each interface/layer. These conditions consist of the continuity of pressure and normal velocity across an interface which does not contain the source. For the source layer, the condition is the continuity of pressure across the interface and discontinuity of normal velocity across the interface is $-S_\delta$. Since the normal velocity is required for the boundary condition matching, this is retrieved from [18] (Eq. 3b) and [14] (Eq. E.21) where gravity is neglected in each case. This leads to the equation for velocity as

$$W = -i\omega^{-1}\varrho_{av}^{-1}P' \tag{4.70}$$

where W is the velocity in the z-direction. Now, one can frame the above boundary conditions as follows:

$$P_j(z_j) = P_{j-1}(z_j), j = 1, 2, ...N \tag{4.71a}$$

$$-i\omega_j^{-1}\varrho_j^{-1}\frac{\partial P_j(z_j)}{\partial z} = -i\omega_{j-1}^{-1}\varrho_{j-1}^{-1}\frac{\partial P_{j-1}(z_{j-1})}{\partial z}, j = 2, ...N \tag{4.71b}$$

$$\frac{\partial P_j(z_j)}{\partial z} = \frac{\partial P_{j-1}(z_{j-1})}{\partial z} - S_\delta, j = s \tag{4.71c}$$

$$\varrho_1^{-1}\frac{\partial P_1(z_1)}{\partial z} = \varrho_0^{-1}\frac{\partial P_0(z_0)}{\partial z}, j = 1 \tag{4.71d}$$

One could follow the above process and solve for constants A_j and B_j at each interface and, then, find the pressure, P_j and its corresponding derivative. However, there is a simpler way which leads to a recursive relation between the pressure and its derivative. We show this below.

Consider the equation for the pressure in the j^{th} layer

$$P_j(z_j) = A_j e^{ik_j z_j} + B_j e^{-ik_j z_j}. \tag{4.72}$$

Its derivative is

$$P_j^{'}(z_j) = ik_j A_j e^{ik_j z_j} - ik_j B_j e^{-k_j z_j}. \tag{4.73}$$

Now, if we want to find the pressure at $P_j(z_j + \triangle z_j)$, we can write

$$P_j(z_j + \triangle z_j) = A_j e^{ik_j(z_j+\triangle z_j)} + B_j e^{-ik_j(z_j+\triangle z_j)}. \tag{4.74}$$

Using Euler's identity, one can expand (4.74) in terms of sine and cosine functions

$$\begin{aligned} P_j(z_j + \triangle z_j) = A_j\left[cos\left\{k_j(z_j + \triangle z_j)\right\} + isin\left\{k_j(z_j + \triangle z_j\right\}\right] \\ +B_j\left[cos\left\{k_j(z_j + \triangle z_j)\right\} + isin\left\{k_j(z_j + \triangle z_j\right\}\right]. \end{aligned} \tag{4.75}$$

Using trignometric identities for the sine and cosine of the sum of two arguments, (4.75) can be expanded further as

$$
\begin{aligned}
P_j(z_j+\triangle z_j) = A_j\left\{\{cos(k_j z_j)cos(k_j\triangle z_j) - sin(k_j z_j)sin(k_j\triangle z_j)\}\right\}\\
+A_j\left\{i\{sin(k_j z_j)cos(k_j\triangle z_j) + cos(k_j z_j)sin(k_j\triangle z_j)\}\right\}\\
+B_j\left\{cos(k_j z_j)cos(k_j\triangle z_j) - sin(k_j z_j)sin(k_j\triangle z_j)\right\}\\
-B_j\left\{i\{sin(k_j z_j)cos(k_j\triangle z_j) + cos(k_j z_j)sin(k_j\triangle z_j)\}\right\}.
\end{aligned}
\tag{4.76}
$$

Collecting various sine and cosine terms in the anticipation of forming different exponential functions, we have

$$
\begin{aligned}
P_j(z_j+\triangle z_j) = A_j\left[\{cos(k_j z_j) + isin(k_j z_j)\}\, cos(k_j\triangle z_j)\right]\\
+A_j\left[i\{cos(k_j z_j) + isin(k_j z_j)\}\, sin(k_j\triangle z_j)\right]\\
+B_j\left[\{cos(k_j z_j) - isin(k_j z_j)\}\, cos(k_j\triangle z_j)\right]\\
-B_j\left[i\{cos(k_j z_j) - isin(k_j z_j)\}\, sin(k_j\triangle z_j)\right].
\end{aligned}
\tag{4.77}
$$

Recombining the terms in braces () into exponential functions, we obtain

$$
\begin{aligned}
P_j(z_j+\triangle z_j) = A_j\left[\left\{e^{(ik_j z_j)}\right\} cos(k_j\triangle z_j) + i\left\{e^{(ik_j z_j)}\right\} sin(k_j\triangle z_j)\right]\\
+B_j\left[\left\{e^{-(ik_j z_j)}\right\} cos(k_j\triangle z_j) - i\left\{e^{-(ik_j z_j)}\right\} sin(k_j\triangle z_j)\right].
\end{aligned}
\tag{4.78}
$$

Now, combining sine and cosine terms, one has

$$
\begin{aligned}
P_j(z_j+\triangle z_j) = \left[A_j\left\{e^{(ik_j z_j)}\right\} cos(k_j\triangle z_j) + B_j\left\{e^{-(ik_j z_j)}\right\} cos(k_j\triangle z_j)\right]\\
+\left[iA_j\left\{e^{(ik_j z_j)}\right\} sin(k_j\triangle z_j) + iB_j\left\{e^{-(ik_j z_j)}\right\} sin(k_j\triangle z_j)\right].
\end{aligned}
\tag{4.79}
$$

This can now be factored and reduced to

$$P_j(z_j+\triangle z_j)=\left[A_j e^{(ik_j z_j)}+B_j e^{-(ik_j z_j)}\right]cos(k_j\triangle z_j)$$
$$+\frac{ik_j}{k_j}\left[A_j e^{(ik_j z_j)}-B_j e^{-(ik_j z_j)}\right]sin(k_j\triangle z_j). \quad (4.80)$$

Identifying the first set of square brackets in (4.80) as P_j from (4.72), and the second set of square brackets as $P_j^{'}$ from (4.73), we have

$$P_j(z_j+\triangle z_j)=P_j(z_j)cos(k_j\triangle z_j)+P_j^{'}\frac{1}{k_j}sin(k_j\triangle z_j). \quad (4.81)$$

A similar derivation can be performed for $P_j^{'}(z_j+\triangle z_j)$. Here, we start with

$$P_j^{'}(z_j)=ik_jA_je^{ik_jz_j}-ik_jB_je^{-k_jz_j}. \quad (4.82)$$

Distributing the exponents of the exponentials as before and writing everything in terms of sines and cosines, one has

$$P_j^{'}(z_j+\triangle z_j)=ik_jA_j\left\{cos(k_jz_j)cos(k_j\triangle z_j)-sin(k_jz_j)sin(k_j\triangle z_j)\right\}$$
$$+ik_jA_j\left[i\left\{sin(k_jz_j)cos(k_j\triangle z_j)+cos(k_jz_j)sin(k_j\triangle z_j)\right\}\right]$$
$$-ik_jB_j\left\{cos(k_jz_j)cos(k_j\triangle z_j)-sin(k_jz_j)sin(k_j\triangle z_j)\right\}$$
$$-ik_jB_j\left[-i\left\{sin(k_jz_j)cos(k_j\triangle z_j)+cos(k_jz_j)sin(k_j\triangle z_j)\right\}\right]. \quad (4.83)$$

Again, forming exponential functions common to the $cos(k_j\triangle z_j)$ and $sin(k_j\triangle z_j)$ terms, we can write

$$P_j^{'}(z_j+\triangle z_j)=ik_jA_j\left[\left\{e^{(ik_jz_j)}\right\}cos(k_j\triangle z_j)+i\left\{e^{(ik_jz_j)}\right\}sin(k_j\triangle z_j)\right]$$
$$-ik_jB_j\left[\left\{e^{-(ik_jz_j)}\right\}cos(k_j\triangle z_j)-i\left\{e^{-(ik_jz_j)}\right\}sin(k_j\triangle z_j)\right]. \quad (4.84)$$

After factoring terms common to the sine and cosine functions, we have

$$P_j^{'}(z_j + \triangle z_j)$$
$$= ik_j \left[A_j e^{(ik_j z_j)} - B_j e^{-(ik_j z_j)}\right] cos(k_j \triangle z_j) \tag{4.85}$$
$$+ \, ik_j \left[iA_j e^{(ik_j z_j)} + iB_j e^{-(ik_j z_j)}\right] sin(k_j \triangle z_j).$$

Multiplying the 2nd bracket ([]) through by i, one can again see that the 1st bracket gives $P_j^{'}(k_j z_j)$ and the 2nd bracket gives $P_j(k_j z_j)$. Hence, the equation for the derivative of P becomes

$$P_j^{'}(z_j + \triangle z_j) = P_j^{'}(z_j) cos(k_j \triangle z_j) - k_j P_j(z_j) sin(k_j \triangle z_j). \tag{4.86}$$

Now that we have obtained a set of recursive equations ((4.81) and (4.86)), there remains the fact that these equations are only correct for the ratio $\frac{P_j^{'}(z_j)}{P_j(z_j)}$. To see this, one needs to examine the boundary conditions under which this model was developed.

As part of this development, reference [14] specifies some very useful boundary conditions on the bottom and top parts of the layer. On the bottom layer, let $B_1 = 1$ in (4.72) and (4.73) and the following boundary value equations are found:

$$P_1(z_1) = R(kz_1) + 1 \tag{4.87}$$

and

$$P_1^{'}(z_1) = ikz_1[R(kz_1) - 1] \tag{4.88}$$

where $R(kz_1)$ is the reflection coefficient at the bottom of layer 1, and kz_1 is the corresponding wave number.

Similarly, another set of boundary conditions is found with the specification that B_{N-1} $= 0$ and $P_{N-1}(z_N)$=1 at the top layer.

Applying these conditions to (4.69), we have

$$P_{N-1}(z_N) = 1 = A_{N-1}e^{ik_{zN}(z_N)}. \tag{4.89}$$

Doing the same for (4.73), we have

$$P'_{N-1}(z_{N-1}) = ikz_N A_{N-1}e^{ik_{zN}z_N}. \tag{4.90}$$

Note that by setting $P_{N-1}(z_N) = 1$ allows one to determine $A_{N-1}e^{ik_{zN}(z_N)} = 1$. This in turn allows $P'_{N-1}(z_{N-1})$ to be found as ikz_N. Or, stated another way, without $A_{N-1}e^{ik_{zN}z_N}$ being known, the ratio $\frac{P'_{N-1}(z_{N-1})}{P_{N-1}(z_N)}$ is correct. That is,

$$\frac{P'_{N-1}(z_{N-1})}{P_{N-1}(z_N)} = \frac{ikz_N A_{N-1}e^{ik_{zN}z_N}}{A_{N-1}e^{ik_{zN}z_N}} = \frac{ikz_N}{1}. \tag{4.91}$$

.

Consequently, all the ratios along the layer interfaces as calculated are correct. And, (4.71c) gives one a manner to arrive at the correct P and P'. Namely, by starting at first layer and proceeding upwards to the source as well as starting at the top layer and working down to the source, we find that (4.71c) must be satisfied at the source.[14] This leads to the relationship

$$\left(\frac{P'_{nu}}{P_{nu}}\right) P_s - \left(\frac{P'_{nl}}{P_{nl}}\right) P_s = -S_\delta, \tag{4.92}$$

where n represents the layer and l and u refer to whether one is in the bottom layer (l) moving upwards, or in the top layer (u) moving downwards. From (4.92), the source value P_s may be determined as

$$P_s = \frac{-S_\delta}{\left(\frac{P'_{nu}}{P_{nu}}\right) - \left(\frac{P'_{nl}}{P_{nl}}\right)}. \tag{4.93}$$

Now that the source value P_s is found, one can scale all the values P_n by multiplying as below:

$$\begin{aligned} &\frac{P_s}{P_{nu}}, \text{ for } z_j > z_s, \\ &\frac{P_s}{P_{nl}}, \text{ for } z_j < z_s. \end{aligned} \tag{4.94}$$

This allows one to find the proper pressure values in all the layers. This completes the development of the FFP model. We now turn to its use in our model comparisons.

4.7 Application of FFP to GF Solutions

4.7.1 Example Results for Green's Functions

Since we are concerned with the pressure field within a finite layer due to a source, it is reasonable to compare the results for the GF and FFP models.. To this end, several plots show the results of applying the two models for the same conditions by overplotting one on the other. Since the GF model is calculated for a δ-function source term, the 2 curves (GF and FFP) are self-normalized to a value of 1 at their source location. They are then over-plotted on each other, such that their source values match at the source location. The outcome is that we are comparing relative pressures in order to achieve an accurate comparison. Below we show 3 examples of these comparisons.

In the figures, the domain of the abscissa is expressed in dimensionless coordinates such that the function extends only to the top of the finite layer as required. A small circle on all of the diagrams indicates the source location. And, the solutions are discontinuous at the source as dictated by the "step" discontinuity for the boundary conditions.

Figure 4.5 shows the comparison for $f_c = 2.0$ with a source location at $Z = 0.265$. There is very good agreement between the FFP and GF from the top of the layer ($Z = 0.43$) until approximately Z=0.1. An explanation for this is that exact boundary condition matches between the 2 methods is not always easy to achieve. One issue that arises is that the FFP formulation is framed in complex notation which has boundary conditions on the derivative of the pressure being imaginary. The GF method yields a completely real result.

