The Cohomology of Right Angled Artin Groups with Group Ring Coefficients

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Recommended Citation
THE COHOMOLOGY OF RIGHT ANGLED ARTIN GROUPS
WITH GROUP RING COEFFICIENTS

C. JENSEN AND J. MEIER

Abstract
We give an explicit formula for the cohomology of a right angled Artin group with group ring coefficients in terms of the cohomology of its defining flag complex.

1. Introduction
Let \( \Gamma \) be a finite simplicial graph and let \( \hat{\Gamma} \) be the induced flag complex, i.e., the maximal simplicial complex whose 1-skeleton is \( \Gamma \). The associated right angled Artin group \( A_\Gamma \) is the group presented by
\[
A_\Gamma = \langle V(\Gamma) \mid vw = wv \text{ if } \{v, w\} \in E(\Gamma) \rangle.
\]
Because one can import topological properties of the associated flag complex \( \hat{\Gamma} \) into the group \( A_\Gamma \), these groups have provided important examples of exotic behavior. (See for example [1], [4] and [12].) Here we refine the understanding of the end topology of right angle Artin groups by giving an explicit formula for the cohomology of \( A_\Gamma \) with group ring coefficients in terms of the cohomology of \( \hat{\Gamma} \) and links of simplices in \( \hat{\Gamma} \).

Definition 1.1. If \( K \) is a simplicial complex let \( S(K) \) denote the set of closed simplices — including the empty simplex — in \( K \). The dimension of a simplex is denoted \( |\sigma| \); the link is denoted \( \text{Lk}(\sigma) \); the star of \( \sigma \) is \( \text{St}(\sigma) \). By definition \( |\emptyset| = -1 \) and \( \text{Lk}(\emptyset) = K \).

Main Theorem. Let \( \Gamma \) be a finite simplicial graph, let \( \hat{\Gamma} \) be the associated flag complex and \( A_\Gamma \) the associated right angled Artin group. As long as \( \hat{\Gamma} \) is not a single simplex,
\[
H^*(A_\Gamma, \mathbb{Z} A_\Gamma) = \bigoplus_{\sigma \in S(\hat{\Gamma})} \left( \bigoplus_{l=1}^{\infty} \mathbb{P}^{-|\sigma|-2} \text{Lk}(\sigma) \right).
\]
If \( \hat{\Gamma} \) is a single simplex then \( A_\Gamma \) is free abelian and \( H^*(A_\Gamma, \mathbb{Z} A_\Gamma) \) is simply \( \mathbb{Z} \) in top dimension.

2000 Mathematics Subject Classification 20F36 (primary), 57M07 (secondary).
Jensen thanks the Louisiana Board of Regents for a Research Competitiveness Subprogram grant.
Meier thanks the American Mathematical Society for the support of a Centennial Research Fellowship and Columbia University for hosting him.
Last revised on 11 December 2003.
Example 1.2. Let \( \hat{\Gamma} \) be \( \mathbb{R}P^2 \). Then the reduced cohomology of \( \text{Lk}(\emptyset) = \mathbb{R}P^2 \) is concentrated in dimension 2 where it is \( \mathbb{Z}_2 \). The link of any other simplex \( \sigma \) is a \((1 - |\sigma|)\)-sphere hence its reduced cohomology is concentrated in dimension \((1 - |\sigma|)\), where it is \( \mathbb{Z}_2 \). Thus \( H^*(A_{\hat{\Gamma}}, \mathbb{Z}_2 \Gamma) \) is trivial except in dimension 3 where it is the sum of a countably generated free abelian group and a countable sum of \( \mathbb{Z}_2 \)'s.

There are at least two approaches to establishing the Main Theorem. One can modify the techniques of [9] that were developed for computing the cohomology of Coxeter groups with group ring coefficients — as well as the cohomology with compact supports of any locally finite building — to compute this cohomology for right angled Artin groups. In fact, the formula given in the Main Theorem is quite similar to the formulas for cohomology with compact supports of locally finite buildings (Theorem 5.8 in [9]). We take a more efficient route, and use the fact that right angled Artin groups are commensurable with certain right angled Coxeter groups [8], and appeal to the formula for the cohomology of a right angled Coxeter group with group ring coefficients ([7] or [9]).

