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PROPER ACTIONS OF AUTOMORPHISM GROUPS OF FREE PRODUCTS OF FINITE GROUPS

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ABSTRACT. If G is a free product of finite groups, let $\Sigma Aut_1(G)$ denote all (necessarily symmetric) automorphisms of G that do not permute factors in the free product. We show that a McCullough-Miller and Gutiérrez-Krstić derived (also see Bogley-Krstić) space of pointed trees is an $\underline{E}\Sigma Aut_1(G)$-space for these groups.

1. INTRODUCTION

We remind the reader (see [8, 11]) that if G is a discrete group, then the contractible G-space $\underline{E}G$ is characterized (up to G-equivariant homotopy) by the property that if H is any subgroup of G then the fixed point subcomplex $\underline{E}G^H$ is contractible if H is finite and empty if H is infinite. These spaces are basic tools in studying the geometry of the group G.

Recall that if G is a free product of n groups $G_1, ..., G_n$, the *symmetric automorphism group* $\Sigma Aut(G)$ of G consists of all automorphisms which send each G_i to a conjugate of some G_j. In this paper, we will assume that each G_i is finite so that $Aut(G) = \Sigma Aut(G)$ by the Kurosh subgroup theorem. We will construct an $\underline{E}\Sigma Aut_1(G)$-space $L(G)$ based on McCullough-Miller's [10] space of trees, which uses rooted trees similar to those found in Gutiérrez-Krstić [6]. Here $\Sigma Aut_1(G)$ is the kernel of the projection $\Sigma Aut(G) \to \Sigma_n$. We will show:

Theorem 1.1. *$L(G)$ is an $\underline{E}\Sigma Aut_1(G)$-space. That is, $L(G)$ is a contractible space which $\Sigma Aut_1(G)$ acts on with finite stabilizers and finite quotient. Moreover, if F is a finite subgroup of $\Sigma Aut_1(G)$, then the fixed point subcomplex $L(G)^F$ is contractible.*

We conjecture that the space $L(G)$ is in fact an $\underline{E}\Sigma Aut(G)$-space in addition to being an $\underline{E}\Sigma Aut_1(G)$-space. We pause to note a few other related papers. In [4] Collins and Zieschang establish the peak reduction methods that underly all of the contractibility arguments here. Gilbert [5] further refines these methods and gives a presentation for $\Sigma Aut(G)$. In [10], McCullough and Miller provide a comprehensive work about symmetric automorphism groups of free products and define McCullough-Miller space. In [3] Bridson and Miller show that every finite subgroup of $\Sigma Aut_1(G)$ fixes a point of McCullough-Miller space $K_0(G)$. In [1] Bogley and Krstić completely calculate the cohomology of $\Sigma Aut(F_n)$. Brady, McCammond, Meier, and Miller [2] use McCullough-Miller space to show that $\Sigma Aut(F_n)$ is a duality group.

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The paper is structured as follows. In section 2, we define the space $L(G)$ and in section 3 we show it is contractible using a standard norm. In section 4 we briefly examine stabilizers of points in $L(G)$. In section 5, we develop many new norms, each of which can be used to show $L(G)$ is contractible. In section 6, we classify fixed point subcomplexes $L(G)^F$ where F is a finite subgroup of $\Sigma Aut_1(G)$ and in section 7 we show that these subcomplexes are contractible.

2. Preliminaries

If $G = G_1 * G_2 * \cdots * G_n$, set $\mathcal{J} = \{G_1, \ldots, G_n\}$ and $\mathcal{J}^0 = \{*, G_1, \ldots, G_n\}$. For each i, choose a $\lambda_i \in G_i - \{1\}$. Let $\mathcal{P}(\mathcal{J}^0)$ be the Whitehead poset constructed in [10]. Elements of $\mathcal{P}(\mathcal{J}^0)$ correspond to labelled bipartite trees, where the $n+1$ labels come from the set $\{*, G_1, \ldots, G_n\}$. Often, as in [10], we will abuse notation and take the labels from the set $\{*, 1, 2, \ldots, n\}$. Given a labelled tree T and a labelled vertex k in the tree, two other labels are equivalent if they are in the same connected component of $T - \{k\}$. This gives us a partition $\underline{\underline{A}}(k)$ of $\{*, 1, 2, \ldots, n\}$. The singleton set $Q(\underline{\underline{A}}(k)) = \{k\}$ is called the operative factor of the partition. Denote by $\underline{\underline{A}}$ the collection of all of these partitions as k ranges over $\{*, 1, 2, \ldots, n\}$. This yields an equivalent notion of elements of $\mathcal{P}(\mathcal{J}^0)$. The poset structure in $\mathcal{P}(\mathcal{J}^0)$ comes from an operation called folding (when the elements are thought of as labelled trees) or by setting $\underline{\underline{A}}(k) \leq \underline{\underline{B}}(k)$ if elements of $\underline{\underline{A}}(k)$ are unions of elements of $\underline{\underline{B}}(k)$. See McCullough and Miller [10] for more details.

Form a deformation retract of $\mathcal{P}(\mathcal{J}^0)$ by folding all edges coming in to $*$ on a labelled tree together, resulting in a labelled tree where $*$ is a valence 1 vertex. Call the resulting poset $P(\mathcal{J})$. Observe that elements of $P(\mathcal{J})$ correspond to pointed trees with labels in $\mathcal{J}$, and that $P(\mathcal{J})$ is $(n-1)$-dimensional. Denote elements of $P(\mathcal{J})$ as pairs $(\mathcal{J}, \underline{\underline{A}})$ as in [10].

Now mimic the construction in section 2 of [10]. That is, we must construct a space out of the posets $P(\mathcal{J})$. Let $\mathcal{B}$ be the set of all bases of G. Define a relation on

$$\{(\mathcal{H}, \underline{\underline{A}}) | \mathcal{H} \in \mathcal{B}, \underline{\underline{A}} \in P(\mathcal{H})\}$$

by relating $(\mathcal{H}, \underline{\underline{A}})$ and $(\mathcal{G}, \underline{\underline{B}})$ whenever there is a product ρ of symmetric Whitehead automorphisms carried by $(\mathcal{H}, \underline{\underline{A}})$ so that $\rho\mathcal{H} = \mathcal{G}$ and $\rho\underline{\underline{A}} = \underline{\underline{A}}$. Denote the equivalence class of $(\mathcal{H}, \underline{\underline{A}})$ by $[\mathcal{H}, \underline{\underline{A}}]$.

An automorphism $(\mathcal{H}, x)$ is *carried* by $(\mathcal{H}, \underline{A})$ if $* \neq Q(\underline{A}) = Q(x)$, x is constant on each petal of $\underline{A}$, and x is the identity on the petal containing $*$. The set of all such equivalence classes forms a poset under the folding operation. Denote both the poset and its geometric realization by $L(G)$. The space $L(G)$ differs from McCullough-Miller space (see [10]) in that it is a moduli space of pointed trees rather than one of trees.

Define an action of $\Sigma Aut(G)$ on $L(G)$ by having $\phi \in \Sigma Aut(G)$ act via $\phi \cdot [\mathcal{H}, \underline{\underline{A}}] = [\phi(\mathcal{H}), \phi(\underline{\underline{A}})]$.

Recall that the *symmetric Fouxe-Rabinovitch* subgroup $\Sigma FR(G)$ is the subgroup generated by all symmetric Whitehead automorphisms which do not conjugate their

operative factor and that

$$\Sigma Aut_1(G) = \Sigma FR(G) \rtimes \Phi, \Sigma Aut(G) = \Sigma Aut_1(G) \rtimes \Omega$$

where $\Phi = \prod Aut(G_i)$ is the subgroup of factor automorphisms and Ω (a product of symmetric groups) permutes the factors. Further note that Φ and Ω are *not* canonical. Throughout this entire paper, we make the convention of choosing them to be with respect to the basis $\mathcal{H}_0 = \{G_1, \dots, G_n\}$.

3. Reductivity lemmas of McCullough and Miller

In this section, we sketch how the work of McCullough and Miller in [10] implies that $L(G)$ is contractible. They show $K(G)$ (see [10] for a thorough definition and treatment of McCullough-Miller space $K(G)$) is contractible by defining a norm on nuclear vertices of $K(G)$ and inductively adding the stars of nuclear vertices (see [10] for definitions) using this norm while insuring that each new intersection in contractible.

We adopt an analogous approach. First, we show that $L(G)$ is contractible using a norm which is directly analogous to that of [10]. In a later section, we will modify this norm along the lines of Krstić and Vogtmann in [9] and show that $L(G)$ can also be shown to be contractible with the modified norm.

If $\mathcal{W}$ is a set of elements of G, we can define a norm on nuclear vertices $[\mathcal{H}, \underline{\underline{0}}]$ of $L(G)$ by setting $\|\mathcal{H}\|_{\mathcal{W}} = \sum_{w \in \mathcal{W}} |w|_{\mathcal{H}}$, where $|w|_{\mathcal{H}}$ is the (non-cyclic) word length of w in the basis $\mathcal{H}$.

