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Plurality of principal angles for a given pseudo-Brewster angle when polarized light is reflected at a dielectric–conductor interface

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The pseudo-Brewster angle $\phi_{PB}$ of minimum reflectance for $p$-polarized light and the principal angle $\phi$ at which incident linearly polarized light of the proper azimuth is reflected circularly polarized are considered as functions of the complex relative dielectric function $\varepsilon$ of a dielectric–conductor interface over the entire complex $\varepsilon$ plane. In particular, the spread of $\phi$ for a given $\phi_{PB}$ is determined, and the maximum difference $(\phi - \phi_{PB})_{\text{max}}$ is obtained as a function of $\phi_{PB}$. The maximum difference $(\phi - \phi_{PB})_{\text{max}}$ approaches $45^\circ$ and $0$ in the limit as $\phi_{PB} \to 0$ and $90^\circ$, respectively. For $\phi_{PB} < 22.666^\circ$, multiple principal angles $\phi_i, i = 1, 2, 3$, as is illustrated by several examples. © 2008 Optical Society of America

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1. INTRODUCTION

The reflection of monochromatic $p$- and $s$-polarized light at an angle $\phi$ by the planar interface between a transparent medium of incidence (ambient) of refractive index $n_0$ and an absorbing medium of refraction (substrate) of complex refractive index $N_1 = n_1 - jk_1$ is governed by the well-known complex-amplitude Fresnel reflection coefficients [1–3]:

\[
eq \frac{\cos \phi - (\varepsilon - \sin^2 \phi)^{1/2}}{\cos \phi + (\varepsilon - \sin^2 \phi)^{1/2}},
\]

\[
r_p = \frac{\cos \phi - (\varepsilon - \sin^2 \phi)^{1/2}}{\cos \phi + (\varepsilon - \sin^2 \phi)^{1/2}},
\]

\[
\varepsilon = N_1^2/n_0^2 = (n - jk)^2 = \varepsilon_r - j\varepsilon_i.
\]

For a given value of the complex relative dielectric function $\varepsilon$, which is characteristic of a given interface at a given wavelength, the amplitude reflectance $|r_p|$ of $p$-polarized light as a function of $\phi$ reaches a minimum at the pseudo-Brewster angle (PBA) $\phi_{PB}$. If the medium of refraction is also transparent, $\varepsilon_i = 0$, the minimum reflectance is zero, $|r_p|_{\text{min}} = 0$, and the PBA $\phi_{PB}$ reverts back to the usual Brewster angle $\phi_B = \tan^{-1}\sqrt{\varepsilon_r}$. Recall that for any $\varepsilon$ the amplitude reflectance $|r_p|$ of $s$-polarized light increases monotonically as a function of $\phi$ between normal and grazing incidence, $0 \leq \phi \leq 90^\circ$.

The first correct derivation of the relation between $\phi_{PB}$ and complex $\varepsilon$ (which replaces Brewster’s law) is believed to be that of Humphreys–Owen [4], as was noted by Holl [5]. Continued interest in this salient feature of the reflection of $p$-polarized light (and other electromagnetic waves) at a dielectric–conductor interface has led to several subsequent derivations [6–9].

Another important and distinct angle of incidence is the principal angle (PA) $\phi_{PA}$ at which incident linearly polarized light of the proper azimuth (called the principal azimuth $\phi_{PA}$) is reflected circularly polarized [1–3,10]. This occurs when the differential reflection phase shift $\Delta$ of $p$- and $s$-polarized light is quarter-wave, i.e.,

\[
\Delta = \delta_p - \delta_s = 90^\circ,
\]

\[
\delta_p = \text{arg}(r_p), \quad \delta_s = \text{arg}(r_s).
\]

The ratio of complex $p$ and $s$ reflection coefficients, also known as the ellipsometric function $\rho$ [2], is obtained from Eqs. (1) and (2) as

\[
\rho = r_p/r_s = \frac{\sin \phi \tan \phi - (\varepsilon - \sin^2 \phi)^{1/2}}{\sin \phi \tan \phi + (\varepsilon - \sin^2 \phi)^{1/2}}.
\]

At the principal angle, $\phi = \phi_{PA}$, $\rho$ becomes pure imaginary,

\[
\rho = \tilde{\rho} = j \tan \phi_{PA}.
\]