In the last section we explain how the formula of the Main Theorem extends results of [4] on the end topology of right angled Artin groups.

2. Background and Definitions

One of the classical approaches to the study of asymptotic properties of a group \( G \) is via its cohomology with \( \mathbb{Z}G \)-coefficients. For example, from Proposition 7.5 and Exercise 4 of [5], if \( G \) is a discrete group and \( X \) is a contractible \( G \)-complex with finite cell stabilizers and finite quotient, then

\[
H^*(G, \mathbb{Z}G) \cong H^c_*(X; \mathbb{Z}),
\]

where \( H^*_c(X; \mathbb{Z}) \) is the cohomology of \( X \) with compact supports. In particular, one can take as \( X \) either of the classifying spaces \( EG \) or \( \overline{EG} \) provided they have finite quotients \( BG \) or \( \overline{BG} \) (cf. [11]). Cohomology with group ring coefficients determines the cohomological dimension of \( G \) [5, VIII.6.7]: If \( G \) is of type FP then

\[
\text{cd } G = \max \{ n : H^n(G, \mathbb{Z}G) \neq 0 \}.
\]

It is also closely related to connectivity at infinity and duality properties as is described at the end of the next section.

Definition 2.1. Right angled Artin groups admit \( \text{CAT}(0) \ K(\pi, 1) \)'s formed as the union of tori. If \( \Gamma \) is a finite simplicial graph, let \( K_{\Gamma} \) be the complex formed by joining tori in the manner described by the flag complex \( \hat{\Gamma} \). That is, for each simplex \( \sigma \subset S(\hat{\Gamma}) \), let \( T_{\sigma} \) be the torus formed by identifying parallel faces of a unit \((|\sigma| + 1)\)-cube. (The torus \( T_{\emptyset} \) is a single vertex.) The complex \( K_{\Gamma} \) is then the union of these tori, subject to \( T_{\sigma} \cap T_{\sigma'} = T_{\sigma''} \) when \( \sigma \cap \sigma' = \sigma'' \) in \( \hat{\Gamma} \). For a proof that these \( K_{\Gamma} \)'s are \( \text{CAT}(0) \) classifying spaces, see [13]. We denote the universal cover of \( K_{\Gamma} \) by \( \tilde{K}_{\Gamma} \).

The complex \( \tilde{K}_{\Gamma} \) is also the Davis complex for an appropriate right angled Coxeter group. Given a finite simplicial graph \( \Gamma \) the right angled Coxeter group \( C_{\Gamma} \) is the
quotient of $A_\Gamma$ formed by declaring that each generator is an involution

$$C_\Gamma = \langle V(\Gamma) \mid vw = wv \text{ if } \{v, w\} \in E(\Gamma) \text{ and } v^2 = 1 \text{ for all } v \rangle.$$  

For a finite simplicial graph $\Gamma$ let $\Gamma'$ be the graph whose vertices are given by $V(\Gamma) \times \{-1, 1\}$ where

$$\{(v, \epsilon), (w, \epsilon)\} \in E(\Gamma') \iff \{v, w\} \in E(\Gamma)$$

and

$$\{(v, \epsilon), (w, -\epsilon)\} \in E(\Gamma') \iff v \neq w$$

for $\epsilon = 1$ or $-1$.

**Theorem 2.2** (Davis-Januszkiewicz [8]). The Artin group $A_\Gamma$ and the Coxeter group $C_\Gamma'$ are commensurable and in fact the complexes $\tilde{K}_\Gamma$ and the Davis complex for $C_\Gamma'$ are identical.

(Because $\tilde{K}_\Gamma$ is the Davis complex for $C_\Gamma'$ we do not actually define the Davis complex for a Coxeter group; see [8] for a definition.)