To avoid re-doing work that McCullough and Miller have already done, we adopt the following conventions. Let G be a free product of n finite groups, as already noted. Let $\bar{G} = G * \langle \lambda_{n+1} \rangle$, where $\langle \lambda_{n+1} \rangle \cong \mathbb{Z}/2$. There is an injective map from $\Sigma Aut(G)$ to $\Sigma Out(\bar{G})$ defined by sending $\phi \in \Sigma Aut(G)$ to $\bar{\phi} \in \Sigma Out(\bar{G})$ where $\bar{\phi}(\lambda_{n+1}) = \lambda_{n+1}$. Moreover, if $v = [\mathcal{H}, \underline{\underline{A}}]$ is a vertex in $L(G)$, we can construct a corresponding vertex $\bar{v}$ in $K(\bar{G})$ by adding λ_{n+1} to $\mathcal{H}$ and relabelling the vertex $*$ in the tree corresponding to $\underline{\underline{A}}$ as $\langle \lambda_{n+1} \rangle$ (or just $n+1$.) Note that if $(\mathcal{H}, x)$ is carried by $[\mathcal{H}, \underline{\underline{A}}]$ then $(\bar{\mathcal{H}}, \bar{x})$ is carried by $(\bar{\mathcal{H}}, \bar{\underline{\underline{A}}})$. Finally, if $\mathcal{W}$ is a set of words in G, we can construct $\bar{\mathcal{W}}$ by sending $w \in \mathcal{W}$ to $\bar{w} = w\lambda_{n+1} \in \bar{\mathcal{W}}$ (cf. Proposition 2.18 in [5], which is the basic idea of what we are doing here in this adjustment.) Then $|w|_v + 1 = |\bar{w}|_{\bar{v}}$ for any $w \in \mathcal{W}$ so that $red_{\mathcal{W}}(\alpha, v) = red_{\bar{\mathcal{W}}}(\bar{\alpha}, \bar{v})$ (see [10] for a definition of the reductivity *red* of a Whitehead move) for any $\alpha \in \Sigma Aut(G)$.

Theorem 3.1. *The space $L(G)$ is contractible.*

Proof. We sketch how the work in Chapters 3 and 4 of McCullough and Miller still applies. Extension of Lemma 3.1 is trivial. For Lemma 3.2, observe that we can define refinements and disjunctions for partitions of $\mathcal{J}^0$ as before, and that property 4 on page 27 of [10] implies that if either $Q(\underline{A}) = *$ or $Q(\underline{S}) = *$, so that the relevant partition is trivial, then the refinement or disjunction of $\underline{A}$ with $\underline{S}$ is just $\underline{A}$. Hence analogs of Lemma 3.2 and Lemma 3.3 follow in the context of $L(G)$.

Lemmas 3.6 and 3.7 of [10] can be used to prove the analogous results in our new context. (The reductive automorphism contructed in Lemma 3.7 still sends λ_{n+1} to λ_{n+1} because of the notions of *constricted* peak reduction in [5].)

For Lemma 3.8 (which McCullough-Miller use to prove their Lemma 4.8 and at the end of their Lemma 4.9) we have symmetric Whitehead automorphisms α, σ at v in $L(G)$ and construct α_0, σ_0 as in [10]. The complication is that α_0, σ_0 might

not be in $L(G)$ because the petal containing $*$ is not conjugated by the identity. We resolve this by conjugating the whole automorphism, if necessary, so that the petal containing $*$ does correspond to the identity. More specifically, construct the corresponding $\bar{\alpha}, \bar{\sigma}, \bar{\alpha}_0, \bar{\sigma}_0$ in $K(\bar{G})$. By McCullough-Miller's Lemma 3.8, we have

$$red_{\bar{\mathcal{W}}}(\bar{\alpha}_0, v) + red_{\bar{\mathcal{W}}}(\bar{\sigma}_0, v) \geq red_{\bar{\mathcal{W}}}(\bar{\alpha}_0, v) + red_{\bar{\mathcal{W}}}(\bar{\sigma}_0, v).$$

Also,

$$red_{\bar{\mathcal{W}}}(\bar{\alpha}_0, v) + red_{\bar{\mathcal{W}}}(\bar{\sigma}_0, v) = red_{\mathcal{W}}(\alpha_0, v) + red_{\mathcal{W}}(\sigma_0, v)$$

by our earlier observations. Now conjugate (in $Aut(\bar{G})$) $\bar{\alpha}_0, \bar{\sigma}_0$ to obtain $\bar{\alpha}'_0, \bar{\sigma}'_0$ which are the identity on the petal containing λ_{n+1}. Since these only differ by conjugation,

$$red_{\bar{\mathcal{W}}}(\bar{\alpha}_0, v) + red_{\bar{\mathcal{W}}}(\bar{\sigma}_0, v) = red_{\bar{\mathcal{W}}}(\bar{\alpha}'_0, v) + red_{\bar{\mathcal{W}}}(\bar{\sigma}'_0, v)$$

and we can take the corresponding α'_0, σ'_0 in $L(G)$ so that

$$red_{\mathcal{W}}(\alpha'_0, v) + red_{\mathcal{W}}(\sigma'_0, v) \geq red_{\mathcal{W}}(\alpha_0, v) + red_{\mathcal{W}}(\sigma_0, v),$$

proving the analog of the lemma.

For Chapter 4, reason as follows. Set $\mathcal{W}_0 = \{\lambda_1, \ldots, \lambda_n\}$ so that the analog of Lemma 4.1 is that there is only one nuclear vertex of minimal height n in $L(G)$. Observe that every nuclear vertex in $K(\bar{G})$ corresponding to a basis of the form $\{\lambda_{n+1}^{i_1}\lambda_1\lambda_{n+1}^{-i_1}, \ldots, \lambda_{n+1}^{i_n}\lambda_n\lambda_{n+1}^{-i_n}, \lambda_{n+1}\}$ is of minimal height $2n$ under the basis $\bar{\mathcal{W}}_0 = \{\lambda_1\lambda_{n+1}, \ldots, \lambda_n\lambda_{n+1}\}$.

Lemmas 4.2 and 4.3 are general lemmas about posets from Quillen [12] and hold without any modification. Theorem 4.4 and Proposition 4.5 are the central theorems, established in section 4.2 by the lemmas 4.6, 4.7, 4.8, and 4.9.

Lemmas 4.6 and 4.7 can be proven using the same proof. Lemma 4.8 can also be proven using the same proof, even though it uses Lemma 3.8 which has been modified slightly. The same holds for the crucial lemma, Lemma 4.9. The basic idea is that we could think of many of the calculations as taking place in $K(\bar{G})$, but just with Whitehead automorphisms whose domain (cf. [5] for notions of domain and constricted peak reduction) does not include G_{n+1}. When we combine and modify these automorphisms, we still obtain ones that are the identity on the last factor. □

4. Finite subgroups

Proposition 4.1. *The stabilizer of a simplex in $L(G)$ under the action of $\Sigma Aut(G)$ is finite.*

Proof. Let $[\mathcal{H}, \underline{\underline{A}}]$ be a vertex of $L(G)$. If $i \neq j$, and both $(\mathcal{H}, x^i)$ and $(\mathcal{H}, x^j)$ are symmetric Whitehead automorphisms carried by $[\mathcal{H}, \underline{\underline{A}}]$, then one of x^i_j or x^j_i must be the identity because at least one of the petal of $\underline{\underline{A}}(i)$ containing j or the petal of $\underline{\underline{A}}(j)$ containing i also contains $*$. Hence Lemma 7.4 of [10] applies to give us that $(\mathcal{H}, x^i)$ and $(\mathcal{H}, x^j)$ commute. Thus the stabilizer of $[\mathcal{H}, \underline{\underline{A}}]$ must be finite. □

Proposition 4.2. *Every finite subgroup of $\Sigma Aut_1(G)$ fixes a point of $L(G)$.*

Proof. From Bridson and Miller in [3], every finite subgroup of $Aut(G)$ fixes a point $v' = [\mathcal{H}', \underline{\underline{A'}}]$ of $K(G)$. From Theorem 7.6 of [10], any finite subgroup F of

$\Sigma Aut_1(G)$ that fixes v' is conjugate by an inner automorphism μ to a subgroup whose elements are of the form

$$\prod(\mathcal{H}, x^i)\phi_i$$

where each $\phi_i \in Aut(G_i)$,the symmetric Whitehead automorphisms $(\mathcal{H}, x^i)$ are carried by v', and each $x^i_i = 1$. Moreover, there is a factor k such that $x^i_k = 1$ for all i and there is an unlabelled vertex r of the tree T' corresponding to $\underline{\underline{A'}}$ such that the petal containing r is always conjugated by the identity in the above Whitehead automorphisms. Let $\mathcal{H} = \mu^{-1}(\mathcal{H}')$. Form a tree T by attaching a free edge with terminal vertex $*$ to T' at the vertex r, and let $\underline{\underline{A}}$ be the vertex type determined by T. Then F fixes the vertex $v = [\mathcal{H}, \underline{\underline{A}}]$ of $K(G)$ and every element of F can be written in the form

$$\prod(\mathcal{H}, x^i)\phi_i$$

where each $\phi_i \in Aut(G_i)$, the symmetric Whitehead automorphisms $(\mathcal{H}, x^i)$ are carried by v, each $x^i_i = 1$, and there is a factor k such that $x^i_k = 1$ for all i. □

5. Better norm

Well-order G as $g_1, g_2, g_3, \ldots$ and order $\mathbb{Z}^G$ lexicographically. For a nuclear vertex v corresponding to a basis $\mathcal{H}$, define a norm $\|v\| \in \mathbb{Z}^G$ by setting $\|v\|_i$ to be the (non-cyclic) length $|g_i|_{\mathcal{H}}$ of g_i in the basis $\mathcal{H}$. This is analogous to the norm used by Krstić and Vogtmann in [9] or Jensen in [7].