For a given complex $\varepsilon$, the PA, $\phi_{PA}$, is determined by solving a cubic equation [10]:

\[
a_3 u^3 + a_2 u^2 + a_1 u + a_0 = 0,
\]

\[
a_0 = \varepsilon_r^2 + \varepsilon_i^2, \quad a_1 = -2(a_0 + \varepsilon_r),
\]

\[
a_2 = a_0 + 4\varepsilon_r + 1, \quad a_3 = -2(\varepsilon_r + 1).
\]
However, as has been noted in [5,10], there exists a small region of fractional optical constants (0 < |ε|, |ε| < 1) within which three distinct PAs exist for each complex ε. This domain of multiple principal angles (MPAs), shown highlighted in Fig. 1, is bounded by the real axis, ε_r=0, and the curve whose parametric equation is given by [10]

\[ ε_r = u + \frac{u^3(u - 2)}{(1 - u)^3}, \]
\[ ε_i = \frac{(2u^6 - 4u^5 + u^4)^{1/2}}{(1 - u)^3}, \]
\[ 0 \leq u \leq 1 - \frac{1}{\sqrt{2}} = 0.293. \]  

Equations (10) represent the locus of all possible values of complex ε for which two of the three principal angles, φ = sin−1(√u), coincide; this locus is represented by the dashed curve in Fig. 1. The cusp point P corresponds to u = 1/4 and is located at ε = (5/27, √2/27). Fractional optical constants are encountered for many materials in the vacuum UV and x-ray spectral regions [11,12] and also in attenuated or total internal reflection when light is incident from an optically dense medium [13].

Because of approximate formulations used in metal optics, the PBA and PA are sometimes erroneously presumed to be the same. In this paper the difference between these two angles, φ − φ_pB, is thoroughly investigated as a function of complex ε. This is accomplished in Section 2 by deliberately holding φ_pB constant and determining all possible values of the associated PA φ. The maximum difference (φ − φ_pB)_{max} is also determined as a function of φ_pB. Unusual results are obtained in the domain of MPAs, as is described in Section 3. Finally, Section 4 gives a brief summary of the paper.

2. RANGE OF PRINCIPAL ANGLES FOR A GIVEN PSEUDO-BREWSTER ANGLE

All possible values of complex ε = (ε_r, ε_i) for which the PBA φ_pB is one and the same are obtained as follows [14]:

\[ ε_r = |ε| \cos θ, \quad ε_i = |ε| \sin θ, \]  
\[ |ε| = \ell \cos (\sqrt[3]{3}), \]
\[ \ell = \frac{2u[1 - (2u/3)]^{1/2}}{(1 - u)}, \]
\[ \zeta = \cos^{-1}\left(1 - \frac{(1 - u)\cos θ}{[1 - (2u/3)]^{3/2}}\right), \]
\[ u = \sin^2 φ_pB, \quad 0 \leq θ \leq 180°. \]

As θ is increased from 0 to 180°, the minimum reflectance \( r_{\text{pB}} \) at the same φ_pB increases monotonically from 0 to 1 [15]. For given φ_pB, and for each θ from 0 to 180° in steps of 1° ε = (ε_r, ε_i) is calculated using Eqs. (11) and (12) and the corresponding values of φ are determined from Eqs. (7)–(9).

In Fig. 2 the difference φ − φ_pB is plotted as a function of θ, 0 ≤ θ = 180°, for constant values of φ_pB from 25° to 85° in equal steps of 5°. For each φ_pB in this range, there is only one PA, φ > φ_pB, and the difference φ − φ_pB increases monotonically as a function of θ. In Fig. 2 the curve for φ_pB = 85° almost coincides with the θ axis.

From Fig. 2 it is also apparent that the maximum difference (φ − φ_pB)_{max} occurs at θ = 180° and that

\[ \partial(φ − φ_pB)/\partial θ = 0, \quad θ = 0, 180° \]  

At the limiting angle θ = 180°, Eqs. (11) and (12) yield ε_i = 0 and ε_r < 0 given by

Fig. 1. Domain of MPAs, shown highlighted, is bounded by the real axis, ε_r=0, and the dashed curve described by Eqs. [10]. Cusp point P is located at ε = (5/27, √2/27) = (0.1852, 0.0524).
\[ e = e_r = -\frac{1}{2} \tan^2 \phi_{pB}[1 + (9 - 8 \sin^2 \phi_{pB})^{1/2}] \]  \hspace{1cm} (14)