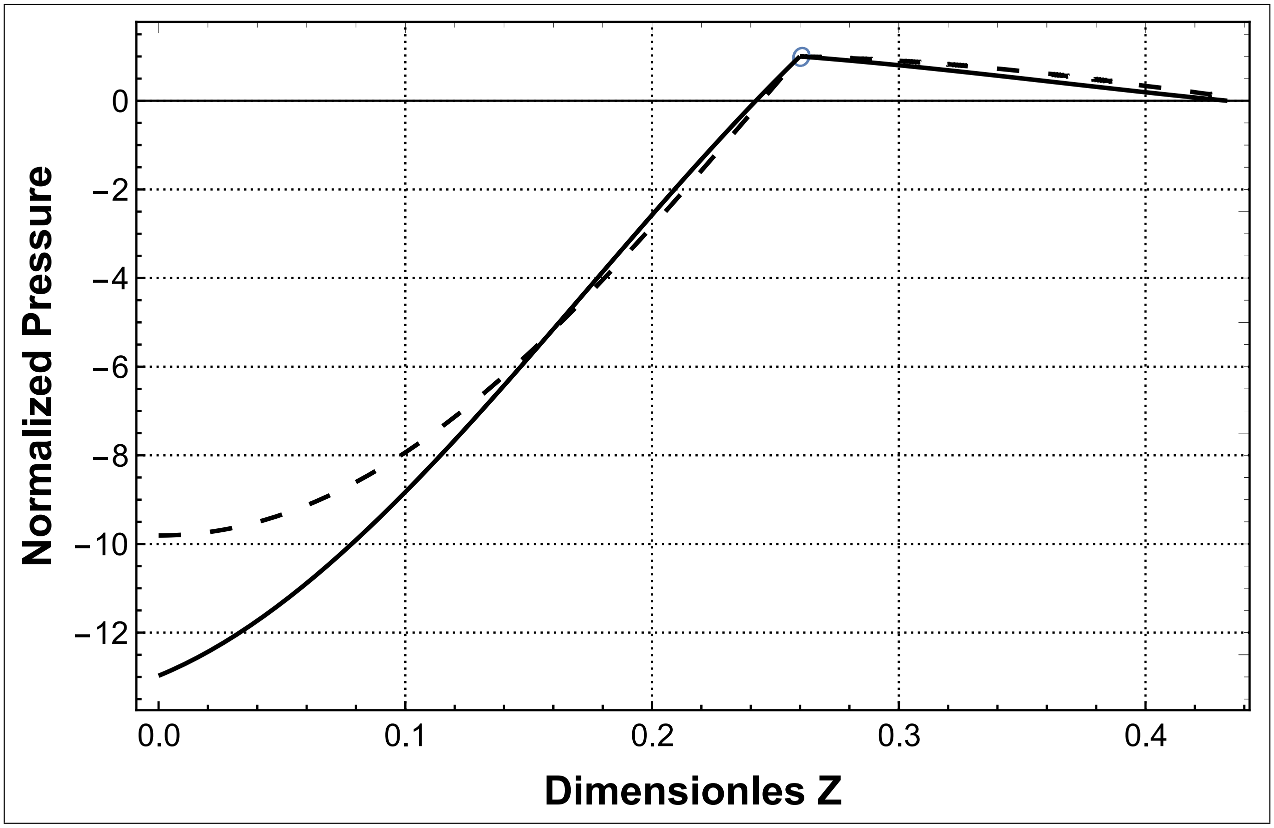

Figure 4.5: Comparison of GF (solid line) and FFP (dash line) Solutions for f_c = 2.0, Z_1=0.265

In figure 4.6, there is another example showing the results between the GF and FFP methods. This comparison corresponds to a value of $f_c = 2.2$ and a source location of $Z = 0.086$. Again, there seems to be reasonable agreement between the 2 curves except at the lower locations, $Z < 0.05$. As before, the discontinuity at the source is observed as well as the GF becoming zero (0) at the top of the layer and the FFP slope at Z=0 as required by the boundary conditions.

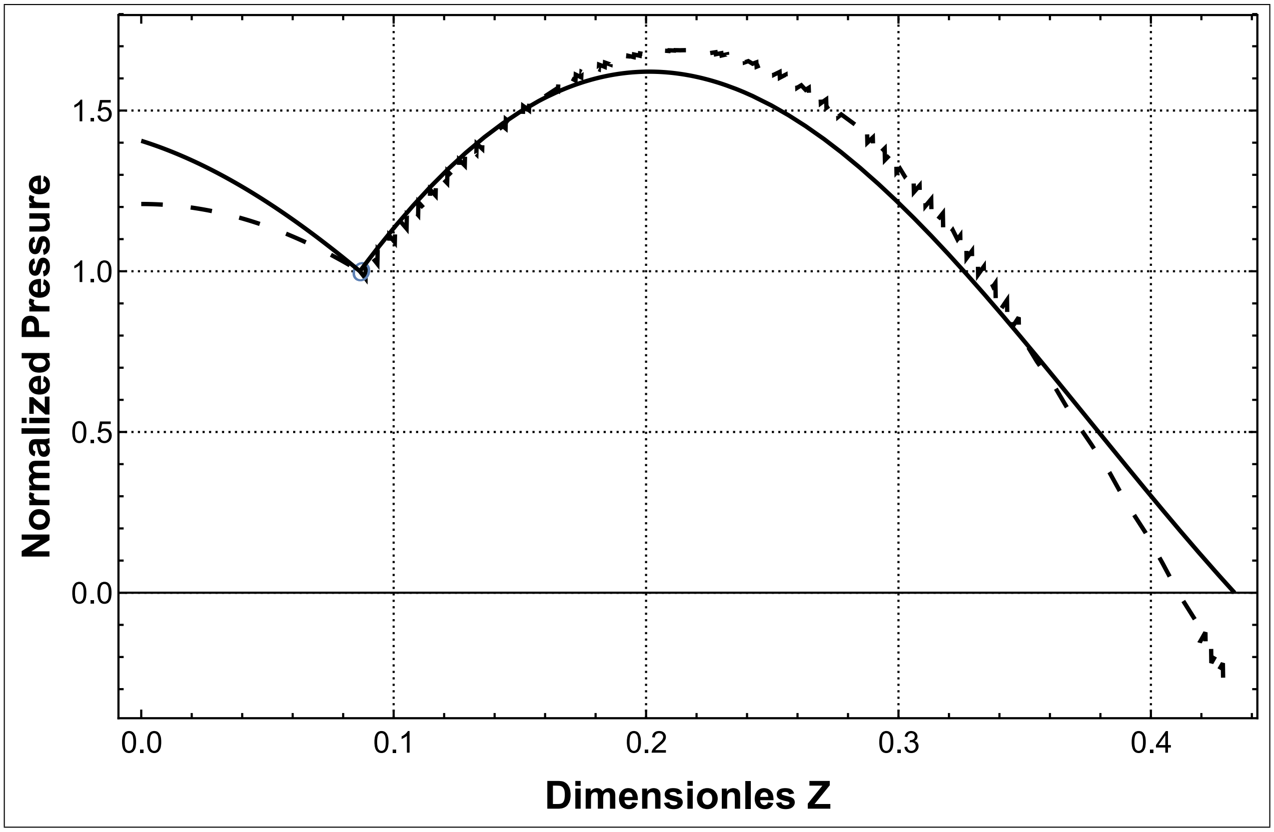

Figure 4.6: Comparison of GF (solid line) and FFP (dash line) Solutions for f_c = 2.2, Z_1=0.086

The final plot in figure 4.7 shows once again a fair comparison between the 2 methods. That is, one observes the discontinuity at the source, as well as, a value of zero (0) for the Green's function at the top. Additionally, the slope of the FFP at $Z = 0$ is zero (0) satisfying its boundary condition. Despite meeting these criteria, there still exists a divergence of the solutions at $Z < 0.05$. Nevertheless, these results are highly encouraging considering that they were achieved by 2 very different, diverse methods.

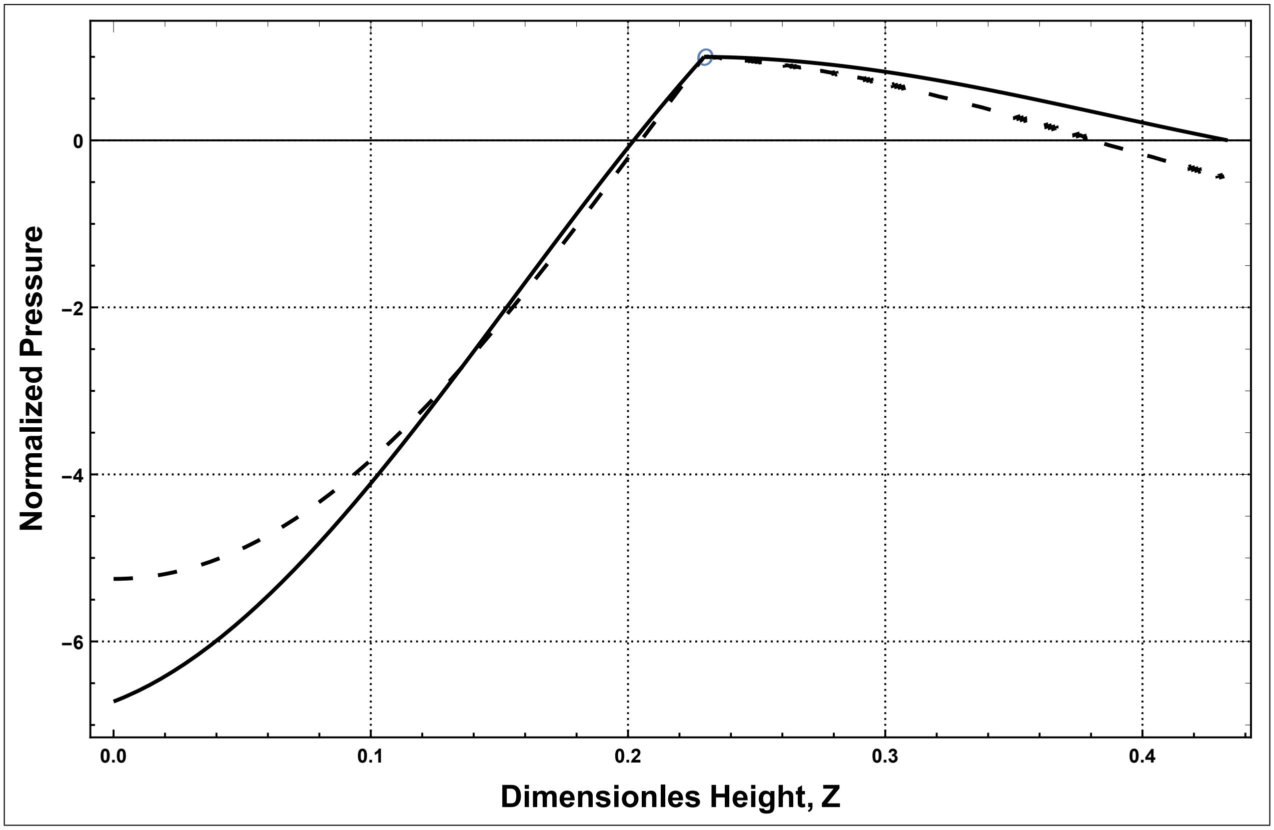

Figure 4.7: Comparison of GF (solid line) and FFP (dash line) Solutions for f_c = 2.4, Z_1=0.2236

4.8 Discussion

Generally, the results shown are of great interest. The reason for this is that we have started from 2 diverse, theoretical origins and shown that they are for most purposes equivalent. That is, the new model (previously developed) began with the linearization of the Continuity, Momentum, and Adiabatic equations in order to derive an expression for a vertically, propagating wave in a non-homogeneous, layered atmosphere. This was represented by a linear combination of Bessel functions of the First and Second Kind of order $-\frac{7}{2}$ with argument $2\Omega\tau$ where Ω is a dimensionless frequency and τ is a dimensionless distance. Based upon this model, a Green's function solution was found for a source arbitrarily located within the finite layer in which the wave propagated. Next, a completely different model was used (FFP) which emulated the solution to the Helmholtz equation in various layers of the non-homogeneous atmosphere. This, of course, resulted in the propagation of plane waves within the multiple layers and was solved by matching the boundary conditions at each layer. Despite the diverse beginnings of these models, they were both aimed at capturing the vertical propagation of sound in the atmosphere. With the reasonable corroboration between the models, it seems that this study was successful in creating, developing, and applying a novel, unique model for vertical propagation in a layered, non-homogeneous atmosphere.

CHAPTER V

Conclusion and Remarks

5.1 Summary and Conclusions

The results of the foregoing study are as follows:

1. From the seminal Euler equations for fluid flow, model differential equations were derived for the vertical propagation of sound in a layered, non-homogeneous media.

2. Atmosphere was modeled as combination of layered adiabatic and isothermal regions. The density and pressure profiles show a decrease in values as a function of height in the adiabatic regions. The sound speed decreases as well as a function of height in this region. However, density and pressure variation is exponential for an isothermal atmosphere and the sound speed remains constant in this region.

3. It was found that the model equation for an adiabatic atmosphere was a Bessel equation of order $-\frac{7}{2}$. The corresponding solutions are Bessel functions of the First and Second Kind. The argument for these functions is found to be $2\Omega\tau$ where Ω is a dimensionless frequency variable and τ is a dimensionless height variable.

