Explicit equations for the polarizing angles of a high-reflectance substrate coated by a transparent thin film

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Explicit equations for the polarizing angles of a high-reflectance substrate coated by a transparent thin film

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Simple explicit equations are derived that determine the angles of incidence at which the parallel and perpendicular polarization components of light are extinguished on reflection from a transparent film coating a high-reflectance (metallic) substrate. The polarizing angles obtained from our approximate expressions are in excellent agreement with those determined by iterative numerical solution of the exact nonlinear equations that govern such angles. For the approximation to be valid, the intensity reflectance of the film-substrate interface, evaluated at the critical angle of the film–ambient interface, must exceed 0.5.

Two observations regarding Fig. 1 make the derivation of accurate explicit equations for the polarizing angles possible:

1. The grazing polarizing angles are small (of the order of a few degrees for a high-reflectance metal substrate). (2) \( R_{12p} \) and \( R_{12s} \) stay virtually constant with \( \theta \) (they change imperceptibly, by <0.24%), as \( \theta \) increases from 0 to 10°.

Correspondingly, two sensible approximations can be made. The first is to replace the exact Fresnel equations of \( R_{ij} \) \((i = p, s)\) by quadratic Taylor-series expansions valid about \( \theta = 0 \). The second is to set

\[ R_{12p}(\theta) = R_{12p}(0), \quad \nu = p, s. \]  

The first three terms of the Taylor-series expansion of the ambient–film Fresnel (amplitude-reflection) coefficients, valid near grazing incidence, can be written as

\[ r_{01p} = -1 + 2(E - 1)^{-1/2} + 2E^2(1 - 1)^{-1/2} + \ldots, \]  
\[ r_{01s} = -1 + 2(E - 1)^{-1/2} + 2(1 - 1)^{-1/2} + \ldots, \]

where

\[ E = N_f^2/N_a^2 \]  

and \( N_a, N_f \) are the ambient and film refractive indices, respectively. By squaring Eqs. (4), the corresponding intensity reflectances are obtained:

\[ R_{01p} = 1 - 4E(E - 1)^{-1/2} + 8E^2(1 - 1)^{-1/2}, \]  
\[ R_{01s} = 1 - 4(E - 1)^{-1/2} + 8(1 - 1)^{-1/2}, \]

where terms including \( \theta_a (a > 2) \) have been dropped.

At the polarizing angles, \( R_{01p} \) and \( R_{01s} \), which appear on the left-hand sides of Eqs. (6a) and (6b), can be replaced by \( R_{12p}(0) \) and \( R_{12s}(0) \), respectively, according to Eqs. (2) and (3). Next, quadratic Eqs. (6) are solved for \( \theta \) to give

\[ \theta_{p2} = (E - 1)^{1/2} \left[ 1 - [2R_{12p}(0) - 1]^{1/2} \right], \]
must be satisfied for the approximate equations to yield meaningful real answers. We have verified that the accuracy of the approximation improves steadily as $R_{12p}(0)$ increases above this lowest limit of 0.5. The polarizing angles from the approximate and exact equations agree to within a few percent for $R_{12p}(0) > 0.7$. This condition is met readily when a high-reflectance metal substrate is used.

For further illustration, the exact and approximate $p$- and $s$-polarizing angles are calculated for a system that consists of a film coating an Al substrate. The exact polarizing angles, obtained by numerical iteration, are given by

$$
\theta_{pe} = 1.335^\circ, \quad \theta_{se} = 2.070^\circ.
$$

The excellent agreement between the approximate and exact polarizing angles verifies the high accuracy of Eqs. (8). From Eqs. (7) and (8) it is evident that the condition

$$
R_{12p}(0) > 0.5 \quad p, s
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Table 1. Comparison between the Approximate and Exact $p (\theta_{pa2}, \theta_{pe})$ and $s (\theta_{sa2}, \theta_{se})$ Grazing Polarizing Angles of Incidence of Transparent Films on Metal Substrates\(^a\)