The maximum PA \( \bar{\phi} \) that corresponds to \( e_r \) of Eq. (14) is given by

\[ \bar{\phi}_{\text{max}} = \sin^{-1} \left\{ \frac{1}{2} [(e_r + 1) + (e_r^2 - 6e_r + 1)^{1/2}] \right\} \]  \hspace{1cm} (15)

The maximum difference \( (\bar{\phi} - \phi_{pB})_{\text{max}} \) calculated from Eqs. (14) and (15) is 24.207°, 15.540°, 7.458°, and 0.073° when \( \phi_{pB} = 30°, 45°, 60°, \) and \( 85° \), respectively. Figure 3

Fig. 2. (Color online) Difference of PA and PBA \( \bar{\phi} - \phi_{pB} \) plotted as a function of the angle \( \theta \) of complex \( e, 0 < \theta < 180° \), for constant values of \( \phi_{pB} \) from 25° to 85° in equal steps of 5°.

Fig. 3. (Color online) Maximum difference \( (\bar{\phi} - \phi_{pB})_{\text{max}} \) as a function of \( \phi_{pB} \) over the entire range \( 0 < \phi_{pB} < 90° \).
shows \((\tilde{\phi} - \phi_B)_{\text{max}}\) plotted versus \(\phi_B\) over the entire range \(0 < \phi_B < 90^\circ\). Notice that \((\tilde{\phi} - \phi_B)_{\text{max}} = 45^\circ\) in the limit as \(\phi_B \to 0\) and that \((\tilde{\phi} - \phi_B)_{\text{max}} = 0\) in the limit as \(\phi_B \to 90^\circ\). The latter limit is approached by metals in the far IR [9].

3. DOMAIN OF MULTIPLE PRINCIPAL ANGLES

MPAs exist when the PBA falls in the range

\[ 0 < \phi_B < 22.666^\circ. \]  

(16)

Figure 4 shows four constant-PBA contours (CPBAC) in the complex \(\varepsilon\) plane that correspond to \(\phi_B = 20^\circ, 21^\circ, 22^\circ\), and \(22.666^\circ\). The CPBAC at \(\phi_B = 22.666^\circ\) passes through the cusp point \(P\).

As an example of MPAs, consider \(\varepsilon = (0.1349, 0.0118)\), which corresponds to \(\theta = 5^\circ\) on the CPBAC \(\phi_B = 20^\circ\). For this value of complex \(\varepsilon\), Fig. 5 shows \(|r_p|, |r_s|\), and \(\Delta\) as functions of the angle of incidence \(\phi\). The minimum reflec-

\[
324u^3 - 80u^2 - 2u + 1 = 0. 
\]  

(17)

Fig. 4. (Color online) Constant-pseudo-Brewster-angle contour (CPBAC) in the complex \(\varepsilon\) plane that corresponds to \(\phi_B = 20^\circ, 21^\circ, 22^\circ\), and \(22.666^\circ\). The CPBAC at \(\phi_B = 22.666^\circ\) passes through the cusp point \(P\).
tance $|r_{min}|$ appears at $\phi = \phi_{PB} = 20^\circ$, and $\Delta = 90^\circ$ occurs at three distinct PAs: $\phi_1 = 39.13^\circ$, $\phi_2 = 24.01^\circ$, and $\phi_3 = 20.49^\circ$. All three PAs $\phi_i$, $i = 1, 2, 3$, are $> \phi_{PB}$, which is true for any complex $\epsilon$.

Figure 6 shows multiple solution branches $\phi_i - \phi_{PB}$, $i = 1, 2, 3$, as functions of $\theta$ for $\phi_{PB} = 20^\circ$, $21^\circ$, and $22^\circ$. For each $\phi_{PB}$ the solid, thin-dashed, and thick-dashed curves correspond to $\phi_1 > \phi_2 > \phi_3$. MPAs exist over a small range of $\theta$, $0 \leq \theta = \theta_{max}$, where $\theta_{max}$ is a function of $\phi_{PB}$. Note that $\phi_3 - \phi_{PB}$ is almost independent of $\phi_{PB}$ for small $\theta (< 7^\circ)$. Also note that Eq. (13) is again satisfied at $\theta = 0$.