One can now derive a formula for $H^*(A_\Gamma, \mathbb{Z}A_\Gamma)$ from known results in the literature. Namely, because

1. cohomology with group ring coefficients can be expressed in terms of cohomology with compact supports of an $EG$, and
2. $\tilde{K}_\Gamma$ is both an $EA_\Gamma$ and an $EC_\Gamma'$, and
3. the cohomology of a Coxeter group with group ring coefficients has been computed, and can be expressed in terms of the cohomology of subcomplexes of links of vertices in the Davis complex ([7] or [9]),

we have the following formula for the cohomology of $A_\Gamma$ with $\mathbb{Z}A_\Gamma$ coefficients.

**Corollary 2.3.** Each $w \in C_\Gamma$ has an associated simplex $\sigma(w) \in S(\hat{\Gamma}')$ such that

$$H^*(A_\Gamma, \mathbb{Z}A_\Gamma) = H^*(C_\Gamma', \mathbb{Z}C_\Gamma') = H^*_c(\tilde{K}_\Gamma, \mathbb{Z}) = \bigoplus_{w \in C_\Gamma'} \prod_{i=1}^{\text{order of } w} (\hat{\Gamma}' - \sigma(w)).$$

Each simplex $\sigma \in S(\hat{\Gamma}') \setminus \{\emptyset\}$ occurs countably many times in this sum, while $\sigma = \emptyset$ occurs exactly once.

Although the formula above is correct, it obfuscates the connection between $H^*(A_\Gamma, \mathbb{Z}A_\Gamma)$ and the cohomology of the flag complex $\hat{\Gamma}$. As a first step toward expressing the right hand side in terms of the flag complex $\hat{\Gamma}$, we give an alternate description of the flag complex $\hat{\Gamma}'$.

For each $v \in V(\Gamma)$ let $\hat{\Gamma}_v$ be the full subcomplex of $\hat{\Gamma}$ induced by the vertices $V(\Gamma) \setminus \{v\}$. Thus $\hat{\Gamma}_v$ is a deformation retract of $\hat{\Gamma}$ with the vertex $v$ removed.

Let $(W, V(\Gamma))$ be the Coxeter system where $W$ is abelian and the generating set has been identified with the vertices of the graph $\Gamma$. Hence $W$ is simply

$$W = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2.$$

Let the $\hat{\Gamma}_v$ be a set of mirrors related to this Coxeter system and form the associated $W$-complex in the following manner. For each $x \in \hat{\Gamma}$ let $W_x$ be the subgroup of $W$ generated by the set of $v \in V(\Gamma)$ such that $x$ belongs to $\hat{\Gamma}_v$. In other words, $W_x$ is
generated by those $v$ such that $x$ is not in the open neighborhood of $v$ in $\hat{\Gamma}$. Define

$$L_{\Gamma} = W \times \hat{\Gamma} / \sim$$

where $(w, x) \sim (v, y)$ if and only if $x = y$ and $w^{-1}v \in W_x$.

The complex $\hat{\Gamma}'$ shows up in the formula of Corollary 2.3 because it is isomorphic to the link of any vertex in $\tilde{K}_\Gamma$. One can find the following result in [8].

**Lemma 2.4.** The complex $\hat{\Gamma}'$ is isomorphic to $L_\Gamma$, and is isomorphic to the link of the vertex in $K_\Gamma$.

If $\sigma \in S(\hat{\Gamma})$ one can form a subcomplex $L_{\sigma} \subset L_{\hat{\Gamma}}$ by defining $W_\sigma$ to be the subgroup of $W$ generated by $\{v \in V(\Gamma) \mid v \notin \sigma\}$, and forming $W_\sigma \times \hat{\Gamma} / \sim$ where as before $(w, x) \sim (v, y)$ if and only if $x = y$ and $w^{-1}v \in W_x$. In particular, if $\sigma = \emptyset$ (the empty simplex) then $L_\emptyset = L_{\hat{\Gamma}}$.