Proposition 5.1. *The norm $\|\cdot\| \in \mathbb{Z}^G$ well orders the nuclear vertices of $L(G)$.*

Proof. Let U be a nonempty subset of nuclear vertices of $L(G)$ and proceed as in [9]. That is, inductively define U_i and d_i by setting d_i to be the minimal length $|g_i|_{\mathcal{H}}$ obtained by all vertices $[\mathcal{H}, \underline{\underline{0}}] \in U_{i-1}$ and letting U_i be all vertices of U_{i-1} which obtain this minimal length. Recall that we chose specific $\lambda_i \in G_i$. Let N be such that $\lambda_i \in \{g_1, g_2, \ldots, g_N\}$ for all i. Let $\mathcal{W}_0 = \{\lambda_1, \ldots, \lambda_n\}$. We claim that U_N is finite. Let $D = \sum_{g_{i_j}=\lambda_i} d_{i_k}$ so that $\|v\|_{\mathcal{W}_0} \le D$ for all $v \in U_N$. Since each G_i is finite, $L(G)$ is locally finite. Hence the analog of the Existence Lemma 3.7 of [10] implies the ball of radius D (using the $\|\cdot\|_{\mathcal{W}_0}$ distance) around $\mathcal{H}_0$ in $L(G)$ is finite. So U_N is finite. Now choose $M \ge N$ large enough so that $\{g_1, g_2, \ldots, g_M\}$ contains a representative from each basis element of each basis corresponding to an element of U_N. Then U_M contains exactly one element, the least element of U. □

Let F be a finite subgroup of $\Sigma Aut_1(G)$. Our goal in the next few sections is to show that the fixed point subspace $L(G)^F$ is contractible. A vertex $[\mathcal{H}, \underline{\underline{A}}]$ of $L(G)^F$ is *reduced* if no element of $L(G)^F$ lies below it in the poset ordering. We will show $L(G)^F$ is contractible by inductively adding stars of reduced vertices and insuring that intersections are always contractible. The essential step will use the fact that $L(G)$ can be shown the be contractible using the above norm, and we will need the flexibility of being able to well-order G in many different ways.

Theorem 5.2. *Given any well order of G, the norm $||v|| \in \mathbb{Z}^G$ defined above on nuclear vertices of $L(G)$ is such that*

$$st(v) \cap (\cup_{u<v} st(u))$$

is contractible for any non-minimal nuclear vertex v, where $st(v)$ is the star of v. Hence $L(G)$ is contractible by induction.

Proof. We sketch how to apply [10] and Theorem 3.1. To prove Lemmas 3.5, 3.6 of [10] with this new norm, simply apply them, by letting $\mathcal{W}$ be a single word, in each coordinate and applying the analogous lemmas mentioned in the proof of Theorem 3.1.

To prove the Existence Lemma 3.7 of [10], suppose $\mathcal{H}$ is a given basis which does not have minimal norm. Suppose that it does have minimal norm on its first m coordinates $\{g_1, g_2, \ldots, g_m\}$ but that the length of g_{m+1} in $\mathcal{H}$ is not minimal. Now apply the Existence Lemma 3.7 of [10] with the set of words defined to be $\{g_1, g_2, \ldots, g_m, g_{m+1}\}$ to get the desired result.

For Lemma 3.8 (the Collins-Zieschang Lemma), note that the result is proven in [10] by showing that the inequality holds coordinate-wise in our norm.

The arguments given in chaper 4 of [10] also carry through, except that they are simplified somewhat because reductive edges now must be strictly reductive. $\square$

6. Fixed point subspaces.

If $[\mathcal{H}, \underline{\underline{A}}]$ is a vertex type, we think of $\underline{\underline{A}}$ as a collection of partitions of $\{*, 1, 2, \ldots, n\}$ rather than a collection of partitions of $\mathcal{H}$, where wG_iw^{-1} in $\mathcal{H}$ is identified with i. For each k, let $I_k(\underline{\underline{A}})$ be the set of labelled vertices that are a distance $2k$ away from $*$ in the tree T corresponding to $\underline{\underline{A}}$. Let $I(\underline{\underline{A}}) = \cup_k I_k(\underline{\underline{A}})$ and define a poset order in $I = I(\underline{\underline{A}})$ by setting $r \leq s$ if r occurs on the minimal path in T from s to $*$. For $i \in I_k = I_k(\underline{\underline{A}})$, define $I(i)$ and $J(i)$ as follows. Let $z_0 = *, z_1, \ldots, z_k = i$ denote the labelled vertices in the unique minimal path from $*$ to i in T and set $J(i) = (z_1, z_2, \ldots, z_k)$. Let a be the unique unlabelled vertex between z_{k-1} and z_k and let $I(i) = I(a)$ denote the set of all labels in $\{1, 2, \ldots, n\}$ at a distance 1 from a in T. That is, $I(i) = \{z_{k-1}\} \cup \{z \in I_k : z_{k-1} < z\}$. (Exception: if $k = 1$, let $I(i) = \{z \in I_k : z_{k-1} = * < z\}$.) Define $J_{<i}(\underline{\underline{A}})$ to be $\{z_1, z_2, \ldots, z_{k-1}\}$. Note that $J_{<i}(\underline{\underline{A}})$ is empty if $i \in I_1(\underline{\underline{A}})$. Define words $w_i \in G$ inductively as follows. For each $i \in I_1$, define $w_i \in G$ such that $H_i = w_iG_iw_i^{-1}$ and so that w_i has minimal length in the basis $\mathcal{H}$ (i.e., if $H_i = wG_iw^{-1}$, then any word wg_i, $g_i \in G_i$ satisfies $H_i = (wg_i)G_i(wg_i)^{-1}$ and we can choose one with minimal length.) For $i \in I_k$, $J(i) = (z_1, \ldots, z_k = i)$, define $w_i = w_{z_k}$ to be the word of minimal length such that

$$H_i = w_{z_1}w_{z_2}\ldots w_{z_k}G_iw_{z_k}^{-1}\ldots w_{z_1}^{-1}w_{z_1}^{-1}.$$

For convenience, let $w(J(i)) = w_{z_1}w_{z_2}\ldots w_{z_k}$ so that $H_i = w(J(i))G_iw(J(i))^{-1}$.

Let a be an unlabelled vertex of the tree T corresponding to $[\mathcal{H}, \underline{\underline{A}}]$ which is at distance $2k+1$ from $*$. If $k = 0$, define the *stem* of a to be $*$. If $k > 0$, define its *stem* to be the first labelled vertex on the unique shortest path from a to $*$. In either case, if i is the stem of a then define $\mathcal{H}(a) = \{w(J(i))^{-1}H_jw(J(i)) : j \in I(a)\}$.

For a given index $i \in I$, let $\pi_i : G \to G_i$ be the canonical projection.

Lemma 6.1. *Let $[\mathcal{H}_0, \underline{\underline{A}}]$ be a vertex type and let $\phi = \prod_j (\mathcal{H}_0, y^j)\psi_j$, where each $(\mathcal{H}_0, y^j)$ is a symmetric Whitehead automorphism, $y_j^j = 1$ for all j, and each ψ_j is a factor automorphism of G_j. Further suppose that $[\mathcal{H}_0, \underline{\underline{A}}]$ is reduced in $L(G)^F$, where $F = \langle\phi\rangle$. Suppose some other vertex type $[\mathcal{H}, \underline{\underline{B}}]$ is also reduced in $L(G)^F$. Write $\phi = \prod_j (\mathcal{H}, x^j)\phi_j$ in this new basis, where $x_j^j = 1$ for all j. Write $H_i = w(J(i))G_iw(J(i))^{-1}$ as above. Then all of the following hold*

(1) *For all* i, r, $i \neq r$,

$$\pi_r(x_i^r) = \pi_r(\phi(w(J(i))))y_i^r \pi_r(w(J(i))^{-1})$$

(2) *For all* i, j, r, $j \neq r$, *if there exists a* $g_r \in G_r$ *such that* $y_i^r = \pi_r(\phi(g_r))y_j^r \pi_r(g_r^{-1})$ *and* $y_j^r \neq 1$ *then* $y_i^r = y_j^r$.

