Single-layer antireflection coatings on absorbing substrates for the parallel and perpendicular polarizations at oblique incidence

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Single-layer antireflection coatings on absorbing substrates for the parallel and perpendicular polarizations at oblique incidence

R. M. A. Azzam

Explicit equations are derived that determine the refractive index of a single layer that suppresses the reflection of \( p \)- or \( s \)-polarized light from the planar interface between a transparent and an absorbing medium at any given angle of incidence. The required layer thickness and the system reflectance for the orthogonal unextinguished polarization also follow explicitly. This generalizes earlier work that was limited to normal incidence or to oblique incidence at dielectric–dielectric interfaces. Specific examples are given of \( p \)- and \( s \)-antireflection layers on Si and Al substrates at \( \lambda = 6328 \) Å at various angles of incidence.

I. Introduction

Single-layer antireflection coatings on dielectric substrates for normally incident monochromatic light are well known.\(^1\) The layer refractive index \( N_1 \) must be chosen as

\[
N_1 = \left( N_0 N_2 \right)^{1/2},
\]

(1)
i.e., equal to the geometric mean of the refractive indices \( N_0 \) of the ambient (incidence medium) and \( N_2 \) of the substrate.

At a general angle of (oblique) incidence \( \phi \), the condition of zero reflection by a transparent film on a transparent substrate, for the parallel \( p \) or perpendicular \( s \) polarization, has also been derived in explicit form,\(^3\) yielding \( N_1 \) as a function of \( N_0, N_2, \) and \( \phi \).

When the substrate is absorbing, antireflection at normal incidence continues to be possible using a transparent film of refractive index

\[
N_1 = \left( N_0 N_2 + \frac{N_2 k_2}{N_0 N_2 - N_0^2} \right)^{1/2},
\]

(2)

where \( N_2 = n_2 - j k_2 \) is the substrate complex refractive index. Equation (2) reduces to Eq. (1) when \( k_2 = 0 \).

In this paper we further generalize these earlier results and derive explicit equations for the refractive index of a transparent film on an absorbing substrate necessary to suppress the reflection of \( p \)- or \( s \)-polarized light at any given angle of incidence. The thickness of the antireflection (polarizing) layer, and the associated unextinguished reflectance (for the orthogonal polarization) of the film–substrate system, are also determined. The results are illustrated by specific examples of antireflection layers on semiconducting (Si) and metallic (Al) substrates at one wavelength (\( \lambda = 6328 \) Å).

II. Basic Relations

In what follows, we will consider the antireflection condition for \( p \)- and \( s \)-polarized light separately. For either polarization, zero reflection by the ambient–film–substrate (0–1–2) system happens if the ambient–film and film–substrate interface reflectances are equal:

\[
|r_{01}| = |r_{12}|, \quad \nu = p, s.
\]

(3)

Equation (3) is basic and has been recognized previously.\(^5\)–\(^7\) However, it appears that no attempt has been made to solve it for the film refractive index \( N_1 \) when the substrate is absorbing.\(^8\) With \( N_2 \) complex \( r_{12} \) is also complex. In this case, manipulating Eq. (3) is simplified considerably by replacing it by the equivalent form

\[
r_{01} = r_{12}^* r_{20},
\]

(4)

where * indicates the complex conjugate.

Once \( N_1 \) that satisfies Eq. (3) or (4) has been determined, the normalized polarizing film thickness is readily obtained as

\[
\xi = -\frac{\arg(X_\nu)}{2\pi}, \quad 0 \leq \xi < 1,
\]

(5)

where

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The least film thickness is

\[ d_\nu = \xi_\nu D_\phi, \]

where

\[ D_\phi = \frac{\lambda}{2} (N_1^2 - N_3^2 \sin^2 \theta)^{-1/2} \]

is the film thickness period and \( \lambda \) is the free-space wavelength of light. Higher polarizing film thicknesses are obtained by adding integral multiples of \( D_\phi \) to \( d_\nu \).