Fig. 6. (Color online) Multiple solution branches of the difference function $\phi_i - \phi_{PB}$, $i = 1, 2, 3$, plotted versus the angle $\theta$ of complex $\epsilon$, for $\phi_{PB} = 20^\circ$, $21^\circ$, and $22^\circ$. For each $\phi_{PB}$ the solid, thin-dashed, and thick-dashed curves correspond to $\phi_1 > \phi_2 > \phi_3$.

Fig. 7. (Color online) CPBAC for $\phi_{PB} = 22.35^\circ$. This curve intersects the boundary of the domain of MPAs at three points, $A$, $B$, and $C$, where $\theta_A = 7.730^\circ$, $\theta_B = 10.763^\circ$, $\theta_C = 14.614^\circ$. Each $\phi_{PB}$ the solid, thin-dashed, and thick-dashed curves correspond to $i = 1, 2, 3$, respectively, where $\phi_1 > \phi_2 > \phi_3$. MPAs exist over a small range of $\theta$, $0 \leq \theta = \theta_{max}$, where $\theta_{max}$ is a function of $\phi_{PB}$. Note that $\phi_3 - \phi_{PB}$ is almost independent of $\phi_{PB}$ for small $\theta (< 7^\circ)$. Also note that Eq. (13) is again satisfied at $\theta = 0$. 

More complex behavior is encountered in a very narrow range of the PBA, 22.339° < \phi_{pB} < 22.5°. Figure 7 shows the CPBAC for \phi_{pB} = 22.35°. This curve intersects the boundary of the region of MPAs at three points \(A, B, C\) where \(\theta_A = 7.730°, \theta_B = 10.763°, \theta_C = 14.614°\).

Figure 8 shows \(\tilde{\phi}_i - \phi_{pB}, i=1,2,3\), as functions of \(\theta\) when \(\phi_{pB} = 22.35°\). For this PBA, MPAs exist for \(0 \leq \theta \leq \theta_A\) and \(\theta_B \leq \theta \leq \theta_C\), whereas one PA appears when \(\theta_A < \theta < \theta_B\) and \(\theta > \theta_C\).

Finally, Fig. 9 shows a composite plot of multiple solution branches of the difference function \(\tilde{\phi}_i - \phi_{pB}, i=1,2,3\), versus \(\theta\) for \(\phi_{pB} = 21°, 22°, 22.3°, 22.35°, 22.5°, \) and 22.666° in the domain of MPAs.
22.666° in the domain of MPA. As in Fig. 6, for each \( \phi_{pB} \) the solid, thin-dashed, and thick-dashed curves correspond to \( \tilde{\phi}_1 > \tilde{\phi}_2 > \tilde{\phi}_3 \). Again, notice that \( \tilde{\phi}_3 - \phi_{pB} \) is almost independent of \( \phi_{pB} \) for small \( \theta < 7° \).

4. SUMMARY

The main conclusions of this work are summarized below:

1. Whereas there is only one unique pseudo-Brewster angle \( \phi_{pB} \) that characterizes a given dielectric–conductor interface, one, two, or three principal angles \( \tilde{\phi}_i > \phi_{pB}, i = 1, 2, 3 \), may exist for the same complex \( \varepsilon \).
2. For a fixed \( \phi_{pB} \) there is a spread of each of the three possible associated principal angles \( \tilde{\phi}_i, i = 1, 2, 3 \).
3. Only one principal angle \( \tilde{\phi}_1 \) exists per each complex \( \varepsilon \) if \( \phi_{pB} > 22.666° \).
4. The maximum difference \( \tilde{\phi} - \phi_{pB} \) for a given \( \phi_{pB} \) occurs when \( \varepsilon \) becomes real negative and is determined by Eqs. (14) and (15). \( \tilde{\phi} - \phi_{pB} \) max \( \rightarrow \) 45° and 0 in the limit as \( \phi_{pB} \rightarrow 0 \) and 90°, respectively.
5. For \( \phi_{pB} \approx 85° \), we find that \( \tilde{\phi} - \phi_{pB} \) max \( \approx \) 0.1°.
6. Complex behavior of the difference function \( \tilde{\phi}_i - \phi_{pB}, i = 1, 2, 3 \), is encountered in the domain of fractional optical constants as is illustrated by Figs. 6 and 9.

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