4. The differential equation representing the isothermal region corresponds to that of a damped, harmonic oscillator.

5. The atmosphere is modeled as a combinations of adiabatic and isothermal regions. The system response is investigated by placing a low-frequency, time-harmonic sound source at various locations: a) at the layer boundaries and b) within the lower, adiabatic layer.

6. Source on a layered boundary was modeled as a boundary value problem.

7. Source within the adiabatic, lower layer was modeled using the Green's function method.

8. In order to test the validity of the GF method, a Fast Field Program (FFP) was implemented to represent the true propagation of plane waves within a mult-layered medium.

9. Matching of the GF and FFP functions were performed through self-normalization with the respective source values and, then, over-plotted on each other. Agreement was generally good, however, there was some divergence of the curves at the lower, dimensionless height of Z ¡ 0.05.

10. The phase velocity rekated to a combination of Bessel functions (Hankel functions) was derived analytically.

In summary, this work allows one to investigate the pressure signatures of a realistic, 2-layered, atmosphere for sources placed at various, arbitrary locations. Since the response for a point sources were studied, the results due to more complex signals may be analyzed utilizing the principle of superposition. Due to the very low frequencies of these signatures, they fall into the category of the low infrasound regime. Hence, remote sensing of the atmosphere may be accomplished to great heights. This is useful in detecting large object in the atmosphere. In particular, such sensing can be used in detecting meteors interacting with the atmosphere in an explosive manner. Additionally, multiple detections could be used to pinpoint impact zones using triangulation. Use of these pressure signals are able to predict the effect that the layered atmosphere will have on such predictions. Additionallly, the model could be used to remotely sense the ground impedance as well as upper-level turbulence layers. This work can be extended to investigate the role of wind speed on acoustic propagation. Furthermore, extended sources can be an interesting topic of investigation.

APPENDICES

APPENDIX A

Detailed Model Derivation for Pressure Equation

A.1 Detailed Model Derivation for Pressure Equation

This appendix provides a more detailed development of the model used in this work. A coarser rendition of the model development is given in Chapter 2. Proceeding from the three Euler equations stated in the introduction, we derive the requisite model. The 3 Euler equations are restated here for convenience.

$$0 = (D\varrho/Dt) - \varrho\nabla \cdot u \qquad \text{Mass Conservation Equation} \qquad \text{(A.1a)}$$

$$\varrho(Du/Dt) = -\nabla p + \varrho b \qquad \text{Momentum Conservation Equation} \qquad \text{(A.1b)}$$

$$Dp/Dt = c^2(D\varrho/Dt)(D\eta/Dt = 0) \qquad \text{Equation of State} \qquad \text{(A.1c)}$$

We begin the derivation with the linearization of the above equations. In order to do this, one needs to express the total pressure, density, and velocity as a sum of an ambient quantity plus a perturbed value. Ambient functions are represented by the subscript av and the perturbed value has no subscript. Additionally, the complete (total) quantity is represented by the subscript a. Hence, the following equations represent the definitions of the total pressure, p_a, total density, ϱ_a, and the fluid velocity (wind) vector,v_a:

$$p_a = p_{av}(z) + p \qquad \text{Linearized Pressure} \tag{A.2a}$$

$$\varrho_a = \varrho_{av}(z) + \varrho \qquad \text{Linearized Density} \tag{A.2b}$$

$$\boldsymbol{v}_a = (u_{av}(z) + u, v_{av}(z) + v, w) \qquad \text{Linearized Velocity} \tag{A.2c}$$

In the above equations, the functions identified with the subscript *"av"* are functions of the vertical z-component only. While the 2nd variable in each equation is a function of all 3 Cartesian components. These components represent the acoustic disturbances in pressure, density, and velocity. Substituting the above equations in the Mass Conservation equation and expanding the Material Derivative, one has:

$$\frac{\partial(\varrho_{av}(z) + \varrho)}{\partial t} + ((u_{av}(z) + u)\mathbf{i}, (v_{av}(z) + v)\mathbf{j}, w\mathbf{k}) \cdot \nabla(\varrho_{av}(z) + \varrho)$$
$$= -(\varrho_{av}(z) + \varrho)\nabla \cdot (u_{av}(z) + u)\mathbf{i}, (v_{av}(z) + v)\mathbf{j}, w\mathbf{k}). \tag{A.3}$$

One can now expand the ∇ operator to get:

$$\frac{\partial(\varrho_{av}(z) + \varrho)}{\partial t} + ((u_{av}(z)+u)\mathbf{i}, (v_{av}(z)+v)\mathbf{j}, w\mathbf{k}) \cdot (\frac{\partial(\varrho_{av}(z) + \varrho)}{\partial x}\mathbf{i} + \frac{\partial(\varrho_{av}(z) + \varrho)}{\partial y}\mathbf{j} + \frac{\partial(\varrho_{av}(z) + \varrho)}{\partial z}\mathbf{k})$$
$$= -(\varrho_{av}(z) + \varrho)(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}) \cdot (u_{av}(z) + u)\mathbf{i}, (v_{av}(z) + v)\mathbf{j}, w\mathbf{k}). \tag{A.4}$$

Expanding the equation above based on the indicated operations, one has

$$\frac{\partial\varrho_{av}(z)}{\partial t} + \frac{\partial\varrho}{\partial t} + (u_{av}(z) + u)\frac{\partial(\varrho_{av}(z) + \varrho)}{\partial x} + (v_{av}(z) + v)\frac{\partial(\varrho_{av}(z) + \varrho)}{\partial y} + w\frac{\partial(\varrho_{av}(z) + \varrho)}{\partial z}$$
$$= -(\varrho_{av}(z) + \varrho)(\frac{\partial(u_{av}(z) + u)}{\partial x} + \frac{\partial(v_{av}(z) + v)}{\partial y} + \frac{\partial w}{\partial z}). \tag{A.5}$$

It is helpful to complete the expansion in order to see more clearly which terms are non-linear and, ultimately, to obtain the total linearized version.

$$
\begin{aligned}
&\frac{\partial \varrho_{av}(z)}{\partial t} + \frac{\partial \varrho}{\partial t} + u_{av}(z)\frac{\partial \varrho_{av}(z)}{\partial x} + u\frac{\partial \varrho_{av}(z)}{\partial x} + u_{av}(z)\frac{\partial \varrho}{\partial x} + u\frac{\partial \varrho}{\partial x} + \\
&v_{av}(z)\frac{\partial \varrho_{av}(z)}{\partial y} + v\frac{\partial \varrho_{av}(z)}{\partial y} + v_{av}(z)\frac{\partial \varrho}{\partial y} + v\frac{\partial \varrho}{\partial y} + w\frac{\partial \varrho_{av}(z)}{\partial z} + w\frac{\partial \varrho}{\partial z} \\
=&-[\varrho_{av}(z)\frac{\partial u}{\partial x} + \varrho\frac{\partial u}{\partial x} + \varrho_{av}(z)\frac{\partial u_{av}}{\partial x} + \varrho\frac{\partial u_{av}}{\partial x} \\
&\varrho_{av}(z)\frac{\partial v_{av}(z)}{\partial y} + \varrho\frac{\partial v_{av}(z)}{\partial y} + \varrho_{av}(z)\frac{\partial v}{\partial y} + \varrho\frac{\partial v}{\partial y} + \varrho_{av}(z)\frac{\partial w}{\partial z} + \varrho\frac{\partial w}{\partial z}]
\end{aligned}
\tag{A.6}
$$

Now the terms which cannot be differentiated with respect to its variable are eliminated. The functions are

$$
\frac{\partial \varrho_{av}(z)}{\partial t} = u_{av}(z)\frac{\partial \varrho_{av}(z)}{\partial x} = u\frac{\partial \varrho_{av}(z)}{\partial x} = v_{av}(z)\frac{\partial \varrho_{av}(z)}{\partial y} = 0 \tag{A.7}
$$

$$
v\frac{\partial \varrho_{av}(z)}{\partial y} = \varrho_{av}(z)\frac{\partial u_{av}}{\partial x} = \varrho\frac{\partial u_{av}}{\partial x} = \varrho_{av}z)\frac{\partial v_{av}(z)}{\partial y} = \varrho\frac{\partial v_{av}(z)}{\partial y} = 0. \tag{A.8}
$$

And, finally, terms which are of 2nd order are dropped. These terms are:

$$
u\frac{\partial \varrho}{\partial x}, w\frac{\partial \varrho}{\partial z}, \varrho\frac{\partial w}{\partial z}. \tag{A.9}
$$

The remaining terms result in the following equation:

$$
\frac{\partial \varrho}{\partial t} + u_{av}(z)\frac{\partial \varrho}{\partial x} + v_{av}(z)\frac{\partial \varrho}{\partial y} + w\frac{\partial \varrho_{av}(z)}{\partial z} + \varrho_{av}(z)[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}] = 0. \tag{A.10}
$$

Next, our attention is turned to the Conservation of Momentum Equation (2.2). A similar process of linearization can be conducted, the details of which are omitted here. However, it should be noted that Eqn. 2.2 is a vector equation, thus resulting in an equation for each vector component **i**, **j**, and **k**. The three equations are:

$$\hat{\mathbf{i}} : \frac{\partial u}{\partial t} + u_{av}(z)\frac{\partial u}{\partial x} + v_{av}(z)\frac{\partial u}{\partial y} + w\frac{\partial u_{av}(z)}{\partial z} + \frac{1}{\varrho_{av}(z)}\frac{\partial p}{\partial x} = 0 \tag{A.11}$$

$$\hat{\mathbf{j}} : \frac{\partial v}{\partial t} + u_{av}(z)\frac{\partial v}{\partial x} + v_{av}(z)\frac{\partial v}{\partial y} + w\frac{\partial v_{av}(z)}{\partial z} + \frac{1}{\varrho_{av}(z)}\frac{\partial p}{\partial y} = 0 \tag{A.12}$$

$$\hat{\mathbf{k}} : \frac{\partial w}{\partial t} + u_{av}(z)\frac{\partial w}{\partial x} + v_{av}(z)\frac{\partial w}{\partial y} + \frac{1}{\varrho_{av}(z)}\frac{\partial p}{\partial z} + \frac{\varrho}{\varrho_{av}(z)} * g = 0. \tag{A.13}$$

As Lamb [10] assumes (and Soloman[14] uses) the fact that the variation in pressure and densisty from equilibrium values are connected by the *adiabatic* relation:

$$Dp_a/Dt = c^2(D\varrho_a/Dt). \tag{A.14}$$

Using the definition of the Material Derivative and expanding the total pressure and density in terms of its components, one has

$$\frac{\partial p}{\partial t} + \mathbf{v}_a \cdot \nabla p_a = c^2(\frac{\partial \varrho}{\partial t} + \mathbf{v}_a \cdot \nabla p_a). \tag{A.15}$$

Expanding this equation and retaining only terms of 1st order, we get [14]:

$$\frac{\partial p}{\partial t} + u_{av}(z)\frac{\partial p}{\partial x} + v_{av}(z)\frac{\partial p}{\partial y} + w\frac{\partial p_{av}(z)}{\partial z} = c^2(\frac{\partial \varrho}{\partial t} + u_{av}(z)\frac{\partial \varrho}{\partial x} + v_{av}(z)\frac{\partial \varrho}{\partial y} + w\frac{\partial \varrho_{av}(z)}{\partial z}). \tag{A.16}$$

Finally, Equations (A.10)–(A.13) with Equation (A.16) form a set of 5 simultaneous differential equations which can be solved for the five variables ϱ, p, u, v, and w. However, our model is developed for a windless condition (no fluid flow). This necessitates that the velocity in the x- and y-directions be set to zero. By doing so, two (2) of the five (5) equations are eliminated and we are now left with only 3 equations to solve. These are equations in the fluctuation variables ϱ, p, and w. The remaining equations are (A.10), (A.13), and (A.16).

In order to put these 3 equations into more tractable form, we write their variables with an explicit, time-harmonic component. This allows one to separate the original, lowercase variable into an uppercase, spatial part and a time-harmonic part. This is shown here:

$$p \to Pe^{-i\omega t}$$

$$\varrho \to \Omega_D e^{-i\omega t}$$

$$w \to We^{-i\omega t}$$

With the above transformed variables substituted into the 3 equations, one arrives at the 3 algebraic equations below. Please observe that the $-i\omega$ terms arise due to the time derivatives of the harmonic part in all 3 equations. Thus, our new equations are

$$\text{-}i\omega\Omega_D + \varrho_{av}^{'} W + \varrho_{av} W^{'} = 0 \quad \text{(A.17a)}$$

$$\text{-}i\omega W + \varrho_{av}^{-1} P^{'} + g\varrho_{av}^{-1}\Omega_D = 0 \quad \text{(A.17b)}$$

$$\text{-}i\omega P - \varrho_{av} g W = \text{-}i\omega c^2 \Omega_D + c^2 \varrho_{av}^{'} W. \quad \text{(A.17c)}$$

These equations contain derivatives, but our notation allows us to write them in an algebraic form which can be solved simultaneously. Once solved, we obtain a 2nd order, differential equation in P. Thus, we commence solving Equations (A.17a)–(A.17c) for the pressure, P.