The complex-amplitude reflection coefficient of the coated substrate for the unextinguished orthogonal polarization \( \nu' \) (if \( \nu' = p, \nu = s \), and vice versa) is

\[ R_{\nu'} = \frac{r_{\nu\nu'} + r_{\nu'\nu} X_\nu}{1 + r_{\nu\nu'} r_{\nu'\nu} X_\nu}, \]

where \( X_\nu \) is given by Eq. (6). The corresponding intensity reflectance is

\[ R_{\nu'} = |R_{\nu'}|^2. \]

### III. Antireflection of the s Polarization

Fresnel's reflection coefficients of the ambient-film (01) and film-substrate (12) interfaces for the s polarization are given by

\[ r_{01s} = (S_0 - S_1)/(S_0 + S_1), \]
\[ r_{12s} = (S_1 - S_2)/(S_1 + S_2), \]

where

\[ S_i = (\epsilon_i - \epsilon_0 \sin^2 \phi)^{1/2}, \]
\[ \epsilon_i = N_i^2, i = 0,1,2. \]

\( \epsilon_i \) is the dielectric constant of medium \( i \). On substituting Eqs. (11) into Eq. (4), the antireflection condition for s-polarized light takes the form

\[ (S_0^2 + S_2^2) \text{Re}(S_0) = S_0(S_0^2 + |S_2|^2), \]

where \( \text{Re} \) indicates the "real part of." To simplify reaching Eq. (14), we used the algebraic fact that if \((A - B)/(A + B) = (C - D)/(C + D),\) then \(A/B = C/D.\) From Eq. (14) we get

\[ S_0^2 = S_0|S_2|^2 - S_0 \text{Re}(S_2)/|\text{Re}(S_0) - S_0|. \]

If \( S_i \) from Eq. (12) are used in Eq. (15), we obtain

\[ \epsilon_1 = \epsilon_0 \left[ \sin^2 \phi + \cos \phi \left[ |\beta_2| - \cos \phi \text{Re}(\beta_2^{1/2}) \right] \right], \]
\[ \beta_2 = (\epsilon_2/\epsilon_0) - \sin^2 \phi. \]

Equation (16) gives the desired dielectric constant of the s-polarization antireflection layer in terms of the ambient dielectric constant \( \epsilon_0 \), substrate complex dielectric constant \( \epsilon_2 = (n_2 - jk_2)^2 \), and angle of incidence \( \phi \). The corresponding refractive index is

\[ N_1 = \epsilon_1^{1/2}. \]

Knittl's result\(^9\) for a transparent film on a transparent substrate is obtained as a special case of Eq. (16) if the imaginary part of \( \epsilon_2 \) is set equal to zero.

As a first example, consider the reflection of light \((\lambda = 6328 \text{ Å})\) in air \((\epsilon_0 = 1)\) by a Si substrate of complex refractive index\(^10\) \( N_2 = 3.85 - j0.02 \). The refractive index \( N_1 \) of the s-polarization antireflection layer was computed from Eqs. (16)-(18) and is plotted in Fig. 1 as a function of \( \phi \) from normal \((\phi = 0)\) to grazing \((\phi = 90^\circ)\) incidence. For \(0 \leq \phi \leq 80^\circ\), we have \(1.9622 \geq N_1 \geq 1.2713\), which correspond to several existing thin-film coating materials.\(^11\),\(^12\) For \( \phi > 80^\circ\), \( N_1 \) is too close to 1 to be realizable by a thin solid film.

Figure 2 shows the normalized and actual film thicknesses \( \xi_\nu \) and \( d_\nu \) \((\text{angstroms})\) thicknesses of the s-polarization antireflection layer on Si at \(\lambda = 6328 \text{ Å}\) vs angle of incidence \( \phi \) (degrees).