(3) $\underline{\underline{A}} = \underline{\underline{B}}$ *(as partitions of* $\{*, 1, 2, \ldots, n\}$*.)*

Proof. For a given index i, write λ_i minimally in the basis $\mathcal{H}$ as

$$\lambda_i = a_r^{-1} \cdot \cdots \cdot a_1^{-1} \cdot (w\lambda_i w^{-1}) \cdot a_1 \cdot \cdots \cdot a_r,$$

where $w = w(J(i)(\underline{\underline{B}}))$. Now $\phi = \prod_j (\mathcal{H}, x^j)$ sends λ_i to

$$\phi(w^{-1})cw\psi_i(\lambda_i)w^{-1}c^{-1}\phi(w)$$

where $c \in *_{j \in J_{<i}(\underline{\underline{B}})} H_j$ comes from symmetric Whitehead moves $(\mathcal{H}, x^j)$ conjugating $(w\lambda_i w^{-1})$. Similarly, $\phi = \prod_j (\mathcal{H}_0, y^j)\psi_j$ sends λ_i to

$$d\psi_i(\lambda_i)d^{-1}$$

where $d \in *_{j \in J_{<i}(\underline{\underline{A}})} G_j$ comes from symmetric Whitehead moves $(\mathcal{H}_0, y^j)$ conjugating λ_i. So

$$\phi(w^{-1})cw\psi_j(\lambda_i)w^{-1}c^{-1}\phi(w) = d\psi_i(\lambda_i)d^{-1}$$

and there exists a $g_i \in G_i$ such that $\phi(w^{-1})cw = dg_i$. Thus

$$\pi_r(x_i^r) = \pi_r(\phi(w(J(i))))y_i^r \pi_r(w(J(i))^{-1})$$

as desired.

By way of contradiction, suppose there exist indices $i, j, r \in I(\underline{\underline{A}})$ and $g_r \in G_r$ such that $y_i^r = \pi_r(\phi(g_r))y_j^r \pi_r(g_r^{-1})$ and $y_i^r \neq y_j^r \neq 1$. Let $S_j = \{k \in I(\underline{\underline{A}}) : y_j^r = y_k^r\}$. Since the S_j petal is not the identity petal, we can conjugate it by g_r. Let $\mathcal{H}'$ be the basis obtained from $\mathcal{H}_0$ by conjugating all of the G_k, $k \in S_j$, by g_r. Then $[\mathcal{H}', \underline{\underline{A}}']$ =$[\mathcal{H}_0, \underline{\underline{A}}]$ and $\underline{\underline{A}} = \underline{\underline{A}}'$ as partitions of $\{*, 1, 2, \ldots, n\}$. However the previous paragraph yields that if $k \in S_j$ then $(y')_k^r = \pi_r(\phi(g_r))y_j^r \pi_r(g_r^{-1})$. In other words if $i \neq r$, we can combine the $S_i = \{k \in I(\underline{\underline{A}}) : y_i^r = y_k^r, k \neq r\}$ and S_j petals of $\underline{\underline{A}}'(r)$. (If $i = r$ then $1 = y_i^r = \pi_r(\phi(g_r))y_j^r \pi_r(g_r^{-1})$. In this case, we can combine the S_j petal with the petal containing $*$.) This contradicts the fact that $[\mathcal{H}_0, \underline{\underline{A}}]$ is reduced in $L(G)^F$.

Finally, we must show that $\underline{\underline{A}} = \underline{\underline{B}}$ as partitions of $\{*, 1, 2, \ldots, n\}$. First, show this under the assumption that for each i, $I_{<i}(\underline{\underline{A}}) = I_{<i}(\underline{\underline{B}})$ (as partitions of $\{*, 1, 2, \ldots, n\}$.) Suppose $x_j^r = x_i^r$ but $y_i^r \neq y_j^r \neq 1$. Since $\pi_r(x_j^r) = \pi_r(x_i^r)$, (1) yields that $y_i^r = \pi_r(\phi(w_i^{-1}w_j))y_j^r \pi_r((w_i^{-1}w_j)^{-1})$ and we can apply (2) to get $y_i^r = y_j^r$. As this is a contradiction, $y_j^r = y_i^r$ whenever $x_j^r = x_i^r$. By symmetry, $x_j^r = x_i^r$ whenever $y_j^r = y_i^r$. So $\underline{\underline{A}} = \underline{\underline{B}}$ in this case.

We are now ready to show $\underline{\underline{A}} = \underline{\underline{B}}$ in general. By way of contradiction, assume $r \in J_{<j}(\underline{\underline{A}})$ but $r \notin J_{<i}(\underline{\underline{B}})$. Then $x_j^r = 1$ and $y_j^r \neq 1$. Let $i \neq r$ be the index for which $y_i^r = 1$. (Abusing notation, we might have to take $i = *$.) From (1), $y_i^r = 1 = \pi_r(x_j^r) = \pi_r(\phi(w(J(j))))y_j^r \pi_r(w(J(j))^{-1})$. By (2), $1 = y_i^r = y_j^r$, which is a contradiction. So $J_{<i}(\underline{\underline{A}}) \subset J_{<i}(\underline{\underline{B}})$. By symmetry, $J_{<i}(\underline{\underline{B}}) \subset J_{<i}(\underline{\underline{A}})$ as well. Now apply the previous case. $\square$

Lemma 6.2. *Let $[\mathcal{H}_0, \underline{\underline{A}}]$ be a vertex type and let $\phi = \prod_j (\mathcal{H}_0, y^j)\psi_j$, where each $(\mathcal{H}_0, y^j)$ is a symmetric Whitehead automorphism, $y^j_j = 1$ for all j, and each ψ_j is a factor automorphism of G_j. Further suppose that $[\mathcal{H}_0, \underline{\underline{A}}]$ is reduced in $L(G)^F$, where $F = \langle \phi \rangle$. Fix an index k and let $d = \prod_{j \in J_{<k}} y^j_k$ (written so that if $j_1 < j_2$ in $J_{<k}$ then $y^{j_1}_k$ occurs before $y^{j_2}_k$ in the product.) Then a word $w \in *G_j$ whose last letter is not in G_k satisfies $\phi(w) = dwg_kd^{-1}$ for some $g_k \in G_k$ if and only if $w \in *_{j \in I(k)} G^\circ_{j,k}$ where the groups $G^\circ_{j,k}$ are defined by $G^\circ_{j,k} = \{g \in G_j : y^j_k g (y^j_k)^{-1} = \psi_j(g)\}$.*

Proof. Let i be the next labelled vertex on a path from k to $*$ in the tree T corresponding to $\underline{\underline{A}}$. If $w \in *_{j \in I(k)} G_j$, $\psi_j(w) = w$ for all $j \in I(k) - \{i\}$, and $\psi_i(g_i) = y^i_k g_i (y^i_k)^{-1}$ for all $g_i \in G_i$ occurring in the normal form of w, it is clear that $\phi(w) = dwd^{-1}$. For the other direction, assume $\phi(w) = dwg_kd^{-1}$ and suppose by way of contradiction that $w \notin *_{j \in I(k)} G_j$. Let S be the set of all indices j for which an element of G_j is a substring of w.

Case 1: *There is an index $r \in S - I(k)$ such that $J_{<r} - J_{<k} \neq \emptyset$ or such that $J_{<r} - J_{<k} = \emptyset$ but $r \notin J_{<k}$.* Choose r satisfying the above condition to be maximal in the poset I. Now choose the first occurance g_r of an element of G_r in w and write $w = u_1 g_r u_2$. Let $c = \prod_{j \in J_{<r}} y^j_r$. Now $\phi(w) = du_1g_ru_2g_kd^{-1} = \phi(u_1)c\psi_r(g_j)c^{-1}\phi(u_2)$. Because r is maximal in I with the given condition, ϕ does not introduce any more words from G_r into w. Hence we must have $du_1 = \phi(u_1)c$ and $g_r = \psi_r(g_r)$. If s is the greatest index in $J_{<r}$, then $y^s_r = \pi_s(\phi(u_1))^{-1} y^s_k \pi_s(u_1)$ and so by (2) of Lemma 6.1, $y^s_r = y^s_k$. This contradicts the fact that $r \notin I(k)$.

Case 2: $S \subseteq I(k) \cup J_{<k}$. Let $\psi = (\prod_{j \in I(k) - J_{<k}} \psi_j) \cdot (\prod_{j \in J_{<k}} (\mathcal{H}_0, y^j)\psi_j)$, ordered so that if $j_1 < j_2$ in I then the automorphisms with index j_1 are evaluated first (i.e., occur later in the listing above.) Note that $\psi(w) = \phi(w)$. Let r be the least index in S (least in the poset I) are let g_r be the first occurance of G_r in w. Write $w = u_1g_ru_2$. Let s be the next labelled vertex on a path from r to k in T and set $c = \prod_{j \in J_{<r}} y^j_s$. After applying the first $|J_{<r}|$ moves $(\mathcal{H}_0, y^j)\psi_j$ of ψ to w, the result is $cu_1g_ru_2c^{-1}$. Moreover, the number of times an element of G_r occurs in the string $cu_1g_ru_2c^{-1}$ is the same as the number of times it occurs in $du_1g_ru_2g_kd^{-1}$. After applying the next move $(\mathcal{H}_0, y^r)\psi_r$ to $cu_1g_ru_2c^{-1}$, we have $cy^r_su_1(y^r_s)^{-1}\psi_r(g_r)y^r_s \ldots$. Let ψ' denote the last $(|J_{<k}| - |J_{<s}|) + |I(k)|$ moves of ψ. Then applying ψ' gives us $cy^r_s\psi'(u_1)(y^r_s)^{-1}\psi_r(g_r)y^r_s \ldots = \phi(u_1)c\psi_r(g_r)y^r_s \ldots$. Because applying ψ' will not introduce any more elements of G_r, we have $\phi(u_1)c = du_1$. This means that $1 = \pi_i(\phi(u_1)^{-1})y^i_k\pi_i(u_1)$ and thus $y^i_k = 1$ by (2) of Lemma 6.1. This is a contradiction.