Rearranging Equation (A.17c), we can obtain an expression for Ω in terms of W and P. This expression can then be used in Equation (A.17b) to arrive at an equation in terms of P, $P^{'}$, and W. This is shown here:

$$-i\omega W + \varrho_{av}^{-1} P^{'} + g\frac{\varrho_{av}^{-1}}{c^2}\left(P + \frac{1}{i\omega}(\varrho_{av} g + c^2 \varrho_{av}^{'}) W\right) = 0 \quad \text{(A.18)}$$

Rearranging terms in Equation (A.18), one can write W in terms of P and $P^{'}$ where

$$W = \frac{\varrho_{av}^{-1} P^{'} + \frac{g\varrho_{av}^{-1}}{c^2} P}{(i\omega - g\frac{\varrho_{av}^{-1}}{c^2}\frac{1}{i\omega}(\varrho_{av} g + c^2 \varrho_{av}^{'}))}. \quad \text{(A.19)}$$

Having found W, one can now solve Equation (A.17c) to obtain Ω in terms of P and $P^{'}$. This leads to the following expression:

$$i\omega\Omega = \frac{P^{'}(c^2\varrho_{av}^{'} + \varrho_{av}g) + P(\underbrace{\tfrac{g}{c^2}(\varrho_{av}g + c^2\varrho_{av}^{'})}_{cancels} - \omega^2\varrho_{av} - \underbrace{\tfrac{g}{c^2}(c^2\varrho_{av}^{'} + \varrho_{av}g)}_{cancels})}{c^2(i\omega\varrho_{av} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{'} + \varrho_{av}g))}. \tag{A.20}$$

It should be noted that Equation (A.20) has 2 terms which cancel. These are the 3rd and 5th terms in the numerator which are marked by the arrows through them. Now that we have $i\omega\Omega$ and W, only $W^{'}$ is needed in order to get an expression for P and $P^{'}$ in Equation (A.17a). Hence, proceeding to find $W^{'}$, one arrives at the equation:

$$W^{'} = \frac{P^{''} + \frac{g}{c^2}P^{'}}{(i\omega\varrho_{av} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{'} + \varrho_{av}g))} - \frac{(P^{'} + \frac{g}{c^2}P)(i\omega\varrho_{av}^{'} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{''} + \varrho_{av}^{'}g))}{(i\omega\varrho_{av} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{'} + \varrho_{av}g))^2}. \tag{A.21}$$

Now one can complete the derivation for P from Equation (A.17a) by substituting the relevant variables. Thus,

$$\begin{aligned} &\frac{-P^{'}(c^2\underbrace{\varrho_{av}^{'} + \varrho_{av}g}_{cancels}) + P\omega^2\varrho_{av}}{c^2(i\omega\varrho_{av} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{'} + \varrho_{av}g))} + \frac{\underbrace{\varrho_{av}^{'}P^{'}}_{cancels} + \varrho_{av}^{'}\frac{g}{c^2}P}{(i\omega\varrho_{av} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{'} + \varrho_{av}g))} \\ &+\frac{\varrho_{av}P^{''} + \underbrace{\varrho_{av}\frac{g}{c^2}P^{'}}_{cancels}}{(i\omega\varrho_{av} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{'} + \varrho_{av}g))} - \frac{\varrho_{av}(P^{'} + \frac{g}{c^2}P)(i\omega\varrho^{'} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{''} + \varrho_{av}^{'}g))}{(i\omega\varrho_{av} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{'} + \varrho_{av}g))^2} = 0. \end{aligned} \tag{A.22}$$

Multiplying by one occurence of the denominator in Equation (A.22) results in the following equation:

$$P\frac{\omega^2}{c^2}\varrho_{av} + \varrho_{av}^{'}\frac{g}{c^2}P + \varrho_{av}P^{''} + \frac{-\varrho_{av}(P^{'} + \frac{g}{c^2}P)(i\omega\varrho_{av}^{'} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{''} + \varrho_{av}^{'}g))}{(i\omega\varrho_{av} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{'} + \varrho_{av}g))} = 0. \tag{A.23}$$

Finally, collecting the like orders of derivatives in Equation (A.23), one gets

$$\begin{aligned} &\varrho_{av}P^{''} - \varrho_{av}P^{'}\left[\frac{(i\omega\varrho_{av}^{'} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{''} + \varrho_{av}^{'}g))}{(i\omega\varrho_{av} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{'} + \varrho_{av}g))}\right] \\ &+P\left[\frac{\omega^2}{c^2}\varrho_{av} + \varrho_{av}^{'}\frac{g}{c^2} + \varrho_{av}\frac{g}{c^2}\frac{(i\omega\varrho_{av}^{'} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{''} + \varrho_{av}^{'}g))}{(i\omega\varrho_{av} - \frac{g}{c^2 i\omega}(c^2\varrho_{av}^{'} + \varrho_{av}g))}\right] = 0. \end{aligned} \tag{A.24}$$

Dividing through by ϱ_{av}, one obtains a 2nd order differential equation whose leading coefficient is 1:

$$P^{''} - P^{'}\left[\frac{(i\omega\varrho^{'}_{av} - \frac{g}{c^2 i\omega}(c^2\varrho^{''}_{av} + \varrho^{'}_{av}g))}{(i\omega\varrho_{av} - \frac{g}{c^2 i\omega}(c^2\varrho^{'}_{av} + \varrho_{av}g))}\right]$$
$$+P\left[\frac{\omega^2}{c^2} + \frac{\varrho^{'}_{av}}{\varrho_{av}}\frac{g}{c^2} + \frac{g}{c^2}\frac{(i\omega\varrho^{'}_{av} - \frac{g}{c^2 i\omega}(c^2\varrho^{''}_{av} + \varrho^{'}_{av}g))}{(i\omega\varrho_{av} - \frac{g}{c^2 i\omega}(c^2\varrho^{'}_{av} + \varrho_{av}g))}\right] = 0. \tag{A.25}$$

In order to remove the imaginary terms from (A.25), we can multiply all fractions containing $i\omega$ by $(\frac{-i\omega}{-i\omega})$ to get:

$$P^{''} - P^{'}\left[\frac{(\omega^2\varrho^{'}_{av} + \frac{g}{c^2}(c^2\varrho^{''}_{av} + \varrho^{'}_{av}g))}{(\omega^2\varrho_{av} + \frac{g}{c^2}(c^2\varrho^{'}_{av} + \varrho_{av}g))}\right]$$
$$+P\left[\frac{\omega^2}{c^2} + \frac{\varrho^{'}_{av}}{\varrho_{av}}\frac{g}{c^2} - \frac{g}{c^2}\frac{(\omega^2\varrho^{'}_{av} + \frac{g}{c^2}(c^2\varrho^{''}_{av} + \varrho^{'}_{av}g))}{(\omega^2\varrho_{av} + \frac{g}{c^2}(c^2\varrho^{'}_{av} + \varrho_{av}g))}\right] = 0. \tag{A.26}$$

Next, in anticipation of making Equation(A.26) more compact and insightful, we multiply terms inside the brackets for P as shown by the factors in braces [{ }]:

$$P^{''} - P^{'}\left[\frac{(\omega^2\varrho^{'}_{av} + \frac{g}{c^2}(c^2\varrho^{''}_{av} + \varrho^{'}_{av}g))}{(\omega^2\varrho_{av} + \frac{g}{c^2}(c^2\varrho^{'}_{av} + \varrho_{av}g))}\right]$$
$$+P\left[\frac{\omega^2}{c^2} + \frac{\varrho^{'}_{av}}{\varrho_{av}}\frac{g}{c^2} - \frac{g}{c^2}\{\frac{c^2}{c^2}\}\frac{(\omega^2\varrho^{'}_{av} + \{\frac{\omega^2}{\omega^2}\}\frac{g}{c^2}(c^2\varrho^{''}_{av} + \varrho^{'}_{av}g))}{(\omega^2\varrho_{av} + \{\frac{\omega^2}{\omega^2}\}\frac{g}{c^2}(c^2\varrho^{'}_{av} + \varrho_{av}g))}\right] = 0. \tag{A.27}$$

Carrying out the implied multiplicative distribution which is shown by the curly brackets ({}, one has:

$$P^{''} - P^{'}\left[\frac{(\omega^2\varrho^{'}_{av} + \frac{g}{c^2}(c^2\varrho^{''}_{av} + \varrho^{'}_{av}g))}{(\omega^2\varrho_{av} + \frac{g}{c^2}(c^2\varrho^{'}_{av} + \varrho_{av}g))}\right]$$
$$+P\left[\frac{\omega^2}{c^2} + \frac{\varrho^{'}_{av}}{\varrho_{av}}\frac{g}{c^2} - \frac{g}{c^2}\frac{(\{c^2\}\omega^2\varrho^{'}_{av} + \{\frac{\omega^2}{\omega^2}\}\frac{g}{c^2}\{c^2\}(c^2\varrho^{''}_{av} + \varrho^{'}_{av}g))}{(\{c^2\}\omega^2\varrho_{av} + \{\frac{\omega^2}{\omega^2}\}\frac{g}{c^2}\{c^2\}(c^2\varrho^{'}_{av} + \varrho_{av}g))}\right] = 0. \tag{A.28}$$

Now an ω^2 is factored from the $P^{'}$ coefficient and an $\omega^2 c^2$ terms is factored from the large fraction of the P coefficient in Equation (A.28). Thus,

$$
\begin{aligned}
&P^{''} - P^{'} \left[\frac{\omega^2(\varrho_{av}^{'} + \frac{g}{c^2\omega^2}(c^2\varrho_{av}^{''} + \varrho_{av}^{'} g))}{\omega^2(\varrho_{av} + \frac{g}{c^2\omega^2}(c^2\varrho_{av}^{'} + \varrho_{av} g))} \right] \\
&+P\left[\frac{\omega^2}{c^2} + \frac{\varrho_{av}^{'}}{\varrho_{av}}\frac{g}{c^2} - \frac{g}{c^2}\frac{c^2\omega^2\left[\varrho_{av}^{'} + \frac{g}{c^2\omega^2}(c^2\varrho_{av}^{''} + \varrho_{av}^{'} g)\right]}{c^2\omega^2\left[\varrho_{av} + \frac{g}{c^2\omega^2}(c^2\varrho_{av}^{'} + \varrho_{av} g)\right]} \right] = 0.
\end{aligned}
\tag{A.29}
$$

One will notice that, for the large fractions in Equation (A.29), the numerators are derivatives of the denominators. Since for a function y, $y/ \; y = \frac{d}{dz}\left[ln(y)\right]$. Using this property and with further manipulation, one arrives at the equation:

$$
\begin{aligned}
&P^{''} - P^{'}\frac{d}{dz}ln\left[\omega^2\varrho_{av}\left[1 + \frac{g}{\omega^2}(\frac{\varrho_{av}^{'}}{\varrho_{av}} + \frac{g}{c^2})\right]\right] \\
&+P\left[\frac{\omega^2}{c^2} + \frac{\varrho_{av}^{'}}{\varrho_{av}}\frac{g}{c^2} - \frac{g}{c^2}\frac{d}{dz}ln\left[(\omega^2 c^2\varrho_{av}\left[1 + \frac{g}{\omega^2}(\frac{\varrho_{av}^{'}}{\varrho_{av}} + \frac{g}{c^2})\right]\right]\right] = 0.
\end{aligned}
\tag{A.30}
$$

Now with the property of the logarithmic function, we can write:

$$
\begin{aligned}
&P^{''} - P^{'}\left\{\frac{d}{dz}ln(\varrho_{av}) + \frac{d}{dz}ln\left[\omega^2\left[1 + \frac{g}{\omega^2}(\frac{\varrho_{av}^{'}}{\varrho_{av}} + \frac{g}{c^2})\right]\right]\right\} \\
&+P\left[\frac{\omega^2}{c^2} + \frac{\varrho_{av}^{'}}{\varrho_{av}}\frac{g}{c^2} - \frac{g}{c^2}\left\{\frac{d}{dz}ln(\varrho_{av}) + \frac{d}{dz}ln\left[(\omega^2 c^2\left[1 + \frac{g}{\omega^2}(\frac{\varrho_{av}^{'}}{\varrho_{av}} + \frac{g}{c^2})\right]\right]\right\}\right] = 0.
\end{aligned}
\tag{A.31}
$$

In this form, it easier to estimate the relevant terms and decide which (if any) can be omitted. Using the fact that in a homogeneous, non-moving atmosphere, the adiabatic sound speed is given by $c = \sqrt{\gamma\frac{p_{av}}{\varrho_{av}}}$ and the fact that the equilibrium equation for the average pressure is $p_{av}^{'} = -\varrho_{av} g$, one arrives at the expression for $\frac{p_{av}^{'}}{p_{av}} = -\frac{g\gamma}{c^2}$. One may also use the Ideal Gas Law via differentian to get:

$$
\frac{\varrho_{av}^{'}}{\varrho_{av}} = \frac{p_{av}^{'}}{p_{av}} - \frac{T^{'}}{T}.
\tag{A.32}
$$

Since $\frac{p'_{av}}{p_{av}}$ is of the order of $10^{-4}m^{-1}$ and the term $\frac{T'}{T}$ is of the order of $0.1m^{-1}$, then an approximate estimate for the term $g(\frac{\varrho'_{av}}{\varrho_{av}} + \frac{g}{c^2})$ is of the order 1. For frequencies of the order of 7 mHz, the outermost brackets ([]) of the 2 log functions reduce to 1, yielding the equation:

$$P'' - P' \left\{ \frac{d}{dz} ln(\varrho_{av}) + \frac{d}{dz} ln\left[[1]\right] \right\}$$
$$+P\left[\frac{\omega^2}{c^2} + \frac{\varrho'_{av}}{\varrho_{av}}\frac{g}{c^2} - \frac{g}{c^2}\left\{\frac{d}{dz}ln(\varrho_{av}) + \frac{d}{dz}ln\left[c^2\left[1\right]\right]\right\}\right] = 0. \quad (A.33)$$

This is further reduced as follows:

$$P'' - P'\left[\frac{\varrho'_{av}}{\varrho_{av}}\right]$$
$$+P\left[\frac{\omega^2}{c^2} + \underbrace{\frac{\varrho'_{av}}{\varrho_{av}}\frac{g}{c^2}}_{cancels} + \left(-\underbrace{\frac{g}{c^2}\frac{\varrho'_{av}}{\varrho_{av}}}_{cancels} - 2\frac{g}{c^2}\left[\frac{c'}{c}\right]\right)\right] = 0. \quad (A.34)$$

As for the ratio $\left|\frac{c'}{c}\right|$, it is normally greatest near the surface of the ground. A safe upper limit is $0.1m^{-1}$. However, this combined with the $\frac{g}{c^2}$ is of the order of 6 x $10^{-4}m^{-2}$. Hence, the term in parentheses in Equation (A.34) may be neglected. This leaves the following, final form of our equation as:

$$P'' - \left(\frac{\varrho'_{av}}{\varrho_{av}}\right)P' + \left(\frac{\omega^2}{c^2}\right)P = 0. \quad (A.35)$$

APPENDIX B

The Layered Atmosphere

B.1 The Layered Atmosphere

Our model atmosphere consists of 2 layers. The first is an adiabatic one which extends from the ground to approximately 13000 meters and the second is isothermal which persists from the top of the adiabatic layer to a height of approximately 30000 meters. Here, we examine the pressure, density, temperature and sound speed functions for each layer.