**Example 2.5.** Let $\Gamma = \hat{\Gamma}$ be the simplicial arc indicated in Figure 1. The group $W$ is then generated by four elements associated with the vertices. Switching to Greek letters we denote these generators as $\alpha, \beta, \gamma$ and $\delta$, where the mirror associated to $\alpha$ is the subgraph induced by $\{b, c, d\}$, and similarly for the other three generators. The complex $L_{\Gamma}$ is then as is indicated in Figure 1. The generator $\alpha$ acts on $L_{\Gamma}$ by exchanging the vertices labeled $a$ and $\alpha(a)$, and leaves all other vertices fixed. Similarly $\beta$ exchanges $b$ and $\beta(b)$, fixing all other vertices, and so on. Finally, if $\sigma = \{b, c\}$ then $L_{\sigma}$ is the bottom complex in Figure 1.

![Figure 1. A defining graph $\Gamma$, the associated complex $L_{\Gamma}$, and subcomplex $L_{\{b,c\}}$.](image)
For any $\sigma \in S(\hat{\Gamma}) \setminus \{\emptyset\}$ let

$$\hat{\Gamma}_{b(\sigma)} = \bigcup_{v \in \sigma^{(0)}} \hat{\Gamma}_v,$$

so that $\hat{\Gamma}_{b(\sigma)}$ is a deformation retract of $\hat{\Gamma}$ with the barycenter of $\sigma$ removed.

**Lemma 2.6.** The cohomology groups of $L_\sigma$ are given by

$$H^*(L_\sigma) = \bigoplus_{\tau \in S(\hat{\Gamma} - \sigma)} H^*(\hat{\Gamma}, \hat{\Gamma}_{b(\tau)}),$$

where in a small abuse of notation we let $S(\hat{\Gamma} - \sigma)$ denote all closed simplices of $\hat{\Gamma}$ except those with non-empty intersection with $\sigma$.

**Proof.** In [6] Mike Davis gives a formula for the homology of a complex on which a Coxeter group acts. One can switch this to a formula for cohomology using universal coefficients, or via a minor rewriting of Davis’s original argument. In our case the formula is rather simple. Since $W_\sigma$ is abelian, each $w \in W_\sigma$ is determined by the set of generators $S(w)$ that are necessary to express $w$. Temporarily following Davis’s notation, define

$$\hat{\Gamma}^{S(w)} = \bigcup_{v \in S(w)} \hat{\Gamma}_v.$$

(If $w = 1$ then $S(w) = \emptyset$ and so $\hat{\Gamma}^{S(w)}$ is empty as well.) Davis’s formula then gives

$$H^*(L_\sigma) \simeq \bigoplus_{w \in W_\sigma} H^*(\hat{\Gamma}, \hat{\Gamma}^{S(w)}).$$

This can be simplified. If $S(w)$ is not the vertex set of a simplex in $\hat{\Gamma}$, then $\hat{\Gamma}^{S(w)} = \hat{\Gamma}$; if $S(w) = \sigma^{(0)}$ for some $\sigma \in S(\hat{\Gamma})$, then $\hat{\Gamma}^{S(w)} = \hat{\Gamma}_{b(\sigma)}$. Thus the formula above can be rewritten as

$$H^*(L_\sigma) \simeq \bigoplus_{\tau \in S(\hat{\Gamma} - \sigma)} H^*(\hat{\Gamma}, \hat{\Gamma}_{b(\tau)}).$$

$\square$

3. **Proof of the Main Theorem**

From Corollary 2.3 we know that

$$H^*(A_{\hat{\Gamma}}, \mathbb{Z}A_{\hat{\Gamma}}) = \mathcal{H}^{-1}(\hat{\Gamma}', \bigoplus_{\sigma \in S(\hat{\Gamma}') \setminus \{\emptyset\}} \left[ \bigoplus_{i=1}^{\infty} \mathcal{H}^{-1}(\hat{\Gamma}' - \sigma) \right]$$

where we know there are infinitely many copies of $\mathcal{H}^{-1}(\hat{\Gamma}' - \sigma)$ since by its construction there are no non-trivial finite conjugacy classes in $C_{\hat{\Gamma}'}$. To arrive at our Main Theorem we need a formula for $\mathcal{H}^{-1}(\hat{\Gamma}' - \sigma)$ where $\sigma$ is any simplex in $S(\hat{\Gamma}')$. Thus our key lemma is:
Lemma 3.1. Let $\sigma \in \mathcal{S}(\hat{\Gamma}')$. Then $\hat{\Gamma}' - \sigma$ is homotopy equivalent to $L_\sigma$ and

$$\bar{H}^*(L_\sigma) = \bigoplus_{\tau \in \mathcal{S}(\hat{\Gamma}-\sigma)} \bar{H}^{*|\tau| - 1}(\text{Lk}(\tau)) .$$