---

\( X_\nu = r_{01\nu}/r_{12\nu}, \)

\( d_\nu = \xi_\nu D_\phi, \)

\( D_\phi = \frac{\lambda}{2} (N_1^2 - N_3^2 \sin^2 \theta)^{-1/2} \)

\( r_{01\nu} = (S_0 - S_1)/(S_0 + S_1), \)
\( r_{12\nu} = (S_1 - S_2)/(S_1 + S_2), \)

\( S_i = (\epsilon_i - \epsilon_0 \sin^2 \phi)^{1/2}, \)
\( \epsilon_i = N_i^2, i = 0,1,2. \)

\( \epsilon_1 = \epsilon_0 \left[ \sin^2 \phi + \cos \phi \left[ |\beta_2| - \cos \phi \text{Re}(\beta_2^{1/2}) \right] \right], \)
\( \beta_2 = (\epsilon_2/\epsilon_0) - \sin^2 \phi. \)

\( N_1 = \epsilon_1^{1/2}. \)

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\( r_{01\nu} = (S_0 - S_1)/(S_0 + S_1), \)
\( r_{12\nu} = (S_1 - S_2)/(S_1 + S_2), \)

\( S_i = (\epsilon_i - \epsilon_0 \sin^2 \phi)^{1/2}, \)
\( \epsilon_i = N_i^2, i = 0,1,2. \)

\( \epsilon_1 = \epsilon_0 \left[ \sin^2 \phi + \cos \phi \left[ |\beta_2| - \cos \phi \text{Re}(\beta_2^{1/2}) \right] \right], \)
\( \beta_2 = (\epsilon_2/\epsilon_0) - \sin^2 \phi. \)

\( N_1 = \epsilon_1^{1/2}. \)

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Fig. 1. Refractive index \( N_1 \) of s-polarization antireflection layer on Si \((N_2 = 3.85 - j0.02)\) as a function of angle of incidence \( \phi \) (degrees). Light \((\lambda = 6328 \text{ Å})\) is assumed to be incident from air \((N_0 = 1)\).

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Fig. 2. Normalized \( \xi_\nu \) and actual \( d_\nu \) \((\text{angstroms})\) thicknesses of the s-polarization antireflection layer on Si at \(\lambda = 6328 \text{ Å}\) vs angle of incidence \( \phi \) (degrees).
Figure 3 presents the unextinguished \( \rho \) reflectance \( \mathcal{R}_\rho \) vs angle of incidence \( \phi \) (degrees) for Si, which is antireflection-coated for the \( s \) polarization. \( \mathcal{R}_p \) and \( \mathcal{R}_s \) are the \( p \) and \( s \) reflectances of the bare Si substrate.

![Figure 3](image_url)

**Table I.** \( s \)-Polarization Antireflection Layers on Si at Five Angles of Incidence

<table>
<thead>
<tr>
<th>( \phi )</th>
<th>( N_1 )</th>
<th>( \xi_s )</th>
<th>( d_s )</th>
<th>( R_{p} )</th>
<th>( \mathcal{R}_{p} )</th>
<th>( \mathcal{R}_{s} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.96218</td>
<td>0.49886</td>
<td>804.4</td>
<td>0</td>
<td>0.3453</td>
<td>0.3453</td>
</tr>
<tr>
<td>30</td>
<td>1.88576</td>
<td>0.49896</td>
<td>868.3</td>
<td>0.0063</td>
<td>0.2933</td>
<td>0.3971</td>
</tr>
<tr>
<td>45</td>
<td>1.78218</td>
<td>0.49909</td>
<td>965.3</td>
<td>0.0361</td>
<td>0.2204</td>
<td>0.4694</td>
</tr>
<tr>
<td>60</td>
<td>1.62041</td>
<td>0.49927</td>
<td>1153.4</td>
<td>0.1333</td>
<td>0.1075</td>
<td>0.5849</td>
</tr>
<tr>
<td>75</td>
<td>1.37754</td>
<td>0.49950</td>
<td>1609.1</td>
<td>0.3710</td>
<td>0.0002</td>
<td>0.7571</td>
</tr>
</tbody>
</table>

\( \phi \) is the angle of incidence in degrees. \( N_1 \) is the layer required refractive index for \( s \)-polarization antireflection, and \( \xi_s \) is its normalized thickness (as a fraction of the thickness period). \( d_s \) is the corresponding actual (least) thickness in angstroms. \( \mathcal{R}_p \) and \( \mathcal{R}_s \) are the reflectances of the film-free air-Si interface for the \( p \) and \( s \) polarizations, respectively. The complex refractive index of Si is taken as \( N_2 = 3.85 - j0.02 \) at \( \lambda = 6328 \) Å.