Before examining specific atmospheres, we define some expressions that will be useful in our derivations. First, we define density, ϱ, as

$$\varrho = \frac{M}{V} = \frac{\frac{M}{M_{mol}}}{V} M_{mol} = \frac{n}{V} M_{mol} \Rightarrow \varrho \propto V^{-1} \tag{B.1}$$

where n is the number of moles of a gas, M is the total mass, M_{mol} is the mass per mole (or molar mass), and V is the volume. For air, $M_{mol} = 28.95 \frac{g}{mol}$.

Next, we turn to the equation for an Ideal gas:

$$PV = nR_{mol}T \tag{B.2}$$

where R_{mol} is the Universal Molar Gas Constant whose value is 8.314 $JK^{-1}mol^{-1}$ and T is the temperature in Kelvin. Rearranging (B.2), we get

$$P = \left[\frac{nM_{mol}}{V}\right]\left[\frac{R_{mol}T}{M_{mol}}\right] = \varrho RT \tag{B.3}$$

where $R = \frac{R_{mol}}{M_{mol}}$ is defined as the *specific* gas constant. The *specific* gas constant for air is $0.287\ \frac{J}{K\,kg}$.

In order to correctly address the sound speed for the various atmospheres, it is necessary to understand its wave origin. In deriving the sound speed, c, it is the proportionality constant between the 2nd time derivative of the change in pressure to the change in density. It is expressed as

$$c_0 = \sqrt{\frac{K_0}{\varrho_0}} \tag{B.4}$$

where K_0 is the Bulk modulus defined generally as

$$K = \frac{dp}{-\frac{dV}{V}} = \frac{dp}{\frac{d\varrho}{\varrho}} = \varrho\frac{dp}{d\varrho} \tag{B.5}$$

and $dM = d(\varrho V) = \varrho dV + V d\varrho = 0$ is used to replace $-\frac{dV}{V}$ with $\frac{d\varrho}{\varrho}$.

With the above definition for K, there arises the issue that there exist different Bulk moduli depending on thermodynamic conditions being used. Thus, for the isothermal condition, one has

$$K_T = \left(\varrho\frac{dp}{d\varrho}\right)\Bigg|_T = \varrho\frac{d\varrho}{d\varrho}R_{mol}T = \varrho\frac{P}{\varrho} = p. \tag{B.6}$$

Thus, the isothermal sound speed becomes

$$c_0 = \sqrt{\frac{p_0}{\varrho_0}}. \tag{B.7}$$

For an isentropic (adiabatic) bulk modulus, one has

$$K_S = \left(\varrho\frac{dp}{d\varrho}\right)\Bigg|_S = \varrho\frac{d[(const)\varrho^\gamma]}{d\varrho} = \varrho(const)\gamma\varrho^{\gamma-1} = (const)\varrho^\gamma = \gamma p. \tag{B.8}$$

And, the adiabatic sound speed is

$$c_0 = \sqrt{\gamma \frac{p_0}{\varrho_0}}. \tag{B.9}$$

B.1.0.1 Atmospheric Scale Height

Atmospheric scale height is useful as it tells one over what distances scale properties of the atmosphere will change. Typically, one uses the value of a height for which there's a pressure change of $1/e$. Sometimes there is confusion with scale heights for various atmospheres as different ones can produce the same scale height. For this reason, these atmospheric models may be confused. In addition to analyzing the properties mentioned above for the various atmospheric layers, we will also investigate its scale height.

B.1.1 Incompressible Atmosphere (Constant Density)

If one examines an incompressible atmosphere (which is not realistic, but instructive), one can develop the equation for the pressure decrease with height (z) as

$$p = p_0 - \varrho_0 g z \tag{B.10}$$

where p_0 is the pressure at the Earth's surface and g is the acceleration of gravity. Solving for the height when the atmospheric pressure is 0 (p=0), one gets

$$p_0 = \varrho_0 g h_0 \Rightarrow h_0 = \frac{p_0}{\varrho_0 g}. \tag{B.11}$$

This may also be written as

$$h_0 = \frac{p_0}{\varrho_0 g} = \frac{RT_0}{g}. \tag{B.12}$$

Here, h_0 is termed a scale height and with p_0=1 atm and ϱ_0 =1.18 $kg\ m^{-3}$, one gets a scale factor of $h_0 = 8.72\,km$.

Turning to finding the sound speed c and using equations (B.6) and (B.7), one has

$$c = \sqrt{\frac{K}{\varrho_0}} = \sqrt{\frac{p}{\varrho_0}} = \sqrt{\frac{p_0 - \varrho_0 g z}{\varrho_0}} = \sqrt{c_0^2 - gz} = c_0\sqrt{1 - \frac{gz}{c_0^2}}. \tag{B.13}$$

Now, from the ideal gas law equation, we can solve for temperature, T, for a constant density, ϱ_0:

$$p = \varrho_0 RT = p_0 - \varrho_0 gz \Rightarrow T = \frac{p_0}{\varrho_0 R} - \frac{\varrho_0 gz}{\varrho_0 R} = \frac{1}{R}\left\{c_0^2 - gz\right\}. \tag{B.14}$$

B.1.2 The Adiabatic Layer

Since the first layer is adiabatic, it may conveniently be characterized by the adiabatic relationship

$$PV^{\gamma} = constant \tag{B.15}$$

where $\gamma = \frac{c_p}{c_v} = \frac{7}{5}$ (for air) is the ratio of specific heats at constant pressure and volume. Since, by (B.1) density, ϱ, and volume, V, are inversely related, (B.15) becomes

$$P = \frac{constant}{V^{\gamma}} \tag{B.16}$$

or

$$P \propto \varrho^{\gamma} \tag{B.17}$$

Using temperature T and the ideal gas law, $PV = nR_{mol}T$, as well as factoring (B.15), one may write

$$PVV^{\gamma-1} = constant \Rightarrow TV^{\gamma-1} = constant. \tag{B.18}$$

Thus,

$$T \propto \varrho^{\gamma-1} \tag{B.19}$$

by (B.1).

For a 'statical relationship' between the pressure and density, one has the expression

$$\frac{\partial P}{\partial z} = -\varrho g \text{ where g is the acceleration due to gravity .} \tag{B.20}$$

Hence, by (B.17) and (B.20), one gets

$$P^{'} \propto \varrho^{\gamma-1}\varrho^{'} \text{ (where } ' \text{ represents the derivative with respect to } z) \tag{B.21a}$$

$$-\varrho g \propto \varrho^{\gamma-1}\varrho^{'} \tag{B.21b}$$

$$g \propto \varrho^{\gamma-2}\varrho^{'}. \tag{B.21c}$$

From (B.19), one finds that the right side of (B.22) is proportional to $T^{'}$. Hence, $T^{'} = constant$. The consequence of this is that T is linear and is of the form:

$$T = mz + b. \tag{B.22}$$

Using $T = T_0$ at z=0, and $T = 0$ at z=l, one arrives at the following equation:

$$T = T_0(1 - \frac{z}{l}) \text{ where } l \text{ is the top of our atmosphere.} \tag{B.23}$$

Note that at $z = 0$, we have the surface temperature T_0 and at the top of the atmosphere $z = l$, the temperature, T, is 0. With the use of (B.19), one has

$$\varrho \propto T^{(\frac{1}{\gamma-1})} \tag{B.24}$$

which when applied to (B.23), one obtains

$$\varrho = \varrho_0(1 - \frac{z}{l})^{\frac{1}{(\gamma-1)}}. \tag{B.25}$$

Combining (B.17) and (B.25), we find the remaining expression for the pressure as

$$p = p_0(1 - \frac{z}{l})^{\frac{\gamma}{(\gamma-1)}}. \tag{B.26}$$

With the expression for adiabatic sound speed from (B.9) and equations (B.25) and (B.26), we have

$$c = \sqrt{\gamma \frac{p_0(1-\frac{z}{l})^{\frac{\gamma}{(\gamma-1)}}}{\varrho_0(1-\frac{z}{l})^{\frac{1}{(\gamma-1)}}}}. \tag{B.27}$$

In our model $\chi = \frac{1}{\gamma-1}$ and (B.28) may be written in terms of χ and c_0 as

$$c = \sqrt{\gamma \frac{p_0}{\varrho_0}} \sqrt{1 - \frac{z}{l}} = c_0 \sqrt{1 - \frac{z}{l}} \tag{B.28}$$

Next, an expression for the scale height is found. Starting with

$$PV^\gamma = const \Rightarrow p(\frac{1}{\varrho})^\gamma = const. \tag{B.29}$$

Or, using the ideal gas equation, one can write

$$p(\frac{1}{\varrho})^\gamma = p(\frac{RT}{p})^\gamma = p^{1-\gamma} T^\gamma = const. \tag{B.30}$$

Since we will be using the hydrostatic equation $\frac{dp}{dz} = -\varrho g = -\frac{p}{RT} g$, we differentiate (B.30) with respect to z:

$$\frac{d}{dz}\left\{p^{1-\gamma} T^\gamma\right\} = \left(\frac{d}{dp} p^{1-\gamma}\right) \frac{dp}{dz} T^\gamma + \left(\frac{d}{dT} T^\gamma\right) \frac{dT}{dz} p^{1-\gamma} \tag{B.31a}$$

$$= (1-\gamma) p^{1-\gamma-1} T^\gamma dp + \gamma T^{\gamma-1} p^{1-\gamma} dT = 0 \tag{B.31b}$$

$$= \frac{(1-\gamma) p^{1-\gamma-1} T^\gamma}{p^{1-\gamma} T^\gamma} dp + \frac{\gamma T^{\gamma-1} p^{1-\gamma}}{p^{1-\gamma} T^\gamma} dT = 0 \tag{B.31c}$$

$$= (1-\gamma) \frac{dp}{p} + \gamma \frac{dT}{T} = 0. \tag{B.31d}$$

Since from above, $\frac{dp}{dz} = -\frac{p}{RT}g$, one may write

$$(1-\gamma)\frac{dp}{p} = -(1-\gamma)\frac{g}{RT}dz \tag{B.32a}$$

$$(1-\gamma)\frac{dp}{p} = -(1-\gamma)\frac{g}{RT}dz = -\gamma\frac{dT}{T}. \tag{B.32b}$$

Or,

$$-(1-\gamma)\frac{g}{R}dz = -\gamma dT \Rightarrow -\frac{g}{R}dz = \frac{\gamma}{\gamma-1}dT. \tag{B.33}$$

Integrating the right-hand side of (B.33) and rearranging terms, we get

$$-\frac{g}{\frac{\gamma}{\gamma-1}R}z = T - T_0. \tag{B.34}$$

Here, we note that $\frac{\gamma}{\gamma-1}R = c_p$, where c_p is the specific heat of the air at constant pressure. Applying this to (B.34), we have

$$T = T_0 - \frac{g}{c_p}z. \tag{B.35}$$

Factoring T_0 and rearranging the result, one has

$$T = T_0\left\{1 - \frac{z}{\frac{c_p T_0}{g}}\right\}. \tag{B.36}$$

Written in this form, it is clear that the scale height is $\frac{c_p T_0}{g}$. For a temperature of 300 K (surface temperature), the scale height for the adiabatic atmosphere is approximately 30.7 km. This value is very close to the scale height of our model (l = 30 km. Hence, its justification in our use of it.

The above equations for temperature, density, pressure, sound speed, and scale height are the equations used in our model for the adiabatic layer.

B.1.3 The Isothermal Layer

For an isothermal atmosphere, we start with the hydrostatic equation coupled with the Ideal Gas law. Thus,

$$\frac{dp}{dz} = -\varrho g. \tag{B.37}$$

For the Ideal Gas law, we have

$$pv = nRT \Rightarrow p = \frac{nRT}{v} \Rightarrow p = \varrho RT. \tag{B.38}$$

Since T is constant, it must equal the temperature at the Earth's suface, which we term T_0. Then substituting ϱ from (B.38) on the right-hand side of (B.37), one gets

$$\frac{dp}{dz} = -\frac{p}{RT_0} g. \tag{B.39}$$

Rearranging (B.39) and integrating, we have

$$\int_{p_0}^{p} \frac{dp}{p} = -\int_{0}^{z} \frac{g}{RT_0} dz. \tag{B.40}$$

This leads to the final pressure equation

$$p = p_0 e^{-\frac{g}{RT_0} z}. \tag{B.41}$$

Since we are considering an isothermal ideal gas, one can write

$$p = \varrho RT_0 \Rightarrow \varrho = \frac{p}{RT_0}. \tag{B.42}$$

Substituting for p from (B.41) into (B.42), one gets

$$\varrho = \left(\frac{p_0}{RT_0}\right) e^{-\frac{g}{RT_0} z} = \varrho_0 e^{-\frac{g}{RT_0} z}. \tag{B.43}$$

Thus, one finds that ϱ and p have the same exponential form.

