Poincaré sphere representation of the fixed-polarizer rotating-retarder optical system

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The combination of a linear polarizer and linear retarder is a versatile optical system that is commonly used for polarization-state generation (PSG) and polarization-state detection. Rotation of these elements around the light beam axis creates a variable elliptic polarizer or analyzer. The retardance (differential phase shift) is often selected as quarter wave. However, it has been shown recently that for PSG, other choices of may be optimal (6).

When the polarizer is set at a fixed azimuth and the retarder is rotated, the point that represents the emergent polarization state in PSG (or the elliptic analyzer in PSM) traces a three-dimensional, nonplanar, figure-8 contour on the Poincaré sphere. A sketch of such a contour on the Poincaré sphere for a rotating quarter-wave retarder (QWR) appears in an early paper by Rajagopalan and Ramaseshan. A computer-generated family of such curves, for different values of , are presented by Sabatke et al. Stereographic projections of the QWR contour onto a tangent (complex) plane are given by Azzam et al.

In this communication, the exact nature of this contour is determined as the line of intersection of a right-circular cylinder with the Poincaré sphere. The cylinder axis is parallel to the polar axis, touches the sphere at the equator (at the point that represents the linear polarization transmitted by the fixed polarizer), and has a radius . (Thus , 1/2, and 1, when , π/2, and π, respectively.) Normal projections of the trajectory in the coordinate planes of the normalized Stokes parameter space are also determined.

Figure 1 shows a PSG system that consists of an ideal linear polarizer with transmission-axis azimuth and a linear retarder of retardance and fast-axis azimuth . and are measured from the x axis of a transverse reference coordinate system for a monochromatic light beam traveling in the direction of the z axis. When the polarizer is fixed, we set for simplicity and without loss of generality. With the Stokes–Mueller calculus, the normalized Stokes parameters of the output light are obtained as

\[
\begin{align*}
 s_0 &= 1, \\
 s_1 &= \cos^2 2C + \sin^2 2C \cos \Delta, \\
 s_2 &= \sin 2C \cos 2C (1 - \cos \Delta), \quad s_3 = \sin 2C \sin \Delta.
\end{align*}
\]

(1)

The normalized Stokes vector \( s = (s_1, s_2, s_3) \) is of unit length, and its tip traces a figure-8 contour on the Poincaré sphere as the retarder is rotated (and is changed) for a given . To reveal the nature of this contour, the normalized Stokes parameters and are rewritten in the following form

\[
\begin{align*}
 s_1 &= \cos^2 (\Delta/2) + \sin^2 (\Delta/2) \cos 4C, \\
 s_2 &= \sin^2 (\Delta/2) \sin 4C.
\end{align*}
\]

(2a) (2b)

\( C \) is readily eliminated between Eqs. (2a) and (2b) so that

\[
[s_1 - \cos^2 (\Delta/2)]^2 + s_2^2 = \sin^4 (\Delta/2).
\]

(3)

Equation (3), which represents the normal projection of the contour in the equatorial plane, is that of a circle with center at

\[
(s_1, s_2) = [\cos^2 (\Delta/2), 0],
\]

(4)

and radius of

\[
r = \sin^2 (\Delta/2).
\]

(5)

It follows that the trajectory of the polarization state at the output of the fixed-polarizer rotating-retarder system of Fig. 1 is the line of intersection with the Poincaré sphere of a right-circular cylinder whose axis and radius are given by Eqs. (4) and (5), respectively.

Figure 2 shows the cylinder–sphere intersection for QWR (60°). For this retardance value the cylinder...
radius is half the sphere radius and the intersection contour passes through the north and south poles of the sphere, which correspond to the right- and left-circular polarizations, respectively. The grid of longitudes and latitudes on the Poincaré sphere represent the equiazimuth and equiellipticity contours, respectively. It is interesting to note that the line of intersection represents the locus of all polarization states with equal azimuth and ellipticity angles, i.e., equal longitude and latitude on the sphere.

Figure 3 shows the normal projection of Fig. 2 in the equatorial ($s_1, s_2$) plane. The circle of intersection of the cylinder with the equatorial plane is represented by Eq. (3), where $\Delta = 90^\circ$.

For completeness, we also determine the two remaining projections of the trajectory of output states in the ($s_1, s_3$) and ($s_2, s_3$) coordinate planes. Elimination of $C$ between these normalized Stokes parameter pairs in Eqs. (1) gives
\[ s_3^2 = [2 \cos^2(\Delta/2)](1 - s_1), \quad (6) \]
\[ s_3^4 - (\sin^2 \Delta)s_3^2 + [4 \cos^2(\Delta/2)]s_2^2 = 0. \quad (7) \]

Equation (6) shows another interesting result, namely, that the trajectory of output states is also the line of intersection of a parabolic cylinder parallel to the \( s_2 \) axis with the Poincaré sphere. Equation (7) is that of the projected figure-8 contour in the \((s_2, s_3)\) plane.

Figures 4 and 5 represent the two normal projections of Fig. 2 in the \((s_1, s_3)\) and \((s_2, s_3)\) coordinate planes, respectively, and show the projected parabola and figure-8 contours for \( \Delta = 90^\circ \).

The foregoing results apply equally when the direction of propagation of light in Fig. 1 is reversed and the system functions as an elliptic analyzer or polarimeter. In this case the normalized Stokes parameters are those of the input elliptic polarization state of maximum transmission through the system for given orientations of the optical elements.

In conclusion, a detailed analysis of the fixed-polarizer rotating-retarder optical system has been presented. The three-dimensional figure-8 locus of (output or input) polarization states characteristic of this common system is shown to be the line of intersection of the Poincaré sphere with a right-circular cylinder whose axis is parallel to the polar axis of the sphere and whose radius is equal to the squared sine of half the retardance of the rotating retarder. These results complement those obtained by others\(^8\)–\(^10\) concerning this locus.

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**REFERENCES**