The unextinguished reflectance \( \mathcal{R}_p \) of the coated Si, computed from Eqs. (6), (9) and (10), plotted vs \( \phi \). We have superimposed on Fig. 3 the reflectances \( \mathcal{R}_p \) and \( \mathcal{R}_s \) of the film-free air-Si interface. At point A (\( \phi_A \) is between 58 and 59°) Abélès’s condition of equal \( \rho \) reflectances of the coated and uncoated substrate is satisfied. For 0 ≤ \( \phi < \phi_A \), the coating that reduces the \( s \) reflectance to zero also diminishes the \( p \) reflectance below its bare-substrate value. At \( \phi = 0 \), the \( p \) and \( s \) polarizations are indistinguishable, and total antireflection of light occurs. For 0 ≤ \( \phi < 45^\circ \), suppression of the \( s \) polarization is accompanied by significant reduction of the \( p \) reflectance, so that excellent (but incomplete) overall antireflection is achieved by one layer.

Table I lists data on \( s \)-polarization antireflection layers on Si (at \( \lambda = 6328 \) Å) at five angles of incidence 0, 30, 45, 60, and 75°. Silicon nitride is a particularly suitable antireflection coating material for any \( \phi \) from 0 to 45°. If prepared by sputtering, its refractive index can be finely tuned within the desired range 1.78 ≤ \( N_1 \) ≤ 1.96 depending on its stoichiometry.

For \( \phi = 45^\circ \), the antireflection layer reduces the \( s \) reflectance of the Si surface from 46.94% to 0 and the \( p \) reflectance from 22.04 to 3.61%. Thus the residual reflectance of the coated Si for unpolarized incident light is only 1.8%. At \( \phi = 75^\circ \), \( \mathcal{R}_p = 37.10\% \) is not sufficiently high to make the film-substrate system an efficient polarizer. (Reflection from bare Si at the pseudo-Brewster angle, \( \phi = 75.44^\circ \), makes a simpler more efficient polarizer.)

As a second example, we consider \( s \)-polarization antireflection layers on an Al substrate with complex refractive index \( N_2 = 1.212 - j6.924 \) at \( \lambda = 6328 \) Å in air (\( c_0 = 1 \)). As \( \phi \) is increased from 0 to 90°, \( N_1 \), computed from Eqs. (16)-(18), decreases from 15.07821 to 1 monotonically. To exclude far from realistic values of the film refractive index, \( N_1(\phi) \) is plotted in Fig. 4 vs \( \phi \) over the restricted range 60° ≤ \( \phi \) ≤ 90°. The corresponding required normalized \( \xi_s \) and actual \( d_s \) film thicknesses are plotted in Fig. 5. Notice that \( \xi_s \) differs appreciably from \( \frac{1}{2} \), so that the layer thickness is no longer close to quarterwave. The unextinguished reflectance \( \mathcal{R}_p \) of the coated surface and the reflectances \( \mathcal{R}_p \) and \( \mathcal{R}_s \) of the bare Al substrate are shown in Fig. 6 as functions of \( \phi \). At A, Abélès condition is satisfied, as before.