<table>
<thead>
<tr>
<th>Substrate ((\lambda))</th>
<th>(N_1)</th>
<th>(\mathcal{R}_{12p}(0))</th>
<th>(\theta_{pa2})</th>
<th>(\theta_{pe})</th>
<th>(\mathcal{R}_{12s}(0))</th>
<th>(\theta_{sa2})</th>
<th>(\theta_{se})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ag, (N_2 = 0.95 - j2.87) ((0.5 \mu m))</td>
<td>1.5</td>
<td>0.9637</td>
<td>0.263</td>
<td>0.26</td>
<td>0.9801</td>
<td>0.322</td>
<td>0.32</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>0.9626</td>
<td>0.237</td>
<td>0.24</td>
<td>0.9736</td>
<td>0.663</td>
<td>0.66</td>
</tr>
<tr>
<td></td>
<td>3.0</td>
<td>0.9625</td>
<td>0.172</td>
<td>0.17</td>
<td>0.9695</td>
<td>1.256</td>
<td>1.26</td>
</tr>
<tr>
<td>Al, (N_2 = 0.64 - j5.497) ((0.492 \mu m))</td>
<td>1.5</td>
<td>0.8561</td>
<td>1.111</td>
<td>1.11</td>
<td>0.9178</td>
<td>1.375</td>
<td>1.37</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>0.8422</td>
<td>1.072</td>
<td>1.08</td>
<td>0.8814</td>
<td>3.141</td>
<td>3.14</td>
</tr>
<tr>
<td></td>
<td>3.0</td>
<td>0.8118</td>
<td>0.947</td>
<td>0.94</td>
<td>0.8549</td>
<td>7.355</td>
<td>7.34</td>
</tr>
<tr>
<td>Rh, (N_2 = 1.62 - j4.633) ((0.564 \mu m))</td>
<td>1.5</td>
<td>0.6093</td>
<td>3.789</td>
<td>3.51</td>
<td>0.7608</td>
<td>4.448</td>
<td>4.40</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>0.5797</td>
<td>3.727</td>
<td>3.36</td>
<td>0.6706</td>
<td>10.320</td>
<td>10.05</td>
</tr>
<tr>
<td></td>
<td>3.0</td>
<td>0.5219</td>
<td>3.558</td>
<td>2.9</td>
<td>0.5710</td>
<td>25.247</td>
<td>23.75</td>
</tr>
</tbody>
</table>

\(^a\) \(N_1\) is the film refractive index, \(N_2\) is the substrate complex refractive index at the indicated wavelength \(\lambda\). \(\mathcal{R}_{12p}(0)\) and \(\mathcal{R}_{12s}(0)\) are the film-substrate intensity reflectances evaluated at exact grazing incidence \((\theta = 0)\). \(\theta_{pa2}\) and \(\theta_{sa2}\) are the approximate polarizing angles calculated from Eqs. (8) and truncated to three decimal places. \(\theta_{pe}\) and \(\theta_{se}\) are the exact polarizing angles as determined by Ruiz-Urbieta and Sparrow. All angles are in degrees.

To test the range of validity of Eqs. (8), Table 1 compares our approximate polarizing angles with the exact ones of Ref. 2 for the low-reflectance metal Rh \((R_0 = 77.1\% \text{ at } \lambda = 0.564 \mu m)\). Poorer agreement is obtained, as expected. The disagreement becomes more pronounced as the reflectances \(\mathcal{R}_{12s}(0)\) approach 0.5, the lowest limit of validity of our approximate equations.

Finally, we should mention that Eqs. (8) continue to hold when the metal substrate is replaced by a high-reflectance multilayer substructure. In this case, \(\mathcal{R}_{12s}(0)\) represents the reflectance of the film-substructure interface at external grazing incidence \((\theta = 0)\) for the \(s\) polarization.

ACKNOWLEDGMENTS

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REFERENCES

3. The Fresnel-interface reflection coefficients are given in many optics books (e.g., Ref. 1) and will not be reproduced here.