As we reached a contradiction in both cases, $w \in *_{j \in I(k)} G_j$. Since w does not end in an element of G_k, $g_k = 1$ and $\phi(w) = dwd^{-1}$. If $g_j \in G_j$ is a letter occuring in the normal form of w, then $\psi_j(g_j) = g_j$ if $j \neq i$ and $y^j_k g_j (y^j_k)^{-1} = \psi_j(g_j)\}$ if $j = i$. Thus $g_j \in G^\circ_{j,k}$ as desired. □

We apologize for the confusing parentheses in $y^j_k g (y^j_k)^{-1}$ above, which denotes conjugating g by y^j_k. Observe that if y^j_k is in the center of G_j (in particular, if G_j is abelian) then $G^\circ_{j,k} = \{g \in G_j : g = \psi_j(g)\}$. Many of the arguments in this paper would be simplified if we were only working with abelian factor groups.

Proposition 6.3. *Let $v_0 = [\mathcal{H}_0, \underline{\underline{A}}]$ be a vertex type and let $\phi = \prod_j (\mathcal{H}_0, y^j)\psi_j$, where each $(\mathcal{H}_0, y^j)$ is a symmetric Whitehead automorphism, $y^j_j = 1$ for all j, and each ψ_j is a factor automorphism of G_j. Further suppose that $[\mathcal{H}_0, \underline{\underline{A}}]$ is reduced in $L(G)^F$, where $F = \langle\phi\rangle$. A necessary and sufficient condition for any other vertex type v to be reduced in $L(G)^F$ is that it have a representative $v = [\mathcal{H}, \underline{\underline{B}}]$ where $\underline{\underline{A}} = \underline{\underline{B}}$ (as partitions of $\{*, 1, 2, \ldots, n\}$) and that that when we write $H_i = w(J(i))G_i w(J(i))^{-1}$ as above we have $w_k \in *_{j\in I(k)} G^\circ_{j,k}$ where the groups $G^\circ_{j,k}$ are defined by $G^\circ_{j,k} = \{g \in G_j : y^j_k g (y^j_k)^{-1} = \psi_j(g)\}$. Moreover, if $v = [\mathcal{K}, \underline{\underline{C}}]$ is any other representative, then*

1. *$\underline{\underline{C}} = \underline{\underline{B}}$ as partitions of $\{*, 1, 2, \ldots, n\}$.*
2. *We can get from $(\mathcal{K}, \underline{\underline{C}})$ to $(\mathcal{H}, \underline{\underline{B}})$ by a series of moves conjugating petals S of various $\underline{\underline{C}}(i)$ by $w(J(i))\pi_i(w_k^{-1})w(J(i))^{-1}$ where $k \in S$ (where the $w(J(i))$ and w_k are taken with respect to the $(\mathcal{K}, \underline{\underline{C}})$ representative of v.)*

Proof. For sufficiency, we note that it is a direct check to see that $(\mathcal{H}_0, y^j)\psi_j \cdot v = v$ for all j if v is as described above. So $\phi = \prod_j (\mathcal{H}_0, y^j)\psi_j$ fixes v as well.

For necessity, suppose that $[\mathcal{H}, \underline{\underline{B}}]$ is a reduced vertex in $L(G)^F$. Then ϕ must fix this vertex type, which means that $[\mathcal{H}, \underline{\underline{B}}]$ is stabilized by a product

$$\prod_j (\mathcal{H}, x^j)\phi_j$$

which equals ϕ, where $x^j_j = 1$ for all j and each ϕ_j is a factor automorphism of H_j. By (3) of Lemma 6.1, $\underline{\underline{A}} = \underline{\underline{B}}$ as partitions of $\{*, 1, 2, \ldots, n\}$.

We show that the w_i have the desired properties by inducting on the distance from i to $*$ in T. If $i \in I_1$, write $w_i = \bar{w}_i \bar{g}_i$, where $\bar{w}_i$ does not end in an element of G_i. then $\phi(\bar{w}_i \lambda_i \bar{w}_i^{-1}) = \phi(\bar{w}_i)\psi_i(\lambda_i)\phi(\bar{w}_i)^{-1} = \phi_i(\bar{w}_i \lambda_i \bar{w}_i^{-1})$. Thus there exists a $g_i \in G_i$ such that $\phi_i(\bar{w}_i) = (\bar{w}_i)g_i$. By Lemma 6.2 $g_i = 1$, $\bar{w}_i \in *_{j\in I(i)} G_j$, and $\psi_j(\bar{w}_i) = \bar{w}_i$ for all j. By way of contradiction, suppose $\bar{g}_i \neq 1$ but $\psi_i(\bar{g}_i) \neq \bar{g}_i$. Write $\bar{w}_i = u_1 \ldots u_s$ in normal form in the basis $\mathcal{H}$, where each u_j comes from an H_{i_j} with $i_j \in I_1$. Because $w_i = \bar{w}_i \bar{g}_i = (\bar{w}_i \bar{g}_i \bar{w}_i^{-1})u_1 \ldots u_s$, has length less than s, we have $u_1 = \bar{w}_i \bar{g}_i^{-1} \bar{w}_i^{-1}$. Then

$$\bar{g}_i^{-1} w_i^{-1} u_2 \ldots u_s = \psi_i(\bar{g}_i^{-1}) w_i^{-1} \phi(u_2 \ldots u_s)$$

Thus if $f = \prod_j \phi_j$ then

$$\phi(u_2 \ldots u_s) = f(u_2 \ldots u_s) = (w_i \psi_i(\bar{g}_i)\bar{g}_i^{-1} w_i^{-1})u_2 \ldots u_s$$

which contradicts the fact that the length of $u_2 \ldots u_s$ in $\mathcal{H}$ is $s - 1$.

For the inductive step, consider an index k and let i be the next labelled vertex on a path from k to $*$ in T. Let $w = w(J(i))$. As in the proof part (1) of Lemma 6.1, we have $c \in *_{j\in J_{<i}(\underline{\underline{B}})} H_j$ coming from symmetric Whitehead moves $(\mathcal{H}, x^j)$ conjugating $(w\lambda_i w^{-1})$ and $d \in *_{j\in J_{<i}(\underline{\underline{A}})} G_j$ coming from symmetric Whitehead moves $(\mathcal{H}_0, y^j)$ conjugating λ_i such that $cw = \phi(w)dg_i$, where $g_i = 1$ by the induction hypothesis. Thus $\pi_i(w) = \pi_i(\phi(w))$. As in the basis step of the induction, write $w_k = \bar{w}_k \bar{g}_k$ where $\bar{w}_k$ does not end in an element of G_k. We have

$$c x^i_k w \bar{w}_k g_k = \phi(w\bar{w}_k) d y^i_k$$

for some $g_k \in G_k$. It follows that

$$\phi(\bar{w}_k) = (dw^{-1}x_k^i w)\bar{w}_k g_k (dy_k^i)^{-1}.$$

By (1) of Lemma 6.1 and the inductive hypothesis,

$$\pi_i(x_k^i) = \pi_i(\phi(w\bar{w}_k))y_k^i \pi_i(w\bar{w}_k)^{-1} = \pi_i(w)\pi_i(\phi(\bar{w}_k))y_k^i\pi_i(\bar{w}_k)^{-1}\pi_i(w)^{-1}.$$

If $\pi_i(\phi(\bar{w}_k))y_k^i\pi_i(\bar{w}_k)^{-1} \neq y_k^i$, change $\mathcal{H}$ by conjugating the entire petal S of $\underline{\underline{B}}$ containing k by $w\pi_i(\bar{w}_k^{-1})w^{-1}$. Then in the new $\mathcal{H}$, $\pi_i(x_j^i) = \pi_i(w)y_k^i\pi_i(w)^{-1}$ for all j in S. Hence $w^{-1}x_k^i w = y_k^i$ and so

$$\phi(\bar{w}_k) = (dy_k^i)\bar{w}_k g_k (dy_k^i)^{-1}.$$

Applying Lemma 6.2, $g_k = 1$ and $\bar{w}_k \in *_{j\in I(k)} G_{j,k}^\circ$. For reasons similar to the base case of the induction, $\psi_k(\bar{g}_k) = \bar{g}_k$ as well. □

Corollary 6.4. *Let $[\mathcal{H}_0, \underline{\underline{A}}]$ be a vertex type and let $\phi = \prod_j (\mathcal{H}_0, y^j)\psi_j$, where each $(\mathcal{H}_0, y^j)$ is a symmetric Whitehead automorphism, $y_j^j = 1$ for all j, and each ψ_j is a factor automorphism of G_j. Further suppose that $[\mathcal{H}_0, \underline{\underline{A}}]$ is reduced in $L(G)^{F_1}$, where $F_1 = \langle\phi\rangle$. Let F_2 be the subgroup generated by all of the $(\mathcal{H}_0, y^j)\psi_j$, so that $F_1 \subseteq F_2$. Then some other vertex type $[\mathcal{H}, \underline{\underline{B}}]$ is reduced in $L(G)^{F_1}$ if and only if it is reduced in $L(G)^{F_2}$.*