To cite a specific numerical result, we give the characteristics of a thin film on Al that suppresses the reflection of the \( s \) polarization at 85°. The required refractive index of the film is \( N_1 = 2.220 \) (rounded to three decimal places) corresponding to ZnS, for example. The normalized and actual film thicknesses are \( \xi_s = \frac{1}{2} \), \( d_s = 15.0782 \) Å.
(4) puts the antireflection condition for the \( p \) polarization in the form
\[
(e_{1} S_{2}^{2} + e_{2} S_{1}^{2}) \text{Re}(e_{2} S_{2}) = S_{0} |e_{0}|^{2} S_{1}^{2} + e_{2} S_{2}^{2}.
\]
By replacing \( S_{1} \) from Eq. (12) into Eq. (20) and rearranging, a quadratic equation
\[
A r_{1} + B r_{1} + C = 0
\]
is obtained, where
\[
A = \cos^{2} \phi \, \text{Re}(\xi_{2}(\sin^{2} \phi) / 2) - \cos \phi \xi_{2} - \sin^{2} \phi),
B = \text{Re}(\xi_{2}(\sin^{2} \phi) / 2) - \cos \phi \xi_{2}^{2},
C = -(\sin^{2} \phi) B;
\]
are normalized film and substrate dielectric constants, respectively. Solving Eq. (21) gives
\[
e_{1} = e_{0} [-B \pm (B^{2} - 4AC)^{1/2}] / 2A,
\]
from which
\[
N_{1} = e_{1}^{-1}.
\]
Equation (18) has been repeated as Eq. (25) for ease of reference.
Equations (21)–(25) give the desired refractive index of the \( p \)-polarization single-layer antireflection coating in terms of the ambient dielectric constant \( e_{0} \), substrate complex dielectric constant \( e_{2} = (n_{2} - jk_{2})^{2} \), and angle of incidence \( \phi \). In the special case of a transparent substrate, \( e_{2} \) is real, and the result reduces to that given by Knittl. Two values of \( N_{1} \) are possible that correspond to the two roots of the quadratic equation. They will be denoted by the additional subscripts + and −, according to the + and − signs that appear in Eq. (24). Because \( N_{1} \) must be real and positive, the following conditions must be satisfied:
\[
B^{2} - 4AC \geq 0,
\]
\[
e_{1} > 0.
\]
We now consider examples of \( p \)-polarization antireflection layers on the same semiconducting (Si) and metallic (Al) substrates (at \( \lambda = 6328 \, \text{Å} \) and in air) as in Sec. III.

Figure 7 shows \( N_{1+} \) and \( N_{1-} \) plotted vs \( \phi \) between \( 0 = 0 \) and \( \phi = 90^\circ \) for Si. Notice the crossover between the two solutions that takes place as \( \phi \) passes through the pseudo-Brewster angle, \( \phi_{PB} = 75.44^\circ \), of Si. If the two solutions of \( N_{1} \) are ordered according to their magnitudes as low and high, \( N_{1+} \) and \( N_{1-} \), we find that \( N_{1+} < 1 \) for \( \phi < \phi_{PB} \) and that \( N_{1+} \) is very slightly >1 for \( \phi > \phi_{PB} + 0.1^\circ \). This low-index branch does not correspond to practical thin-film coating materials and will not be pursued further.

Figure 8 shows the normalized and actual film thicknesses \( \xi_{p} \) (= \( \xi_{p+} \) or \( \xi_{p-} \)) and \( d_{p} \), respectively, of the \( p \)-polarization antireflection layer on Si; \( d_{p} \) is associated with and calculated from \( N_{1+} \) and \( N_{1-} \) the higher of the two refractive indices \( N_{1+} \) and \( N_{1-} \). Significant deviation of the thickness from a quarterwave occurs in the vicinity of the pseudo-Brewster angle.17

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**Fig. 5.** Normalized \( L_{p} \) and actual \( R_{p} \) (angstroms) thicknesses of the \( s \)-polarization antireflection layer on Al at \( \lambda = 6328 \, \text{Å} \) vs angle of incidence \( \phi \) (degrees).

**Fig. 6.** Unextinguished \( p \) reflectance \( R_{p} \) vs angle of incidence \( \phi \) (degrees) for Al, which is antireflection-coated for the \( s \) polarization.

---

0.41425 and \( d_{p} = 661 \, \text{Å} \). Such a layer reduces the reflectance of Al for the \( s \) polarization from the bare-surface value of 99.17\% to zero, while enhancing its \( p \) reflectance from 73.03 to 95.26\%. Thus this film-substrate system functions as an excellent reflection polarizer.

### IV. Antireflection of the \( p \) Polarization

Fresnel’s reflection coefficients of the ambient-film and film-substrate interfaces for the \( p \) polarization are given by
\[
r_{0p} = (\xi_{1} S_{0} - e_{0} S_{1})/(\xi_{1} S_{0} + e_{0} S_{1}), \quad (19a)
\]
\[
r_{12p} = (e_{2} S_{1} - \xi_{1} S_{2})/(e_{2} S_{1} + \xi_{1} S_{2}), \quad (19b)
\]
where \( e_{1} \) and \( S_{1} \) are the same as previously defined in Eqs. (12) and (13). Substitution of Eqs. (19) into Eq.
FIG. 7. Refractive indices $N_{1+}$ and $N_{1-}$ of $p$-polarization antireflection layers on Si ($N_2 = 3.85 - j0.02$) as functions of angle of incidence $\phi$ (degrees). Light ($\lambda = 6328 \text{ Å}$) is assumed to be incident from air ($N_0 = 1$).