Proof. The complex $\hat{\Gamma}$ embeds in $\hat{\Gamma}'$ in a number of ways. Let the standard embedding $\hat{\Gamma} \hookrightarrow \hat{\Gamma}'$ have image the subcomplex induced by $\{(v, 1) \mid v \in V(\Gamma)\}$. Define $\hat{\Gamma}'^{\text{op}}$ to be the subcomplex induced by $\{(v, -1) \mid v \in V(\Gamma)\}$. If $\sigma$ is a simplex in $\mathcal{S}(\hat{\Gamma}')$ then $\sigma$ is defined by a set of vertices in $\Gamma$ along with choices of $\pm 1$. If

$$\sigma \sim \{(a, 1), (b, 1), \ldots, (c, 1), (x, -1), \ldots, (y, -1), (z, -1)\}$$

then the automorphism $\alpha \beta \cdot \gamma$ takes $\sigma$ to the simplex

$$\sigma' \sim \{(a, -1), (b, -1), \ldots, (c, -1), (x, -1), \ldots, (y, -1), (z, -1)\} .$$

(Here we have used the same convention on naming generators of $W$ as in Example 2.5.) Thus in discussing the topology of $\hat{\Gamma}' - \sigma$ for $\sigma \in \mathcal{S}(\hat{\Gamma}')$, we may without loss of generality assume $\sigma \subset \hat{\Gamma}'^{\text{op}} \subset \hat{\Gamma}'$. But the space formed by removing the closed simplex $\sigma \subset \hat{\Gamma}'^{\text{op}}$ from $\hat{\Gamma}'$ deformation retracts onto the subcomplex formed by making all possible reflections of $\hat{\Gamma}$ that do not involve the generators of $W$ that correspond to vertices of $\sigma$. In other words, $\hat{\Gamma}' - \sigma$ deformation retracts onto $L_\sigma$, which implies our first claim.

From Lemma 2.6 we know $\bar{H}^{*}(L_\sigma) = \bigoplus_{\tau \in \mathcal{S}(\hat{\Gamma}-\sigma)} \bar{H}^{*}(\hat{\Gamma}, \hat{\Gamma}_{b(\tau)})$, thus it suffices to establish

$$\bar{H}^{*}(\hat{\Gamma}, \hat{\Gamma}_{b(\tau)}) \simeq \bar{H}^{*-|\tau| - 1}(\text{Lk}(\tau)) .$$

First, if $\tau = \emptyset$, $\text{Lk}(\tau) = \hat{\Gamma}$ and $|\tau| = -1$, so we get $\bar{H}^{*-|\tau| - 1}(\text{Lk}(\tau)) = \bar{H}^{*}(\hat{\Gamma})$. If $\tau \neq \emptyset$ then by excision, $\bar{H}^{*}(\hat{\Gamma}, \hat{\Gamma}_{b(\tau)}) \simeq \bar{H}^{*}(\text{St}(\tau), S^{[\tau]}[\text{Lk}(\tau)])$ where $\text{St}(\tau)$ is the closed star of $\tau$ and $S^{[\tau]}$ denotes the $i^{th}$ suspension. Because the star $\text{St}(\tau)$ is contractible, the long exact sequence in cohomology shows

$$\bar{H}^{*}(\text{St}(\tau), S^{[\tau]}[\text{Lk}(\tau)]) = \bar{H}^{*-1}(S^{[\tau]}[\text{Lk}(\tau)]) .$$