Proof. Since $F_1 \subseteq F_2$, $L(G)^{F_2} \subset L(G)^{F_1}$. However, if v is a nuclear vertex of $L(G)^{F_1}$, then from Proposition 6.3 every element of F_2 fixes it (i.e, $G_{j,k}^\circ = \{g \in G_j : y_k^j g (y_k^j)^{-1} = \psi_j(g)\}$ only depends on $(\mathcal{H}_0, y^j)\psi_j$.) □

Let $v_0 = [\mathcal{H}_0, \underline{\underline{A}}]$ be a vertex type, F be a finite subgroup of $\Sigma Aut_1(G)$ that fixes v_0, and suppose v_0 is reduced in $L(G)^F$. Let $\phi = \prod_j (\mathcal{H}_0, y^j)\psi_j \in F$, where each $(\mathcal{H}_0, y^j)$ is a symmetric Whitehead automorphism, $y_j^j = 1$ for all j, and each ψ_j is a factor automorphism of G_j. Define $\pi_j(\phi) = (\mathcal{H}_0, y^j)\psi_j$. Define the groups $G_{j,k}^\circ$ by

$$G_{j,k}^\circ = \cap_{\phi\in F}\{g \in G_j : y_k^j g(y_k^j)^{-1} = \psi_j(g) \text{ where } \pi_j(\phi) = (\mathcal{H}_0, y^j)\psi_j\}.$$

A representative $(\mathcal{H}, \underline{\underline{B}})$ of some other vertex $v = [\mathcal{H}, \underline{\underline{B}}]$ is *F-standard* if all of the following hold

- $\underline{\underline{A}} = \underline{\underline{B}}$ as partitions of $\{*, 1, 2, \ldots, n\}$.
- When we write $H_i = w(J(i))G_i w(J(i))^{-1}$ then $w_i \in *_{j\in I(i)} G_{j,k}^\circ$.

Theorem 6.5. *Let $[\mathcal{H}_0, \underline{\underline{A}}]$ be a vertex type and suppose that $F \subseteq \Sigma Aut_1(G)$ fixes $[\mathcal{H}_0, \underline{\underline{A}}]$. Further suppose that $[\mathcal{H}_0, \underline{\underline{A}}]$ is reduced in $L(G)^F$. A necessary and sufficient condition for any other vertex type v to be reduced is that it have an F-standard representative.*

Proof. Let F_+ be the subgroup generated by $\{\pi_j(\phi) : j \in I, \phi \in F\}$ so that $F \subseteq F_+$. Now $L(G)^{F_+} = L(G)^F$ by Corollary 6. From Proposition 6.3, each reduced vertex $v = [\mathcal{H}, \underline{\underline{B}}]$ in $L(G)^{F_+}$ must have $\underline{\underline{A}} = \underline{\underline{B}}$ because the structure of $\underline{\underline{B}}$ depends only on the symmetric Whitehead moves $(\mathcal{H}_0, y^j)$ occuring in $\phi \in F$. In particular, $\underline{\underline{B}}(j)$ is the wedge (see [10], page 29) of all of the full carriers of the $(\mathcal{H}_0, y^j)$ occuring in $\phi \in F$.

To show that each $w_i \in *_{j\in I(i)} G^\circ_{j,k}$, note that (2) of Proposition 6.3 means that if the letters from G_j in a particular w_k are already in

$$\cap_{\phi\in F_1}\{g \in G_j : y^j_k g(y^j_k)^{-1} = \psi_j(g) \text{ where } \pi_j(\phi) = (\mathcal{H}_0, y^j)\psi_j\},$$

then if we take a $\xi \notin F$ and conjugate petals again to get the letters in

$$\{g \in G_j : y^j_k g(y^j_k)^{-1} = \psi_j(g) \text{ where } \pi_j(\xi) = (\mathcal{H}_0, y^j)\psi_j\},$$

then they are still in the previous group and hence in the intersection of the two groups. □

7. Contractibility of fixed point subspaces.

For this entire section, let F be a finite subgroup of $\Sigma Aut_1(G)$ that fixes $[\mathcal{H}_0, \underline{\underline{A}}]$ and suppose that $[\mathcal{H}_0, \underline{\underline{A}}]$ is reduced in $L(G)^F$. If $[\mathcal{H}, \underline{\underline{B}}]$ is any other reduced vertex and $\mathcal{H} = \{H_1, \dots, H_n\}$ then for all j, k define $H^\circ_{j,k} = w(J(i))G^\circ_{j,k}w(J(i))^{-1}$. If a is the next unlabelled vertex on a path from k to $*$, set $G^\circ_{j,a} = G^\circ_{j,k}$ and $H^\circ_{j,a} = H^\circ_{j,k}$. In addition, define $G^{\circ\circ}_{j,k}$ to be $G^\circ_{j,k}$ if it is nontrivial and $\mathbb{Z}/2$ otherwise. Define $H^{\circ\circ}_{j,k}$, etc., analogously. For each j, a choose $1 \neq \lambda_{j,a} \in G^{\circ\circ}_{j,a}$. Let $G_a = *_{j\in I(a)} G^{\circ\circ}_{j,a}$.

Note that if $[\mathcal{H}, \underline{\underline{A}}] \in L(G)^F$ with $(\mathcal{H}, \underline{\underline{A}})$ F-standard and $j \in I_1$, then $*_{k\in I_1} H_k = *_{k\in I_1} G_k$. This follows by letting N be the normal closure of $*_{k\notin I_1} G_k$ and considering $G/N \cong *_{k\in I_1} G_k$. Observe that $*_{k\notin I_1} H_k \subseteq N$. If $j \in I_1$, then $H_j \subseteq *_{k\in I_1} G_k$ because $H_j = w_j G_j w_j^{-1}$ with $w_j \in *_{k\in I_1} G_k$. Therefore, $*_{k\in I_1} H_k \subseteq *_{k\in I_1} G_k$. Furthermore, since $\mathcal{H}$ is a basis of G, if $j \in I_1$ and $g_k \in G_k$, then we can write $g_k = v_1 v_2 \dots v_s$ in the basis $\mathcal{H}$. Taking the quotient by N, this yields a way of writing g_k in $*_{k\notin I_1} H_k$. It follows that $*_{k\in I_1} H_k = *_{k\in I_1} G_k$, as desired. More generally, one can verify that $*_{k\in I(a)} H_k = *_{k\in I(a)} w(J(i))G_k w(J(i))^{-1}$. A direct induction argument now yields that $w(J(i)) \in *_{k\in I(a)\cup J_{<i}} G_k$.

With the same setup and hypothesis of Theorem 6.5 and where $[\mathcal{H}, \underline{\underline{A}}] \in L(G)^F$ with $(\mathcal{H}, \underline{\underline{A}})$ F-standard, we have

Lemma 7.1. *Let a be an unlabelled vertex of T with stem i. Then for each $h \in *_{k\in I(a)} G_k$,*

$$|h|_{\mathcal{H}} = 2|w(J(i))|_{\mathcal{H}} + |h|_{\mathcal{H}(a)}.$$

Proof. Let $w = w(J(i))$. We show the result by induction on the distance d from a to $*$ in T. Assume $d \geq 3$, as the basis step of $d = 1$ is immediate.

If m is the next labelled vertex on a path from i to $*$ and y is the unlabelled vertex between i and m then $w = w' w_i$, $w' = w(J(m))$. By our inductive hypothesis, and $|w_i|_{\mathcal{H}} = 2|w(J(m))|_{\mathcal{H}} + |w_i|_{\mathcal{H}(y)}$. In other words, if $w' = v_{t+1} \dots v_s$ in $\mathcal{H}$ and $w_i = (w'^{-1} v_1 w')(w'^{-1} v_2 w') \dots (w'^{-1} v_t w')$ is a minimal way of writing w_i in $\mathcal{H}(y)$, then $w_i = v_s^{-1} \dots v_{t+1}^{-1}(v_1 v_2 \dots v_t)v_{t+1} \dots v_s$ is a minimal way of writing the length $t+s$ word w_i and has no cancellations in $\mathcal{H}$. Thus $w = w' w_i = (v_1 v_2 \dots v_t)v_{t+1} \dots v_s$ is a minimal way of writing the length s word w.

If $h = u_1 u_2 \dots u_r$ is a minimal way of writing h in the basis $\mathcal{H}(a)$ then

$$h = v_s^{-1} \dots v_1^{-1}(w u_1 w^{-1}) \dots (w u_r w^{-1}) v_1 \dots v_s.$$

No cancellation occurs among the $w u_j w^{-1}$ by themselves or the v_j by themselves. We must verify that no cancellation occurs at the stages $v_1^{-1}(w u_1 w^{-1})$ or $(w u_r w^{-1}) v_1$ because the v_i are not in H_l for $l \in I(a)$. This follows because if

$v_1 = wu_1w^{-1}$, then $v_1 \in H_i$, $u_1 \in G_i$. But recall that we chose w_i to have minimal length among all w_ig, $g \in G_i$, and

$$v_s^{-1} \dots v_{t+1}^{-1}(v_2 \dots v_t)v_{t+1} \dots v_s = w_iu_1^{-1}$$

has smaller length. So this does not occur and similarly no other cancellations occur. $\square$

Let a be an unlabelled vertex of the tree T corresponding to some $v = [\mathcal{H}, \underline{\underline{A}}]$ with $(\mathcal{H}, \underline{\underline{A}})$ F-standard and let i be the stem of a.

