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R. M. A. Azzam
University of New Orleans, razzam@uno.edu

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Maximum minimum reflectance of parallel-polarized light at interfaces between transparent and absorbing media

R. M. A. Azzam
Department of Electrical Engineering, University of New Orleans, Lakefront, New Orleans, Louisiana 70148

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The pseudo-Brewster angle \( \phi_{PB} \) of minimum reflectance \( R_{pm} \) for the parallel (p) polarization, of an interface between a transparent and an absorbing medium, is given by \( \phi_{PB} = \tan^{-1} \epsilon^{1/2} \), where \( \epsilon = \epsilon_1/\epsilon_0 \). When the medium of refraction is absorbing (and \( \epsilon \) becomes complex), the reflectance of the interface for incident p-polarized light is nonzero at all angles of incidence but reaches a minimum at the so-called pseudo-Brewster angle \( \phi_{PB} \). In this Letter we derive a new equation for \( \phi_{PB} \) in terms of complex \( \epsilon \). The correct relation, which replaces Eq. (1), between \( \phi_{PB} \) and the complex relative refractive index

\[
N = N_1/N_0 = \epsilon^{1/2}
\]

was first found by Humphreys-Owen after it had eluded others for many years.  

Subsequently we present and analyze a condition of maximum minimum parallel reflectance for interfaces between transparent and absorbing media. We were led to this condition by the following reasoning. Let

\[
r = |r| e^{i\delta}
\]

be the interface normal-incidence complex reflection coefficient. For a given value of the amplitude reflectance, \( |r| = \), the oblique-incidence parallel reflectance \( R_p \) goes to zero at exact Brewster angles \( \phi_B(0) \) and \( \phi_B(\pi) \) when the normal-incidence phase shift \( \delta \) equals 0 and \( \pi \), respectively. This represents light reflection from opposite sides of a given interface between two transparent media; in this case, \( \phi_B(\pi) = 90^\circ - \phi_B(0) \). When \( \delta \neq 0 \) or \( \delta \neq \pi \), \( R_p \) reaches a nonzero minimum \( R_{pm} \) at a pseudo-Brewster angle \( \phi_{PB} \). If we allow \( \delta \) to vary continuously from 0 to \( \pi \) with \( |r| \) constant, \( R_{pm} \) must go from 0 (at \( \delta = 0 \)) to a maximum \( R_{p_{mm}} \) at a certain \( \delta = \delta_{mm} \) and back to 0 (at \( \delta = \pi \)). The subscript mm denotes maximum minimum here and throughout.

The condition of maximum minimum parallel reflectance is verified by direct computation assuming different values of \( |r| \). \( R_{pm} \) and \( \phi_{PB} \) are determined as functions of \( \delta \) and \( |r| \), and \( R_{p_{mm}}, \delta_{mm}, \) and \( \phi_{PB_{mm}} \) are computed and plotted versus \( |r| \).

We adopt the \( e^{jwt} \) time dependence and the Nebraska (Muller) conventions. At normal incidence the reflection coefficients for the p and s polarizations differ in sign, i.e., \( r_s = -r_p = r \).

2. NEW DERIVATION FOR THE PSEUDO-BREWSTER ANGLE \( \phi_{PB} \) OF AN INTERFACE WITH KNOWN \( \epsilon \)

The simplicity of the following derivation of the pseudo-Brewster angle \( \phi_{PB} \) in terms of the complex relative dielectric constant \( \epsilon = \epsilon_1/\epsilon_0 \) of an interface, compared with that of Ref. 3, results from stating the condition of minimum parallel reflectance in terms of the complex amplitude-reflection coefficient \( r_p \), instead of the real intensity reflectance

\[
R_p = |r_p|^2.
\]

Thus, if we write

\[
r_p = |r_p| e^{i\theta_p}
\]

at any angle of incidence \( \phi \) and take the derivative with respect to \( \theta \) (indicated by a prime superscript) of the natural logarithm of both sides, we get

\[
r'_p r_p = (|r_p|^2 |r_p|) + j\delta'_p.
\]

The condition for minimum parallel reflectance is that

\[
R'_p = 0,
\]

or, equivalently,
if Eq. (5) is used. Because \( |p| \neq 0 \) at any angle of incidence when \( \varepsilon \) is complex, Eq. (9) requires that

\[
|p|' = 0. \tag{10}
\]

With \( |p| \neq 0 \), the condition for minimum \( R_p \) takes its most convenient form when Eq. (10) is substituted into Eq. (7), namely,

\[
\text{Re}(r_p')/r_p = 0. \tag{11}
\]

Equation (11) locates the pseudo-Brewster angle, \( \phi = \phi_{PB} \).

To proceed from Eq. (11), we must write \( r_p \) as a function of \( \varepsilon \) and \( \phi \):

\[
r_p = (1 - X)/(1 + X), \tag{12}
\]

\[
X = (\varepsilon - \sin^2 \phi)^{1/2}/\varepsilon \cos \phi. \tag{13}
\]

Differentiation of Eqs. (12) and (13) gives

\[
r_p'/r_p = -2X'/((1 - X^2) \sin \phi
\]

\[
= \frac{2\varepsilon(1 - \varepsilon)\sin \phi}{(\varepsilon - \sin^2 \phi)^{1/2}(\varepsilon^2 \cos^2 \phi - \varepsilon + \sin^2 \phi)}. \tag{14}
\]

By substituting Eq. (14) into Eq. (11) and using the convenient change of variable

\[
u = \sin^2 \phi, \tag{15}
\]

we obtain

\[
\text{Re} \left[ (\varepsilon - u)^{1/2} \left( 1 - \frac{\varepsilon + 1}{\varepsilon} \right) \right] = 0. \tag{16}
\]

In reaching Eq. (16) we used the fact that, if \( \text{Re}(rz) = 0 \), where \( r \) and \( z \) are real and complex, respectively, \( \text{Re}(1/z) = 0 \).