For the sound speed $c = \sqrt{\frac{p}{\varrho}}$, one finds upon substituing (B.41) and (B.43) for p and ϱ, one gets

$$c = \sqrt{\frac{p_0 e^{-\frac{g}{RT_0}z}}{\varrho_0 e^{-\frac{g}{RT_0}z}}} = \sqrt{\frac{p_0}{\varrho_0}} = c_0. \tag{B.44}$$

Thus, for an atmosphere with an exponentially, decreasing density, the sound speed is a constant!

Next we consider the scale height for this atmosphere. Considering (B.43), one sees that the density, ϱ, may be rewritten as

$$\varrho = \varrho_0 e^{-\frac{g}{RT_0}z} = \varrho_0 e^{-\frac{z}{\frac{RT_0}{g}}}. \tag{B.45}$$

By inspection, one sees that ϱ achieves a value of $\varrho_0 e^{-1}$ when $z = \frac{RT_0}{g}$. Therefore, the scale height is

$$\frac{RT_0}{g} = h_0 \tag{B.46}$$

Note that this is the same scale height that was found for a constant density atmosphere in (B.12) which is 8.72 km.

APPENDIX C

Determining Boundary Value Constants A, B, and C for Model 2

C.1 Determining Bondary Value Constants A, B, and C for Model 2

Here, we will combine the results for the general solutions in the lower and upper layers in order to get the exact equation for the pressure from a time-harmonic source. To begin, we note that the Bessel's function derived for the lower layer was given as Equation (2.31) and is repeated here for convenience:

$$\tau^2 \frac{d^2 X}{d\tau^2} + \tau \frac{dX}{d\tau} + (4\Omega^2\tau^2 - \chi^2)X = 0. \tag{C.1}$$

The solution to this equation are the two Bessel functions of First and Second Kind with order χ and argument $2\Omega\tau$. Hence,

$$P = A\frac{J_\chi(2\Omega\tau)}{\tau^\chi} + B\frac{Y_\chi(2\Omega\tau)}{\tau^\chi} \tag{C.2}$$

where A and B are constants to be determined from the boundary conditions. We remind the reader that $\tau = \sqrt{1 - Z}$, where Z is a dimensionless variable such that $Z = \frac{z}{l}$. For the purposes of our derivations, τ will be recast in this light. Thus, our general solution in the lower layer becomes

$$P_1 = A\frac{J_\chi(2\Omega(1-Z)^{\frac{1}{2}})}{(1-Z)^{\frac{\chi}{2}}} + B\frac{Y_\chi(2\Omega(1-Z)^{\frac{1}{2}})}{(1-Z)^{\frac{\chi}{2}}}. \tag{C.3}$$

In a similar vein, Equation (3.8) can be rewritten in terms of the dimensionless Zs as:

$$P_2 = Ce^{-\left\{\frac{\alpha}{2}[Z-Z_2]\right\}}cos\left\{\frac{\alpha}{2}\sqrt{\left[\frac{4\Omega^2}{\alpha^2}-1\right]}\,[Z-Z_2]\right\} \tag{C.4}$$

where, again, the constant C is to be determined by the boundary conditions.

In order to solve for the constants A, B, and C, 3 boundary conditions are necessary. These are now stated below as:

$$P_1|_{Z=0} = P_0 \tag{C.5a}$$

$$P_1|_{Z=Z_2} = P_2|_{Z=Z_2} \tag{C.5b}$$

$$P_1'\Big|_{Z=Z_2} = P_2'\Big|_{Z=Z_2}. \tag{C.5c}$$

In Equation (C.5a), the pressure at the ground level, $Z = 0$, is set to an arbitrary, unspecified constant, P_0 for the moment.

We will now apply the boundary conditions to our solutions for the appropriate medium. For Equation (C.5a), we have

$$A\frac{J_\chi(2\Omega(1-Z)^{\frac{1}{2}})}{(1-Z)^{\frac{\chi}{2}}}\Bigg|_{Z=0} + B\frac{J_\chi(2\Omega(1-Z)^{\frac{1}{2}})}{(1-Z)^{\frac{\chi}{2}}}\Bigg|_{Z=0} = P_0 \tag{C.6}$$

or

$$AJ_\chi(2\Omega) + BY_\chi(2\Omega) = P_0. \tag{C.7}$$

It should be remembered that $\chi = -7/2$.

Turning to the 2nd boundary condition in Equation (C.5b), we have

$$A\frac{J_\chi(2\Omega(1-Z)^{\frac{1}{2}})}{(1-Z)^{\frac{\chi}{2}}}\bigg|_{Z=Z_2} + B\frac{Y_\chi(2\Omega(1-Z)^{\frac{1}{2}})}{(1-Z)^{\frac{\chi}{2}}}\bigg|_{Z=Z_2} = Ce^{-\left\{\frac{\alpha}{2}[Z-Z_2]\right\}}cos\left\{\frac{\alpha}{2}\sqrt{\left[\frac{4\Omega^2}{\alpha^2}-1\right]}[Z-Z_2]\right\}\bigg|_{Z=Z_2}. \tag{C.8}$$

Applying the boundary condition at $Z = Z_2$, we have

$$A\frac{J_\chi\left(2\Omega\left[1-Z_2\right]^{\frac{1}{2}}\right)}{(1-Z_2)^{\frac{\chi}{2}}} + B\frac{Y_\chi\left(2\Omega\left[1-Z_2\right]^{\frac{1}{2}}\right)}{(1-Z_2)^{\frac{\chi}{2}}} = C. \tag{C.9}$$

With some rearrangement and substituting the value $\chi = -\frac{7}{2}$, we have

$$(1-Z_2)^{\frac{7}{4}}\left[AJ_\chi\left(2\Omega\left[1-Z_2\right]^{\frac{1}{2}}\right) + BY_\chi\left(2\Omega\left[1-Z_2\right]^{\frac{1}{2}}\right)\right] = C. \tag{C.10}$$

This now leaves us with finding an expression for the 3rd boundary condition in Equation (C.5c). To this end, we will need the derivatives of Equations (C.3) and (C.4). We commence with finding the derivative of Equation (C.3). Using a general form of Equation (C.10) for Z, we have

$$\begin{aligned}\frac{\partial(1-Z)^{\frac{7}{4}}\left[AJ_{-\frac{7}{2}}\left(2\Omega\left[1-Z\right]^{\frac{1}{2}}\right) + BY_{-\frac{7}{2}}\left(2\Omega\left[1-Z\right]^{\frac{1}{2}}\right)\right]}{\partial Z} &= P_1'\bigg|_{Z=Z_2} \\ = \frac{\partial(1-Z)^{\frac{7}{4}}}{\partial Z}\left[AJ_{-\frac{7}{2}}\left(2\Omega\left[1-Z\right]^{\frac{1}{2}}\right) + BY_{-\frac{7}{2}}\left(2\Omega\left[1-Z\right]^{\frac{1}{2}}\right)\right] \\ +(1-Z)^{\frac{7}{4}}\frac{\partial}{\partial Z}\left[AJ_{-\frac{7}{2}}\left(2\Omega\left[1-Z\right]^{\frac{1}{2}}\right) + BY_{-\frac{7}{2}}\left(2\Omega\left[1-Z\right]^{\frac{1}{2}}\right)\right] &= P_1'\bigg|_{Z=Z_2}\end{aligned} \tag{C.11}$$

$$\begin{aligned}= -\frac{7}{4}(1-Z)^{\frac{3}{4}}\left[AJ_{-\frac{7}{2}}\left(2\Omega\left[1-Z\right]^{\frac{1}{2}}\right) + BY_{-\frac{7}{2}}\left(2\Omega\left[1-Z\right]^{\frac{1}{2}}\right)\right] \\ +(1-Z)^{\frac{7}{4}}\left[A\left(-\Omega\left[1-Z\right]^{-\frac{1}{2}}\right)J'_{-\frac{7}{2}} + B\left(-\Omega\left[1-Z\right]^{-\frac{1}{2}}\right)Y'_{-\frac{7}{2}}\right] = P_1'\bigg|_{Z=Z_2}\end{aligned} \tag{C.12}$$

$$
\begin{aligned}
&-(1-Z_2)^{\frac{3}{4}}\left\{\frac{7}{4}\left[AJ_{-\frac{7}{2}}\left(2\Omega\left[1-Z_2\right]^{\frac{1}{2}}\right)+B\,Y_{-\frac{7}{2}}\left(2\Omega\left[1-Z_2\right]^{\frac{1}{2}}\right)\right]\right\} \\
&-\Omega(1-Z_2)^{\frac{5}{4}}\left\{\left[AJ'_{-\frac{7}{2}}\left(2\Omega\left[1-Z_2\right]^{\frac{1}{2}}\right)+B\,Y'_{-\frac{7}{2}}\left(2\Omega\left[1-Z_2\right]^{\frac{1}{2}}\right)\right]\right\}=P'_1\Big|_{Z_2}.
\end{aligned}
\tag{C.13}
$$

Then, for the derivative of P_2 from Equation (C.4), we have

$$
\begin{aligned}
&C\left(\frac{-\alpha}{2}\right)e^{-\left\{\frac{\alpha}{2}[Z-Z_2]\right\}}cos\left\{\frac{\alpha}{2}\sqrt{\left[4\left(\frac{\Omega}{\alpha}\right)^2-1\right]}\,[Z-Z_2]\right\}\Bigg|_{Z=Z_2} \\
&-C\left(\frac{\alpha}{2}\sqrt{4\left(\frac{\Omega}{\alpha}\right)^2-1}\right)e^{-\left\{\frac{\alpha}{2}[Z-Z_2]\right\}}sin\left\{\frac{\alpha}{2}\sqrt{\left[4\left(\frac{\Omega}{\alpha}\right)^2-1\right]}\,[Z-Z_2]\right\}\Bigg|_{Z=Z_2}=P'_2\Big|_{Z=Z_2}.
\end{aligned}
\tag{C.14}
$$

When Equation (C.12) is evaluated at $Z = Z_2$, one gets the simple result:

$$
-C\frac{\alpha}{2}=P'_2\Big|_{Z=Z_2}. \tag{C.15}
$$

One may now eliminate C between Equations (C.10) and (C.15) to get B in terms of A. Thus, we proceed as follows:

$$
\begin{aligned}
&-\frac{\alpha}{2}(1-Z_2)^{\frac{7}{4}}\left[AJ_{-\frac{7}{2}}\left(2\Omega\left[1-Z_2\right]^{\frac{1}{2}}\right)+B\,Y_{-\frac{7}{2}}\left(2\Omega\left[1-Z_2\right]^{\frac{1}{2}}\right)\right] \\
&=-(1-Z_2)^{\frac{3}{4}}\left\{\frac{7}{4}\left[AJ_{-\frac{7}{2}}\left(2\Omega\left[1-Z_2\right]^{\frac{1}{2}}\right)+B\,Y_{-\frac{7}{2}}\left(2\Omega\left[1-Z_2\right]^{\frac{1}{2}}\right)\right]\right\} \\
&-\Omega(1-Z_2)^{\frac{5}{4}}\left\{\left[AJ'_{-\frac{7}{2}}\left(2\Omega\left[1-Z_2\right]^{\frac{1}{2}}\right)+B\,Y'_{-\frac{7}{2}}\left(2\Omega\left[1-Z_2\right]^{\frac{1}{2}}\right)\right]\right\}.
\end{aligned}
\tag{C.16}
$$

Factoring terms common to A and B, we arrive at a relationship between the 2 constants:

$$B = A\frac{\left\{\left[-\frac{\alpha}{2}(1-Z_2)+\frac{7}{4}\right]J_{-\frac{7}{2}}\left[2\Omega(1-Z_2)^{\frac{1}{2}}\right]+\Omega(1-Z_2)^{\frac{1}{2}})J'_{-\frac{7}{2}}\left[2\Omega(1-Z_2)^{\frac{1}{2}}\right]\right\}}{\left\{\left[\frac{\alpha}{2}(1-Z_2)-\frac{7}{4}\right]Y_{-\frac{7}{2}}\left[2\Omega(1-Z_2)^{\frac{1}{2}}\right]-\Omega(1-Z_2)^{\frac{1}{2}})Y'_{-\frac{7}{2}}\left[2\Omega(1-Z_2)^{\frac{1}{2}}\right]\right\}}. \tag{C.17}$$

To avoid any further cumberson notation, we will designate the denominator in (C.14) as E and the numerator as F. Thus (C.14) reduces to

$$B = A\frac{F}{E}. \tag{C.18}$$

We can now use the expression for B in Equation (C.7) to obtain A. Hence,

$$A = \frac{EP_0}{J_{-\frac{7}{2}}(2\Omega)E + Y_{-\frac{7}{2}}(2\Omega)F}. \tag{C.19}$$

We now have all the constants to form the solutions from Equations (C.3) and (C.4) (The equation for C is determined by Equation (C.9).)