Let

$$\mathcal{H}_a = \{w_jG_{j,a}^{\circ\circ}w_j^{-1} : j \in I(a) - \{i\}\} \cup \{G_{i,a}^{\circ\circ}\} = \{w(J(i))^{-1}H_{j,a}^{\circ\circ}w(J(i)) : j \in I(a)\}.$$

Thus $w_jG_{j,a}^{\circ\circ}w_j^{-1} \in \mathcal{H}_a$ iff $w_jG_jw_j^{-1} \in \mathcal{H}(a)$. By Theorem 6.5, each $w_j \in G_a = *_{j \in I(a)} G_{j,a}^{\circ\circ}$.

Let A be the set of unlabelled vertices in the tree T corresponding to $\underline{\underline{A}}$. Well order A so that if a is on the unique shortest path from b to $*$, then $a < b$. Choose a well order for each G_a that puts $\lambda_{i,a}$, where i is the stem of a first, then the other letters $\lambda_{j,a}$, $j \in I(a) - \{i\}$, and finally all of the other words. Well order $\cup_{a \in A} G_a$ so that: (i) If $g < h$ in G_a then $g < h$ in $\cup_{a\in A}G_a$; and (ii) If $a < b$, i is the stem of a, and j is the stem of b, then every element of $G_a - G_{i,a}^{\circ\circ}$ occurs before every element of $G_b - G_{j,b}^{\circ\circ}$.

Order $\mathbb{Z}^{\cup_{a\in A}G_a}$ lexicographically and define a norm

$$\|(\mathcal{H}, \underline{\underline{A}})\| \in \mathbb{Z}^{\cup_{a\in A}G_a}$$

on the F-standard pair $(\mathcal{H}, \underline{\underline{A}})$ representing a nuclear vertex of $L(G)^F$ by setting the gth coordinate to be $|g|_{\mathcal{H}}$, the length of the word g in the basis given by $\mathcal{H}$. As stated, this does not define a norm on nuclear vertices of $L(G)^F$ because there is more than one way to write the vertex type $v = [\mathcal{H}, \underline{\underline{A}}]$ in an F-standard basis. We solve this problem by definining

$$\|v\| = \min_{[\mathcal{K}, \underline{\underline{A}}] = v, (\mathcal{K}, \underline{\underline{A}})\ F\text{-standard}} \|(\mathcal{H}, \underline{\underline{A}})\|.$$

Observe that given any representative $(\mathcal{H}, \underline{\underline{A}})$ there is an easy algorithm to construct the minimal representative by proceeding inductively through A. If a_0 is the least element of A (the vertex adjacent to $*$ in T), then we cannot change the values in the range G_{a_0} at all. Supposing we have minimized all values less than a particular $a \in A$, we let i be the stem of a. We now conjugate all of $I(a) - \{i\}$ by a single element of $w(J(i))G_{i,a}^{\circ}w(J(i))^{-1}$, if necessary, to reduce the norm restricted to G_a.

Proposition 7.2. *The norm*

$$\|v\| \in \mathbb{Z}^{\cup_{a\in A}G_a}$$

well orders the nuclear vertices of $L(G)^F$.

Proof. Let U be a nonempty subset of nuclear vertices of $L(G)$. Inductively define U_g and d_g by setting d_g to be the minimal length $|g|_{\mathcal{H}}$ obtained by all vertices $[\mathcal{H}, \underline{\underline{A}}] \in \cap_{h<g} U_h$ and letting U_g be all vertices of $\cap_{h<g}U_h$ which obtain this minimal length.

We show by induction that if a is an unlabelled vertex of T then any $\mathcal{H}, \mathcal{K} \in \cap_{g \in G_a} U_g$ satisfy $\mathcal{H}_a = \mathcal{K}_a$. For the basis step, let a_0 denote the unlabelled vertex adjacent to $*$ and note that $\| \cdot \| \in \mathbb{Z}^{G_{a_0}}$ well orders the nuclear vertices of $L(G_{a_0})$ by Proposition 5.1.

For the inductive step, consider an unlabelled vertex $a \neq a_0$ of T and suppose $\mathcal{H}, \mathcal{K} \in \cap_{g \in G_a} U_g$. By induction, for all $b < a$, $\mathcal{H}_b = \mathcal{K}_b$. In particular, if i is the stem of a, then $w(J_{\mathcal{H}}(i)) = w(J_{\mathcal{K}}(i))$ and $|w(J_{\mathcal{H}}(i))|_{\mathcal{H}} = |w(J_{\mathcal{K}}(i))|_{\mathcal{K}}$. Now use Lemma 7.1 and Proposition 5.1 applied to $L(G_a)$ to get that $\mathcal{H}_a = \mathcal{K}_a$.

So $\mathcal{H} = \mathcal{K}$ and we are done. $\square$

Observe that $[\mathcal{H}_0, \underline{\underline{A}}]$ is the unique minimal vertex (with $(\mathcal{H}_0, \underline{\underline{A}})$ as its minimal F-standard representative) of $L(G)^F$ in the above norm. A strictly reductive symmetric Whitehead move at a non-minimal nuclear vertex $[\mathcal{H}, \underline{\underline{A}}]$ is a symmetric Whitehead automorphism carried by some vertex type in the ascending star of $[\mathcal{H}, \underline{\underline{A}}]$.

The well order in $\cup_{a \in A} G_a$ restricts to a well order on G_a so that we have an induced norm in $\mathbb{Z}^{G_a}$ on $L(G_a)$. Let $v = [\mathcal{H}, \underline{\underline{A}}]$, where we assume for the remainder of the section that whenever we write v this way, we have chosen $(\mathcal{H}, \underline{\underline{A}})$ to be a minimally-normed F-standard pair representing v using the algorithm stated before Proposition 7.2. If $\alpha = (\mathcal{H}_a, y^k)$ is a reductive Whitehead move at $v_a = [\mathcal{H}_a, \underline{\underline{0}}]$ in $L(G_a)$, then let $\alpha^a = (\mathcal{H}, x^k)$ be defined as follows: If $j \leq i$ in I or j is not comparable with i in I, then define $x^k_j = 1$. On the other hand, if $j \geq l$ for some $l \in I(a) - \{i\}$, then set $x^k_j = w(J(i)) y^k_l w(J(i))^{-1}$. Suppose α is carried by $[\mathcal{H}_a, \underline{\underline{B}}]$ and T_a is the tree for $\underline{\underline{B}}$. Define T^a by first cutting out a and all edges attached to a from T, and then glueing T_a in to the resulting hole, by attaching the vertex j of T_a to the vertex j of $T - \{a\}$. Let $[\mathcal{H}, \underline{\underline{B}}^a]$ be the vertex type corresponding to the tree T^a and observe that α^a is carried by $[\mathcal{H}, \underline{\underline{B}}^a]$.

Lemma 7.3. *If $\alpha = (\mathcal{H}_a, y^k)$ is a reductive Whitehead move at $v_a = [\mathcal{H}_a, \underline{\underline{0}}]$ in $L(G_a)$ carried by $[\mathcal{H}_a, \underline{\underline{B}}]$ as described above, then $\alpha^a = (\mathcal{H}, x^k)$ is a reductive Whitehead move at $v = [\mathcal{H}, \underline{\underline{A}}]$ and is carried by $[\mathcal{H}, \underline{\underline{B}}^a]$.*

Proof. Recall that i is the stem of a. Let $w(J(i)) = w$. Note that we can assume $y^k_i = 1$ since $|\lambda_{i,a}|_{v_a} = 1$ is minimal. By way of contradiction, suppose the index k is one for which $G^\circ_{k,a} = \langle 1 \rangle$. Recall that by Theorem 6.5, $w_j \in *_{r \in I(a)} G^\circ_{r,a}$ for all $j \in I(a) - \{i\}$. So when we write each w_j in the basis $\mathcal{H}_a$, we need not use any letters from $w(J(i))^{-1} H^{\circ\circ}_{k,a} w(J(i))$. In addition, if $j \neq k$, when we write $\lambda_{j,a}$ in the basis $\mathcal{H}_a$, we need not use any letter from $w(J(i))^{-1} H^{\circ\circ}_{k,a} w(J(i))$. If $j = k$, the normal form of $\lambda_{j,a}$ in the basis $\mathcal{H}_a$ uses exactly one letter from $w(J(i))^{-1} H^{\circ\circ}_{k,a} w(J(i))$ and the rest from other elements of $\mathcal{H}_a$.

Let $g = u_{1,r_1} \dots u_{t,r_t}$ be the normal form in the basis v_a of some element of $*_{j \in I(a)} G^\circ_{j,a}$, where $u_{1,r_j} \in G^\circ_{r_j,a}$. Then the normal form of g in the basis $\alpha(v_a)$ is the product

$$[(y^k_{r_1})^{-1}][(y^k_{r_1} u_{1,r_1} (y^k_{r_1})^{-1}][y^k_{r_1} (y^k_{r_2})^{-1}][y^k_{r_2} u_{2,r_2} (y^k_{r_2})^{-1}] \\ \dots [y^k_{r_{t-1}} (y^k_{r_t})^{-1}][y^k_{r_t} u_{t,r_t} (y^k_{r_t})^{-1}][y^k_{r_t}]$$

which has length greater than or equal to t. Furthermore, the length is equal to t only when $y^k_{r_j} = 1$ for every $j = 1, 2, \dots t$.