FIG. 8. Normalized ($\zeta_p = \zeta_{p-} = \zeta_p$) and actual ($d_p$, angstroms) thicknesses of the $p$-polarization antireflection layer on Si at $\lambda = 6328 \text{ Å}$ vs angle of incidence $\phi$ (degrees). $d_p$ corresponds to the higher of the two refractive indices $N_{1+}$ and $N_{1-}$ of Fig. 7.

Figure 9 shows the unextinguished reflectance $R_s$ of Si coated with the $p$-antireflection layer of index $N_{1h}$. The bare Si reflectances $R_p$ and $R_s$ for the $p$ and $s$ polarizations are also indicated as functions of $\phi$.

Table II summarizes data on $p$-antireflection layers on Si at $\lambda = 6328 \text{ Å}$ for the same five angles of incidence $\phi = 0, 30, 45, 60,$ and $75^\circ$ considered in Table I. Only the high-index solution is included. The first lines of Tables I and II (at $\phi = 0$) are identical, as expected.
For any given interface between (linear, nonmagnetic, and optically isotropic) transparent and absorbing media, the refractive index \( N_1 \) of an intermediate transparent layer can be found that allows suppression of the reflection of \( p \)- or \( s \)-polarized light at a specified angle of incidence \( \phi \). The explicit equations that determine \( N_1 \) are Eqs. (16)-(18) for the \( s \) polarization, and Eqs. (21)-(25) for the \( p \) polarization. \( N_1 \) must correspond to the refractive index of an existing thin-film coating material for the mathematical solution to be physically realizable by a single film. Thus \( p \) and \( s \) antireflection will often be possible only over limited ranges of \( \phi \), depending on the substrate and ambient optical constants. The thickness of the antireflection layer and the reflectance of the coated substrate for the unextinguished orthogonal polarization are calculated, also explicitly, from Eqs. (5)-(8) and Eqs. (9) and (10), respectively. The method is applied to Si and Al substrates at \( \lambda = 6328 \) Å, and the results appear graphically and in tables.

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References

8. For given optical constants of the ambient, film, and substrate, Eq. (3) becomes a transcendental equation in the angle of incidence \( \phi \) that can be solved by iteration. For high-reflectance (metal) substrates, explicit equations for the polarizing angles, that are approximate but accurate, can be obtained.
9. See, for example, R. M. A. Azzam and N. M. Bashara, Ellipsometry and Polarized Light (North-Holland, Amsterdam, 1977), Chap. 4.
17. Examining the behavior of the solutions in the immediate neighborhood (within \( \pm 0.01^\circ \)) of the pseudo-Brewster angle \( \phi_\text{PB} \) reveals some interesting detail. For example, there is a gap in this already very narrow range of \( \phi \) where no solutions exist. The gap is approximately bounded by the angles \( \phi_1 = 75.4397^\circ \) and \( \phi_2 = 75.4401^\circ \) and includes \( \phi_\text{PB} \). The solutions \( N_{1+} \) and \( N_{1-} \) merge at the upper edge of the gap to form one continuous curve. Furthermore, the concident \( \theta_\text{PB} \) and \( \theta_\text{ext} \) tend to zero as the lower edge of the gap is approached from the left and tend to one as the upper gap edge is approached from the right.
18. Elsewhere we consider directly and in detail the conditions for the extinction of the \( p \) and \( s \) polarizations at the same angle of incidence by a transparent film on an absorbing substrate along with interesting applications; see R. M. A. Azzam, “Extinction of the \( p \) and \( s \) Polarizations of a Wave on Reflection at the Same Angle from a Transparent Film on an Absorbing Substrate: Applications to Parallel-Mirror Crossed Polarizers and a Novel Integrated Polarimeter,” J. Opt. Soc. Am. A 2, in press (1985).