APPENDIX D

Determining Boundary Value Constants A, B, and C for Model 3

D.1 Determining Constants A,B, and C for Model 3

The environment for Model 3 is exactly the same as that for Model 2 with the exception that the source is now located at the top of the bottom layer i.e. on the boundary between the lower and upper layer. Hence, the pressure equation used for the 2nd model still applies here. Thus, we start with developing the correct values of the boundary values with the pressure solution repeated here

$$P = A\frac{J_\chi(2\Omega\tau)}{\tau^\chi} + B\frac{Y_\chi(2\Omega\tau)}{\tau^\chi}. \tag{D.1}$$

If P_1 and P_2 are the pressures in the 1st (bottom) and 2nd (top) layers, respectively, then boundary conditions for this model may be stated as follows:

$$P_1'\Big|_{Z=0} = 0 \quad \text{(D.2a)}$$

$$P_2|_{Z=Z_2} = P_0 \quad \text{(D.2b)}$$

$$P_1|_{Z=Z_2} = P_2|_{Z=Z_2}\,, \quad \text{(D.2c)}$$

where again prime (') denotes the derivative with respect to Z.

Taking the boundary conditions in order, we start with (D.2a). This we write as below:

$$A\left[\frac{J_{-\frac{7}{2}}(2\Omega(1-Z)^{\frac{1}{2}})}{(1-Z)^{\frac{-\frac{7}{2}}{2}}}\right]'\Bigg|_{Z=0} + B\left[\frac{Y_{-\frac{7}{2}}(2\Omega(1-Z)^{\frac{1}{2}})}{(1-Z)^{\frac{-\frac{7}{2}}{2}}}\right]'\Bigg|_{Z=0} = 0, \quad \text{(D.3)}$$

where τ has been expanded in terms of its definition in Z and the prime (') indicates derivative with respect to Z. Implementing the derivative, we have

$$\begin{aligned}&-\frac{7}{4}(1-Z)^{\frac{3}{4}}\left[AJ_{-\frac{7}{2}}\left(2\Omega\,[1-Z]^{\frac{1}{2}}\right)+B\,Y_{-\frac{7}{2}}\left(2\Omega\,[1-Z]^{\frac{1}{2}}\right)\right]\Bigg|_{Z=0}\\&+(1-Z)^{\frac{7}{4}}\left[A\left(-\Omega\,[1-Z]^{-\frac{1}{2}}\right)J'_{-\frac{7}{2}}+B\left(-\Omega\,[1-Z]^{-\frac{1}{2}}\right)Y'_{-\frac{7}{2}}\right]\Bigg|_{Z=0}=0.\end{aligned} \quad \text{(D.4)}$$

Solving (D.4) at $Z = 0$, we find

$$-\frac{7}{4}\left[AJ_{-\frac{7}{2}}\left(2\Omega\right)+B\,Y_{-\frac{7}{2}}\left(2\Omega\right)\right]+\left[A\left(-\Omega\right)J'_{-\frac{7}{2}}+B\left(-\Omega\right)Y'_{-\frac{7}{2}}\right]=0. \quad \text{(D.5)}$$

Collecting terms and solving for A in terms of B, we get for A:

$$A = -B\left[\frac{\frac{7}{4}Y_{-\frac{7}{2}}\left(2\Omega\right)+\Omega\,Y'_{-\frac{7}{2}}(2\Omega)}{\frac{7}{4}J_{-\frac{7}{2}}\left(2\Omega\right)+\Omega J'_{-\frac{7}{2}}(2\Omega)}\right]. \quad \text{(D.6)}$$

The next boundary condition (D.2b) is, perhaps, the simplest to solve

$$P_2|_{Z=Z_2} = P_0 = Ce^{-\left\{\frac{\alpha}{2}[Z-Z_2]\right\}} cos\left\{\frac{\alpha}{2}\sqrt{\left[\frac{4\Omega^2}{\alpha^2}-1\right]}[Z-Z_2]\right\}\Bigg|_{Z=Z_2}. \tag{D.7}$$

Evaluating (D.7) at $Z = Z_2$ yields the following equation for C

$$C = P_0. \tag{D.8}$$

Using (D.8) in the last boundary condition (D.2c), we have

$$P_0 = \left\{A\frac{J_{-\frac{7}{2}}(2\Omega\tau)}{\tau^{-\frac{7}{2}}} + B\frac{Y_{-\frac{7}{2}}(2\Omega\tau)}{\tau^{-\frac{7}{2}}}\right\}\Bigg|_{Z=Z_2}. \tag{D.9}$$

Substituting for τ at Z_2 and $\chi = -\frac{7}{2}$ and substituting for A from (D.6), we have

$$P_0 = B\left\{-\left[\frac{\frac{7}{4}Y_{-\frac{7}{2}}(2\Omega) + \Omega Y'_{-\frac{7}{2}}(2\Omega)}{\frac{7}{4}J_{-\frac{7}{2}}(2\Omega) + \Omega J'_{-\frac{7}{2}}(2\Omega)}\right]\frac{J_{-\frac{7}{2}}(2\Omega\tau(Z_2))}{[\tau(Z_2)]^{-\frac{7}{2}}} + \frac{Y_{-\frac{7}{2}}(2\Omega\tau(Z_2))}{[\tau(Z_2)]^{-\frac{7}{2}}}\right\}. \tag{D.10}$$

Now, B can be formally found as

$$B = \frac{P_0}{\left\{-\left[\frac{\frac{7}{4}Y_{-\frac{7}{2}}(2\Omega)+\Omega Y'_{-\frac{7}{2}}(2\Omega)}{\frac{7}{4}J_{-\frac{7}{2}}(2\Omega)+\Omega J'_{-\frac{7}{2}}(2\Omega)}\right]\frac{J_{-\frac{7}{2}}(2\Omega\tau(Z_2))}{[\tau(Z_2)]^{-\frac{7}{2}}} + \frac{Y_{-\frac{7}{2}}(2\Omega\tau(Z_2))}{[\tau(Z_2)]^{-\frac{7}{2}}}\right\}}. \tag{D.11}$$

Now that B is found in terms of all known quantities, A is determined and C has previously been found. Thus, all the constants are now determined for the 3rd Model.

APPENDIX E

Green's Function Derivation

E.1 Determining Boundary Value Constants A, B, C, and D for the Green's Function

As in the previous case for a harmonic source, we now must solve another boundary condition problem for an impulsive source. However, here we must follow the boundary conditions set forth in finding the Green's functions. Here, we are only concerned with the lower layer as the complete solution is found in the main text. Again, we are concerned with Bessel's equation as previously derived and restated below:

$$\tau^2 \frac{d^2 X}{d\tau^2} + \tau \frac{dX}{d\tau} + (4\Omega^2\tau^2 - \chi^2)X = 0. \tag{E.1}$$

This, again, has, as our starting point, the general solution:

$$X = AJ_\chi\left(2\Omega\sqrt{1-Z}\right) + BY_\chi\left(2\Omega\sqrt{1-Z}\right) \tag{E.2}$$

where A and B are constants to be determined by boundary conditions and the reader is reminded that $\tau = \sqrt{1 - \frac{z}{l}}$ and $Z = \frac{z}{l}$.

Now as is the nature of the Green's function method, we must solve the above equation (with different coefficients) in the region below and above the source location. Hence, the lower layer is split by the location of the source into 2 layers in which Equation (E.2) is solved in each layer. Thus, we will state here the general boundary conditions which must be solved to determine constants A, B, C, and D. It should also be recognized that the boundary conditions at the top and bottom of the 1st layer are chosen to be homogeneous. In order to solve for the constants A, B, C, and D, 4 boundary conditions are necessary. These are now stated below where the subscript "1" refers to the layer below the source location and "2" refers to the layer above the source location. We'll refer to the source location as $Z = Z_1$.

$$X_1'\Big|_{Z=0} = 0 \tag{E.3a}$$

$$X_1\big|_{Z=Z_1} = X_2\big|_{Z=Z_1} \tag{E.3b}$$

$$X_1'\Big|_{Z=Z_1} - X_2'\Big|_{Z=Z_1} = \frac{1}{\tau_1^2} \tag{E.3c}$$

$$X_2\big|_{Z=Z_2} = 0. \tag{E.3d}$$

Since we will be needing the derivative of X, we will find this expression first. Using the chain rule on (E.2), we find the derivatives to be:

$$\frac{d}{dZ}\left[J_\chi\left(2\Omega\tau\right)\right] = -\frac{\Omega}{\tau} J_\chi'\left(2\Omega\tau\right) \tag{E.4a}$$

$$\frac{d}{dZ}\left[Y_\chi\left(2\Omega\tau\right)\right] = -\frac{\Omega}{\tau} Y_\chi'\left(2\Omega\tau\right). \tag{E.4b}$$

Using the expressions for the Bessel function and its derivatives in Equations (E.3a)–(E.3d), we have the following substitutions for the Green's functions:

$$1.\ \left\{A(-\frac{\Omega}{\tau})J'_\chi(2\Omega\tau) + B(-\frac{\Omega}{\tau})Y'_\chi(2\Omega\tau)\right\}\Big|_{Z=0} = 0 \tag{E.5a}$$

$$2.\ \{AJ_\chi(2\Omega\tau) + BY_\chi(2\Omega\tau)\}|_{Z=Z_1} = \{CJ_\chi(2\Omega\tau) + DY_\chi(2\Omega\tau)\}|_{Z=Z_1} \tag{E.5b}$$

$$\begin{aligned} 3.\ &\left\{C(-\frac{\Omega}{\tau})J'_\chi(2\Omega\tau) + D(-\frac{\Omega}{\tau})Y'_\chi(2\Omega\tau)\right\}\Big|_{Z=Z_1} \\ -&\left\{A(-\frac{\Omega}{\tau})J'_\chi(2\Omega\tau) + B(-\frac{\Omega}{\tau})Y'_\chi(2\Omega\tau)\right\}\Big|_{Z=Z_1} = \frac{1}{\tau^2}\Big|_{Z=Z_1} \end{aligned} \tag{E.5c}$$

$$4.\ \{CJ_\chi(2\Omega\tau) + DY_\chi(2\Omega\tau)\}|_{Z=Z_2} = 0. \tag{E.5d}$$

Solving (1) above in Equation (E.5a), we find

$$B = -A\frac{J'_\chi(2\Omega)}{Y'_\chi(2\Omega)} = -Ag \tag{E.6}$$

where here we are using lower case letters to represent the ratio of various quantities. This will be quite useful later on in effecting a solution as these expressions become more and more unwielding.

Next we rearrange the terms in (2) in Equation (E.5b) to get

$$[A - C] = [D - B]\,\frac{Y_\chi(2\Omega\tau(Z_1))}{J_\chi(2\Omega\tau(Z_1))}. \tag{E.7}$$

Rewriting Equation (E.5c) in (3) above, we have

$$-[A - C]\,J'_\chi(2\Omega\tau(Z_1)) + [D - B]\,Y'_\chi(2\Omega\tau(Z_1)) = \frac{-\tau}{\Omega\tau^2(Z_1)} = \frac{-1}{\Omega\tau(Z_1)}. \tag{E.8}$$

Substituting Equation (E.7) into Equation (E.8), one has

$$-[D - B]\,\frac{Y_\chi(2\Omega\tau(Z_1))}{J_\chi(2\Omega\tau(Z_1))}J'_\chi(2\Omega\tau(Z_1)) + [D - B]\,Y'_\chi(2\Omega\tau(Z_1)) = \frac{-1}{\Omega\tau(Z_1)}. \tag{E.9}$$

Factoring and multiplying by $J_\chi(2\Omega\tau(Z_1))$, one has

$$[D - B]\left\{-Y_\chi(2\Omega\tau(Z_1))J'_\chi(2\Omega\tau(Z_1)) + Y'_\chi(2\Omega\tau(Z_1))J_\chi(2\Omega\tau(Z_1))\right\} = \frac{-J_\chi(2\Omega\tau(Z_1))}{\Omega\tau(Z_1)}. \tag{E.10}$$

The term in braces is the Wronskian $[W]$ of $J_\chi(2\Omega\tau(Z_1))$ and $Y_\chi(2\Omega\tau(Z_1))$. That is,

$$\begin{aligned} &W[J_\chi(2\Omega\tau(Z_1)), Y_\chi(2\Omega\tau(Z_1))] \\ &= \left\{-Y_\chi(2\Omega\tau(Z_1))J'_\chi(2\Omega\tau(Z_1)) + Y'_\chi(2\Omega\tau(Z_1))J_\chi(2\Omega\tau(Z_1))\right\}. \end{aligned} \tag{E.11}$$

Thus, Equation (E.10) may be rewritten as

$$[D - B] = \frac{-J_\chi(2\Omega\tau(Z_1))}{W[J_\chi(2\Omega\tau(Z_1)), Y_\chi(2\Omega\tau(Z_1))]\,\Omega\tau(Z_1)} = e. \tag{E.12}$$

A similar process may be used with Equations (E.7) and (E.8) to solve for $[A - C$ resulting in:

$$[A - C] = \frac{-Y_\chi(2\Omega\tau(Z_1))}{W[J_\chi(2\Omega\tau(Z_1)), Y_\chi(2\Omega\tau(Z_1))]\,\Omega\tau(Z_1)} = f. \tag{E.13}$$

Finally, the last boundary condition in (4) of Equation (E.5d) becomes

$$D = -C\frac{J_\chi(2\Omega\tau(Z_2))}{Y_\chi(2\Omega\tau(Z_2))} = -Ch. \tag{E.14}$$