But the cohomology of a suspension is just a shifted copy of the cohomology of the original complex

$$\bar{H}^{*-1}(S^{[\tau]}[\text{Lk}(\tau)]) = \bar{H}^{*-|\tau| - 1}(\text{Lk}(\tau))$$

and the result follows. \qed

Example 3.2. In Example 2.5 we considered $\Gamma = \hat{\Gamma} = \text{a simplicial arc, and two associated complexes, } L_\Gamma$ and $L_{\{b,c\}}$. (The first claim of Lemma 3.1 states that $L_{\{b,c\}}$ is homotopy equivalent to $L_\Gamma$ with the closed edge $\{\beta(b), \gamma(c)\}$ removed.) The formula of Lemma 3.1 says, for example, that

$$\bar{H}^1(L_\Gamma) = \bigoplus_{\sigma \in \mathcal{S}(\hat{\Gamma})} \bar{H}^{1-|\sigma|-1}(\text{Lk}(\sigma)) = \bigoplus_{\sigma \in \mathcal{S}(\hat{\Gamma})} \bar{H}^{-|\sigma|}(\text{Lk}(\sigma)) .$$

This then becomes

$$H^1(L_\Gamma) = \bar{H}^1(\text{Lk}(\emptyset)) \oplus \bar{H}^0(\text{Lk}(a)) \oplus \bar{H}^0(\text{Lk}(b)) \oplus \bar{H}^0(\text{Lk}(c)) \oplus \bar{H}^0(\text{Lk}(d)) \oplus \bar{H}^{-1}(\text{Lk}(\{a,b\})) \oplus \bar{H}^{-1}(\text{Lk}(\{b,c\})) \oplus \bar{H}^{-1}(\text{Lk}(\{c,d\}))$$
\[ H^*(A_\Gamma, \mathbb{Z}A_\Gamma) = H^1(\hat{\Gamma}) \oplus \left( H^0(\bullet) \right)^2 \oplus \left( H^0(\bullet \bullet) \right)^2 \oplus \left( H^{-1}(\emptyset) \right)^3 = \mathbb{Z}^5, \]

using the convention that $H^{-1}(\emptyset) = \mathbb{Z}$.

In the case of $L_{\{b,c\}}$ one drops all the terms involving $b$ or $c$, which are precisely the non-trivial terms above, hence $H^1(L_{\{b,c\}}) = 0$.

We can now prove our Main Theorem.

**Theorem 3.3.** Let $\Gamma$ be a finite simplicial graph, let $\hat{\Gamma}$ be the associated flag complex and $A_\Gamma$ the associated right angled Artin group. As long as $\hat{\Gamma}$ is not a single simplex,

\[ H^*(A_\Gamma, \mathbb{Z}A_\Gamma) = \bigoplus_{\sigma \in \mathcal{S}(\hat{\Gamma})} \left[ \bigoplus_{i=1}^{\infty} \mathbb{H}^{-|\sigma|-2}(Lk(\sigma)) \right]. \]

If $\hat{\Gamma}$ is a single simplex then $A_\Gamma$ is free abelian and $H^*(A_\Gamma, \mathbb{Z}A_\Gamma)$ is simply $\mathbb{Z}$ in top dimension.

**Proof.** From Corollary 2.3 we have

\[ H^*(A_\Gamma, \mathbb{Z}A_\Gamma) = \bigoplus_{w \in C_{\hat{\Gamma}'}} \mathbb{H}^{-1}(\hat{\Gamma} - \sigma(w)). \]

By Lemma 3.1 this gives

\[ H^*(A_\Gamma, \mathbb{Z}A_\Gamma) = \bigoplus_{w \in C_{\hat{\Gamma}'}} \left[ \bigoplus_{\tau \in \mathcal{S}(\hat{\Gamma} - \sigma(w))} \mathbb{H}^{-|\tau|-2}(Lk(\tau)) \right]. \]

If $\hat{\Gamma}$ is not a single simplex, then each $\tau \in \mathcal{S}(\hat{\Gamma})$ will show up in the product inside the square brackets for infinitely many $w \in C_{\hat{\Gamma}'}$, and the formula in the theorem follows.