Taking g to be $\lambda_{j,a}$ for $k \neq j \in I(a)$, we see that α cannot reduce any of the lengths $|\lambda_{j,a}|$ Taking g to be w_k, we also see that α cannot reduce $|\lambda_{k,a}|$. But α is reductive by hypothesis and the first coordinates of G_a are the $|\lambda_{j,a}|$ for $j \in I(a)$. By the above paragraph, each $y^k_j = 1$ and α is the identity map. This is a contradiction. So $G^\circ_{k,a}$ is nontrivial.

Since α^a was defined by letting $x_j^k = 1$ if $j \leq i$ in I or j is not comparable with i in I, we know that α^a does not change any coordinate $|g|_{\mathcal{H}}$ with $g \in G_b$ for some $a \neq b \in X$ where (i) b is on the path from i the $*$ in T or (ii) where a is not on the path from b to $*$. Let $h \in G_a$ give the first coordinate where α is reductive. By Lemma 7.1, α^a is not reductive on any coordinate of $\cup_{b \in X} G_b$ up to h, and it is as reductive as α is on the h coordinate.

For a particular $l \in I(a) - \{i\}$, x_j^k is constant on the branch of T given by taking $j \geq l$. Hence α^a is carried by $[\mathcal{H}, \underline{\underline{B}}^a]$ in the ascending star of $v = [\mathcal{H}, \underline{\underline{A}}]$. □

Note: Observe that there are some cases where a non-reductive move $\alpha = (\mathcal{H}_a, y^k)$ at v_a still induces a well defined (but non-reductive) move α^a at v. In particular, we must have $G^{\circ}_{k,a}$ nontrivial and $y_i^k = 1$.

Next we investigate how a reductive move at v defines moves at various v_as. Let $\alpha = (\mathcal{H}, y^k)$ be a reductive move at $v = [\mathcal{H}, \underline{\underline{A}}]$ carried by $[\mathcal{H}, \underline{\underline{B}}]$ and let a be an unlabelled vertex of T which is adjacent to k. Suppose that Y is the tree for $\underline{\underline{B}}$ We can define the move $\alpha_a = (\mathcal{H}_a, x^k)$ at $v_a = [\mathcal{H}_a, \underline{0}]$ carried by $[\mathcal{H}, \underline{\underline{B_a}}]$ as follows:

Case 1: *a is the next vertex on a path from k to $*$ in T.* Let i be the stem of a. Set $x^k = w(J(i))^{-1} y^k w(J(i)))$. The tree T_a for $\underline{\underline{B_a}}$ is given by looking at the subtree of Y spanned by vertices in $I(a)$.

Case 2: *k is the stem of a.* Set $x^k = w(J(i))^{-1} y^k w(J(i)))$. As in the previous case, the tree T_a for $\underline{\underline{B_a}}$ is given by looking at the subtree of Y spanned by vertices in $I(a)$.

Lemma 7.4. *If $\alpha = (\mathcal{H}, y^k)$ is a reductive Whitehead move at $v = [\mathcal{H}, \underline{\underline{A}}]$ in $L(G)^F$ carried by $[\mathcal{H}, \underline{\underline{B}}]$ as described above, then there is some a adjacent to k in the tree T determined by $\underline{\underline{A}}$ such that $\alpha_a = (\mathcal{H}_a, x^k)$ is a reductive Whitehead move at $v_a = [\mathcal{H}_a, \underline{0}]$ in $L(G_a)$ carried by $[\mathcal{H}_a, \underline{\underline{B_a}}]$.*

Moreover, if $a_0, a_1, \ldots, a_m$ is a complete list of vertices adjacent to k in T then

$$\alpha = (\alpha_{a_0})^{a_0} (\alpha_{a_1})^{a_1} \ldots (\alpha_{a_m})^{a_m}$$

and all of the (not necessarily reductive) terms in the product commute.

Proof. The last assertion of the lemma follows directly. It remains to show that at least one α_{a_j} is reductive. Let a_0 be the vertex adjacent to k in T which is on the path from k to $*$. Let $a_1, \ldots, a_m$ be the other vertices adjacent to T, ordered so that if $a_r < a_s$ in X then $r < s$ as integers. Assume as always that $y_k^k = 1$. Let t be the least index such that $y_j^k \neq 1$ for some $j \in I(a_t)$. We show that α_{a_t} is reductive.

Since $y_j^k \neq 1$ for some $j \in I(a_t)$, α must change the norm of some letter in G_{a_t}. By the minimality of t, α does not change the norm of any letter in G_a for $a < a_t$. Therefore, since α is reductive, it must be reductive on G_{a_t}. Lemma 7.1 now yields that α_{a_t} is reductive. □

Theorem 7.5. *Let F be a finite subgroup of $\Sigma Aut_1(G)$ that fixes $[\mathcal{H}_0, \underline{\underline{A}}]$, and suppose that $[\mathcal{H}_0, \underline{\underline{A}}]$ is reduced in $L(G)^F$. Then $L(G)^F$ is contractible.*

Proof. We do this by induction, adding the ascending stars of nuclear vertices $v = [\mathcal{H}, \underline{\underline{A}}]$ in $L(G)^F$ step by step according to the norm of Proposition 7.2, always insuring that the reductive part of the star $st(v)$ (of v in $L(G)^F$) is contractible.

We follow the discussion of McCullough and Miller on pages 36-37 of [10]. Namely, we first let R_1 be the reductive part of the star of v. We then let R_2

denote the full subcomplex of R_1 spanned by all vertices each of whose nontrivial based partitions $\underline{\underline{B}}(i)$ is reductive. (Note that in the context of vertices in the star of $v = [\mathcal{H}, \underline{\underline{A}}]$ in $L(G)^F$, a *trivial* based partition $\underline{\underline{B}}(i)$ is one which is equal to $\underline{\underline{A}}(i)$.) Define a poset map $f_1 : R_1 \to R_2$ as follows: If $[\mathcal{H}, \underline{\underline{B}}]$ is in R_1, then send it to $[\mathcal{H}, \underline{\underline{C}}]$, where each nontrivial based partition $\underline{\underline{B}}(j)$ with negative reductivity is replaced by the trivial based partition with the same operative factor. Since $f_1(w) \leq w$ for all $w \in R_1$, Quillen's Poset Lemma [12] yields that R_2 is a deformation retract of R_1.

Let $\underline{\underline{B}}(k)$ denote a partition corresponding to some $[\mathcal{H}, \underline{\underline{B}}]$ in R_2 Suppose that $a_0, \ldots, a_m$ are the vertices adjacent to k in the tree T corresponding to $\underline{\underline{A}}$ (cf. the proof of Lemma 7.4.) Then $\underline{\underline{B}}(k)$ is *admissible* if each $(\underline{\underline{B}}_{a_j})^{a_j}(k)$ is either trivial (that is, equal to $\underline{\underline{A}}(k)$) or reductive. Now let R_3 denote the full subcomplex of R_2 spanned by all vertices each of whose nontrivial reductive based partitions is admissible. Define a map $f_2 : R_2 \to R_3$ by combining all petals of $\underline{\underline{B}}(k)$ containing elements of $I(a_j) - \{k\}$ for each j where $(\underline{\underline{B}}_{a_j})^{a_j}(k)$ is nontrivial and not reductive. As before, $f_2(w) \leq w$ for all $w \in R_2$ so that R_2 deformation retracts to R_3.

For each $a \in A$, let $R_2(st(v_a))$ denote the reductive portion of the star of v_a in $L(G_a)$ where where each based partition is either trivial or reductive. Each nonempty $R_2(st(v_a))$ is contractible by Theorem 5.2. Let $\bar{A} = \{a \in A : R_2(st(v_a)) \neq \emptyset\}$. Recall that if P_1 and P_2 are posets, we can form their join $P_1 \star P_2$ as the poset with elements $P_1 \cup (P_1 \times P_2) \cup P_2$. If $p_1, p_1' \in P_1$, $p_1 < p_1'$ in P_1 $p_2, p_2' \in P_2$, and $p_2 < p_2'$ in P_2, then in the poset $P_1 \star P_2$ we have $(p_1, p_2) \geq (p_1', p_2')$, $(p_1, p_2) \geq p_1$, $(p_1, p_2) \geq p_2$, $p_1 \geq p_1'$, and $p_2 \geq p_2'$. This coincides with the more usual definition of the join of two topological spaces

$$X \star Y = \frac{X \times [0,1] \times Y}{(x, 0, y) \sim (x, 0, y'), (x', 1, y) \sim (x, 1, y)}$$

in the sense that the realization of $P_1 \star P_2$ is homeomorphic to the join of the realization of P_1 with that of P_2. However, from Lemmas 7.3 and 7.4 there is a poset isomorphism

$$f_3 : R_3 \to \star_{a \in \bar{A}} R_2(st(v_a))$$

given by

$$f(\underline{\underline{B}}) = \prod_{a \in \bar{A}, \underline{\underline{B_a}} \neq \underline{\underline{A_a}}} \underline{\underline{B_a}}.$$

Since each poset in the join is contractible, $\star_{a \in \bar{A}} R_2(st(v_a))$ is contractible. $\square$

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