Equations (E.6), (E.12), (E.13), and (E.14) now form a reduced system of 4 equations for the 4 unknowns A, B, C, and D. These can be solved in terms of the lower case variables and the results are:

$$A = \left[\frac{e - fh}{g - h}\right] \tag{E.15a}$$

$$B = \left[\frac{e - fh}{h - g}\right] g \tag{E.15b}$$

$$C = \left[\frac{e - fg}{g - h}\right] \tag{E.15c}$$

$$D = \left[\frac{e - fg}{h - g}\right] h. \tag{E.15d}$$

Below we expand the expressions in Equations (E.15a)–(E.15d) in terms of their representative functions. Thus, we have

$$e = \frac{-J_\chi(2\Omega\tau(Z_1))}{W[J_\chi(2\Omega\tau(Z_1)),\, Y_\chi(2\Omega\tau(Z_1))]\,\Omega\tau(Z_1)} \tag{E.16a}$$

$$f = \frac{-Y_\chi(2\Omega\tau(Z_1))}{W[J_\chi(2\Omega\tau(Z_1)),\, Y_\chi(2\Omega\tau(Z_1))]\,\Omega\tau(Z_1)} \tag{E.16b}$$

$$g = \frac{J'_\chi(2\Omega)}{Y'_\chi(2\Omega)} \tag{E.16c}$$

$$h = \frac{J_\chi(2\Omega\tau(Z_2))}{Y_\chi(2\Omega\tau(Z_2))}. \tag{E.16d}$$

The constants written in full are shown below. However, we have omitted the 2Ω argument from the Bessel functions for brevity and replaced $\tau(Z_{1,2})$ with $Z_{1,2}$ for the same reason.

$$A = \frac{\frac{-J_\chi(Z_1)}{W[J_\chi(Z_1), Y_\chi(Z_1)]\Omega\tau(Z_1)} + \left\{\frac{Y_\chi(Z_1)}{W[J_\chi(Z_1), Y_\chi(Z_1)]\Omega\tau(Z_1)}\right\}\left\{\frac{J_\chi(Z_2)}{Y_\chi(Z_2)}\right\}}{\left[\frac{J'_\chi(2\Omega)}{Y'_\chi(2\Omega)}\right] - \left[\frac{J_\chi(Z_2)}{Y_\chi(Z_2)}\right]} \tag{E.17a}$$

$$B = \left\{\frac{\frac{-J_\chi(Z_1)}{W[J_\chi(Z_1), Y_\chi(Z_1)]\Omega\tau(Z_1)} + \left\{\frac{Y_\chi(Z_1)}{W[J_\chi(Z_1), Y_\chi(Z_1)]\Omega\tau(Z_1)}\right\}\left\{\frac{J_\chi(Z_2)}{Y_\chi(Z_2)}\right\}}{\left[\frac{J_\chi(Z_2)}{Y_\chi(Z_2)}\right] - \left[\frac{J'_\chi(2\Omega)}{Y'_\chi(2\Omega)}\right]}\right\}\left[\frac{J'_\chi(2\Omega)}{Y'_\chi(2\Omega)}\right] \tag{E.17b}$$

$$C = \frac{\frac{-J_\chi(Z_1)}{W[J_\chi(Z_1), Y_\chi(Z_1)]\Omega\tau(Z_1)} + \left\{\frac{Y_\chi(Z_1)}{W[J_\chi(Z_1)\, Y_\chi(Z_1)]\Omega\tau(Z_1)}\right\}\left\{\frac{J'_\chi(2\Omega)}{Y'_\chi(2\Omega)}\right\}}{\left[\frac{J'_\chi(2\Omega)}{Y'_\chi(2\Omega)}\right] - \left[\frac{J_\chi(Z_2)}{Y_\chi(Z_2)}\right]} \tag{E.17c}$$

$$D = \left\{\frac{\frac{-J_\chi(Z_1)}{W[J_\chi(Z_1)\, Y_\chi(Z_1)]\Omega\tau(Z_1)} + \left\{\frac{Y_\chi(Z_1)}{W[J_\chi(Z_1), Y_\chi(Z_1)]\Omega\tau(Z_1)}\right\}\left\{\frac{J'_\chi(2\Omega)}{Y'_\chi(2\Omega)}\right\}}{\left[\frac{J_\chi(Z_2)}{Y_\chi(Z_2)}\right] - \left[\frac{J'_\chi(2\Omega)}{Y'_\chi(2\Omega)}\right]}\right\}\left[\frac{J_\chi(Z_2)}{Y_\chi(Z_2)}\right] \tag{E.17d}$$

APPENDIX F

The Complementary Solution

F.1 The Complementary Solution

As with all differnetial equations, there is usually a particular solution and a complementary solution. The complementary solution is usually the most frequently encountered equation as it is homogeneous with regards to a source term. In this appendix, we solve for the complementary solution in the bottom layer. This will require again solving a boundary value problem in which 3 constants are to be determined. These we will name E, F, and H. Here F and H are the coefficients for the Bessel function of the First and Second Kind which constitute our Bessel solution and G is the constant for the exponentially damped function in the top layer (Appendix D, (C.4)). The 3 conditions which will determine the boundary value constants are:

1. Derivative of the Pressure at Z=0 is 0.
2. Continuity of Pressure at $Z = Z_2$.
3. Continuity of the Derivative of the Pressure at $Z = Z_2$.

As found previously, solutions in the lower medium require Bessel functions of the form:

$$X = EJ_\chi\left(2\Omega\tau\right) + FY_\chi\left(2\Omega\tau\right) \tag{F.1}$$

where E and F are the constants we seek. However, while (F.1) is the Bessel function solution, it does not represent the full pressure equation. For that we must divide Equation (F.1) by τ^χ . So the general pressure solution looks like

$$P_C = E\frac{J_\chi(2\Omega\tau)}{\tau^\chi} + F\frac{Y_\chi(2\Omega\tau)}{\tau^\chi} \tag{F.2}$$

where P_C represents the pressure corresponding to the complementary solution.

Following from Equation (F.2), we have

$$\frac{\partial P_C}{\partial Z} = \frac{\partial\tau^{-\chi}}{\partial Z}\left[EJ_\chi\left(2\Omega\tau\right) + FY_\chi\left(2\Omega\tau\right)\right] + \tau^\chi\frac{\partial}{\partial Z}\left[EJ_\chi\left(2\Omega\tau\right) + FY_\chi\left(2\Omega\tau\right)\right]. \tag{F.3}$$

In the interest of being clear in the derivation of these constants, we will use the following abbreviations for the pressure function and its derivatives:

$$J_\tau \equiv \frac{J_\chi}{\tau^\chi} \tag{F.4a}$$

$$dJ_\tau \equiv \frac{\partial}{dZ}(\frac{J_\chi}{\tau^\chi}) \tag{F.4b}$$

$$Y_\tau \equiv \frac{Y_\chi}{\tau^\chi} \tag{F.4c}$$

$$dY_\tau \equiv \frac{\partial}{dZ}(\frac{Y_\chi}{\tau^\chi}). \tag{F.4d}$$

Using this terminology and writing Equations (F.2) and (F.3) in terms of Equations (F.4a)–(F.4d), we have

$$P_C = EJ_\tau + F\,dY_\tau \tag{F.5}$$

and

$$\frac{\partial}{\partial Z}(P_C) = EdJ_\tau + FdY_\tau \tag{F.6}$$

where for convenient reference, the reader is reminded of the definitions below:

$$\frac{d}{dZ}(\frac{1}{\tau^\chi}) = \chi \frac{1}{\tau^{\chi+1}} \tag{F.7a}$$

$$\frac{d}{dZ}[J_\chi(2\Omega\tau)] = -\frac{\Omega}{\tau} J'_\chi(2\Omega\tau) \tag{F.7b}$$

$$\frac{d}{dZ}[Y_\chi(2\Omega\tau)] = -\frac{\Omega}{\tau} Y'_\chi(2\Omega\tau)\,. \tag{F.7c}$$

It should be noted that the complementary solution cannot be found on its own merit from the conditions listed previously. In particular, the 2nd condition is not complete. That is, one does not have a value for the pressure at Z_2 for the complementary equation. To remedy this shortcoming, one must consider the total pressure when considering conditions 2 and 3 above. The total pressure consists of the particular solution plus the complementary function. In our case, the particular solution is the Green's function found previously (see Appendix E). The Green's functions, or more appropriately the pressure function formed from the Green's functions, completes the conditions that are needed to find the complementary solution. To write this more clearly, we write the total pressure as

$$P_T = P_G + P_C \tag{F.8}$$

where P_T is the total pressure, P_G is the pressure derived from the Green's Function, and P_C is the complementary pressure function as shown in (F.5).

Applying the first boundary condition above to (F.6), one can write

$$\left.\frac{\partial}{\partial Z}(P_C)\right|_{Z=0} = \{EdJ_\tau + FdY_\tau\}|_{Z=0} = 0 \tag{F.9}$$

where, upon solving, we have

$$E = -F\frac{dY_\tau|_{Z=0}}{dJ_\tau|_{Z=0}}. \tag{F.10}$$

This equation was sufficient in relating the constants E and F, since the Green's function derivative (and, consequently, its corresponding pressure derivative) was already applied at Z=0. This was done in order to satisfy one of the homogeneous boundary conditions needed to find the Green's function.

Next, we apply the 2nd condition above for the total pressure at $Z = Z_2$. This can be expressed generically as

$$P_T|_{Z=Z_2} = \cancelto{0}{P_G|_{Z=Z_2}} + P_C|_{Z=Z_2}\,. \tag{F.11}$$

Here we see that the pressure term from the Green's function is 0 as this was the other homogeneous condition that was used in finding the Green's function coefficients. Hence, we are left with the following equation for the continuity of pressure at Z_2:

$$EJ_\tau|_{Z=Z_2} + F\ dY_\tau|_{Z=Z_2} = H \tag{F.12}$$

where the equation in the top layer at $Z = Z_2$ (Appendix D, (C.4)) evaluates to H.

Finally, from the 3rd condition above (which expresses the continuity of the derivative of the pressure), we have

$$\left.\frac{\partial P_T}{\partial Z}\right|_{Z=Z_2} = \left.\frac{\partial P_G}{\partial Z}\right|_{Z=Z_2} + \left.\frac{\partial P_C}{\partial Z}\right|_{Z=Z_2} = -\frac{\alpha}{2}G \tag{F.13}$$

where again we evaluated the expression for the top layer at $Z = Z_2$.

Substituting for the various derivatives in (F.13), we have

$$\left.\frac{\partial P_T}{\partial Z}\right|_{Z=Z_2} = CdJ_\tau|_{Z=Z_2} + DdY_\tau|_{Z=Z_2} + EdJ_\tau|_{Z=Z_2} + FdY_\tau|_{Z=Z_2} = -\frac{\alpha}{2}H. \tag{F.14}$$

Here it should be noted that the coefficients C and D are the coefficients from the Green's function pressure solution and they are evaluated at $Z = Z_1$.

Multiplying (F.12) by $\frac{\alpha}{2}$ and adding it to (F.14), we have

$$C|_{Z=Z_1}\, dJ_\tau|_{Z=Z_2} + D|_{Z=Z_1}\, dY_\tau|_{Z=Z_2} + E\left\{\frac{\alpha}{2}J_\tau + dJ_\tau\right\}\Big|_{Z=Z_2} + F\left\{\frac{\alpha}{2}Y_\tau + dY_\tau\right\}\Big|_{Z=Z_2} = 0. \tag{F.15}$$

Substituting for E from (F.10) in (F.15) and factoring, we find the expression for F to be:

$$F = \frac{C|_{Z=Z_1}\, dJ_\tau|_{Z=Z_2} + D|_{Z=Z_1}\, dY_\tau|_{Z=Z_2}}{\frac{dY_\tau}{dJ_\tau}\Big|_{Z=0}\left\{\frac{\alpha}{2}J_\tau + dJ_\tau\right\}\big|_{Z=Z_2} - \left\{\frac{\alpha}{2}Y_\tau + dY_\tau\right\}\big|_{Z=Z_2}}. \tag{F.16}$$

Once F is known, the expression for E is found to be:

$$E = -\frac{C|_{Z=Z_1}\, dJ_\tau|_{Z=Z_2} + D|_{Z=Z_1}\, dY_\tau|_{Z=Z_2}}{\frac{dY_\tau}{dJ_\tau}\Big|_{Z=0}\left\{\frac{\alpha}{2}J_\tau + dJ_\tau\right\}\big|_{Z=Z_2} - \left\{\frac{\alpha}{2}Y_\tau + dY_\tau\right\}\big|_{Z=Z_2}}\,\frac{dY_\tau|_{Z=0}}{dJ_\tau|_{Z=0}}. \tag{F.17}$$

Now that E and F are found, H is easily found from (F.12) which is repeated here for convenience:

$$H = E J_\tau|_{Z=Z_2} + F\, dY_\tau|_{Z=Z_2}\,. \tag{F.18}$$

All of the constants are now known for P_G and P_C, hence, the total pressure function, P_T can be calculated and studied.

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List of Publications

1. Yoerger, Edward J. and McDaniel, Suzanne T. (1996), *Vertical Spatial Coherence for a Transient Signal Forward-Scattered from the Sea Surface*, IEEE Journal of Oceanic Engineering, Vol. 21, No.1, pages 24-36.

2. Yoerger, Edward J. and Puri, Ashok (To be submitted), Vertical Acoustic Propagation in a Non-Homogeneous Media for a Time-harmonic Source.

3. Yoerger, Edward J. and Puri, Ashok (To be submitted),Acoustic Response in a Non-Homogeneous, Layered Atmosphere for a Time-Harmonic, Distributed Source.

VITA

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