On the other hand, if $\hat{\Gamma}$ is a single simplex $\sigma$, then $\sigma$ only occurs in the summand corresponding to $1 \in C_{\hat{\Gamma}'}$. All other simplices occur infinitely often, but if $\tau \neq \sigma$ then $Lk(\tau)$ is contractible, and $\mathbb{H}^{-2}(Lk(\tau))$ is zero. Thus $H^*(A_\Gamma, \mathbb{Z}A_\Gamma) = H^{*-|\sigma|-2}(\emptyset)$, consistent with the fact that $A_\Gamma = \mathbb{Z}|\sigma|+1$. \qed

As was alluded to in the previous section, cohomology with group ring coefficients is closely related to asymptotic properties. A group $G$ that admits a finite $K(G,1)$ is \textit{n-acyclic at infinity} if roughly speaking, complements of compact sets in the universal cover have trivial homology through dimension $n$ (see [9] for a precise definition.) It was from this perspective that Brady and Meier determined when a right angled Artin group was \textit{n-acyclic at infinity}. Their approach was via a combinatorial Morse theory argument using the $K_\Gamma$ complexes. However, there is an algebraic characterization that says a group $G$ is $n$-acyclic at infinity if and only if $H^i(G, \mathbb{Z}G) = 0$ for $i \leq n+1$ and $H^{n+2}(G, \mathbb{Z}G)$ is torsion-free (see [10]). The group $G$ is an \textit{n-dimensional duality group} if there is a dualizing module $D$ such that $H_i(G, M) \simeq H^{n-i}(G, M \otimes D)$ for all $i$ and all $G$-modules $M$. This too can be recast in terms of cohomology with group ring coefficients: $G$ is an $n$-dimensional duality group if its cohomology with group ring coefficients is torsion-free and con-
centrated in dimension $n$ [2]. Thus our Main Theorem implies three results of [4]. It is important to remember that $\emptyset \in S(\hat{\Gamma})$, and the formal dimension of $\emptyset$ is $-1$.

**Corollary 3.4** (Prop. 4.1 in [4]). A right angled Artin group $A_\Gamma$ is $n$-acyclic at infinity if and only if for all $\sigma \in S(\hat{\Gamma})$, $Lk(\sigma)$ is $(n-|\sigma|-1)$-acyclic.

**Proof.** Since $Lk(\sigma)$ is $(n-|\sigma|-1)$-acyclic it follows by universal coefficients that its cohomology is trivial up to dimension $n-|\sigma|-1$ and that $H^{n-|\sigma|}(Lk(\sigma))$ is zero for $i \leq n+1$ and $H^{n+2}(A_\Gamma, \mathbb{Z}A_\Gamma)$ is torsion-free.

**Corollary 3.5** (Theorem C in [4]). A right angled Artin group $A_\Gamma$ is a duality group if and only if $\hat{\Gamma}$ is Cohen-Macaulay.

**Proof.** A simplicial complex $K$ is Cohen-Macaulay if for any simplex $\sigma \in S(K)$, the cohomology of $Lk(\sigma)$ is concentrated in top dimension (and is torsion free). It follows from the formula of the Main Theorem that $H^\ast(A_\Gamma, \mathbb{Z}A_\Gamma)$ is torsion free and concentrated in top dimension if and only if $\hat{\Gamma}$ is Cohen-Macaulay.

Recall that an $n$-dimensional duality group is called a Poincaré duality group if and only if $H^n(G, \mathbb{Z}G) = \mathbb{Z}$ [3]. After the statement of Theorem C in [4] it was remarked that a Theorem of Strebel combined with Theorem C implies that a right angled Artin group $A_\Gamma$ is a Poincaré duality group if and only if $A_\Gamma$ is free abelian. This characterization follows directly from the formula in our Main Theorem.

**Corollary 3.6.** A right angled Artin group $A_\Gamma$ is a Poincaré duality group if and only if it is free abelian.

**Proof.** The Main Theorem implies that $H^n(A_\Gamma, \mathbb{Z}A_\Gamma)$ is not finitely generated — in particular it is not equal to $\mathbb{Z}$ — unless $\hat{\Gamma}$ is a simplex and hence $A_\Gamma$ is free abelian.

**References**

\[ H^*(A_r, \mathbb{Z}A_r) \]


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