For a given complex \( \varepsilon \) (of a given interface), the \( u \) that satisfies Eq. (16) determines the pseudo-Brewster angle

\[
\phi_{PB} = \sin^{-1} u^{1/2}. \tag{17}
\]

In the special case when \( \varepsilon \) is real (i.e., for an interface between two transparent media), Eq. (16) has the following solution for \( u \):

\[
u = \varepsilon/(\varepsilon + 1). \tag{18}
\]

From Eqs. (18) and (15), we get

\[
\varepsilon = u/(1 - u) = \tan^2 \phi, \tag{19}
\]

which is the correct Brewster law, as expected.

An alternative preferable form of the equation for the pseudo-Brewster angle is

\[
\text{Im} \left[ (\varepsilon - u) \left( 1 - \frac{\varepsilon + 1}{\varepsilon} \right) \right] = 0. \tag{20}
\]

To obtain Eq. (20) from Eq. (16) we used the fact that, when \( \text{Re} \varepsilon = 0 \), \( \text{Im} \varepsilon^2 = 0 \).

Equation (20) can be used to prove the existence of the pseudo-Brewster angle for any interface, i.e., for any given \( \varepsilon \). We write Eq. (20) as \( \text{Im}[F(u)] = 0 \). A solution of this equation for \( u \) (hence for the pseudo-Brewster angle) exists if the trajectory of \( F(u) \) in the complex plane, as \( u \) increases from 0 to 1, intersects the real axis. Such intersection is guaranteed because the end points of this trajectory, \( F(0) = \varepsilon \) and \( F(1) = \varepsilon^{-1} - 1 \), lie on opposite sides of the real axis for any complex \( \varepsilon \).

If we substitute

\[
\varepsilon = \varepsilon_c + j \varepsilon_i, \quad (21a)
\]

\[
1/\varepsilon = \bar{\varepsilon} = \bar{\varepsilon}_c + j \bar{\varepsilon}_i, \quad (21b)
\]

where \( \bar{\varepsilon}_c = \varepsilon_c/(\varepsilon_c^2 + \varepsilon_i^2) \), \( \bar{\varepsilon}_i = -\varepsilon_i/(\varepsilon_c^2 + \varepsilon_i^2) \) (21c) and \( \varepsilon_i \) is negative in the Nebraska (Muller) conventions.\(^6\) Eq. (20) can be expanded to give a cubic equation in \( u \):

\[
\alpha_0 u^3 + \alpha_2 u^2 + \alpha_1 u + \alpha_0 = 0, \tag{22}
\]

with coefficients

\[
\alpha_0 = -\varepsilon_i, \quad \alpha_1 = 2\varepsilon_i, \quad \alpha_2 = -2\varepsilon_i + 3\bar{\varepsilon}_i, \quad \alpha_3 = 2\bar{\varepsilon}_i + 2|\varepsilon_i|^2, \tag{23}
\]

Use of Eqs. (21c) and multiplication by a common factor lead to the following equivalent set of coefficients:

\[
\alpha_0 = |\varepsilon|^2, \quad \alpha_1 = -2|\varepsilon|^2, \quad \alpha_2 = |\varepsilon|^4 - 3|\varepsilon|^2, \quad \alpha_3 = 2|\varepsilon|^4 + 2|\varepsilon|^2, \tag{24}
\]

where \( |\varepsilon|^2 = \varepsilon_c^2 + \varepsilon_i^2 \). Equation (22), with coefficients given by Eqs. (24), becomes identical to the corresponding cubic equation derived by Hymphreys-Owen\(^5\) when the substitution \( \varepsilon = N^2 = (n - jk)^2 \) is made.

3. CONDITION OF MAXIMUM MINIMUM PARALLEL REFLECTANCE

In terms of the complex normal-incidence reflection coefficient, \( |\varepsilon| e^{j\delta} \), the complex relative dielectric constant \( \varepsilon \) of the interface is given by

\[
\varepsilon = \left[ 1 - |\varepsilon| e^{j\delta} \right]^2. \tag{25}
\]

For a given value of the normal-incidence amplitude reflectance, \( |\varepsilon| = 0.1, 0.2, \ldots, 0.9 \), we let the associated normal-incidence phase shift \( \delta \) take values from 0 to 180° in equal steps of 1°. \( \varepsilon \) is computed from Eq. (25) and the coefficients of the cubic equation are determined by Eqs. (24). The cubic Eq. (22) is solved explicitly and exactly,\(^8\) and only one real root is always found in the interval 0 \( \leq u \leq 1 \), from which the pseudo-Brewster angle \( \phi_{PB} \) is calculated by using Eq. (17). Figure 1 shows \( \phi_{PB} \) as a function of \( \delta \) with \( |\varepsilon| \) as a parameter marked by each curve. As we have already noted in Section 1, for a given \( |\varepsilon| \), \( \delta = 0 \) and \( \delta = 180° \) represent the limiting cases of internal and external reflection, respectively, at a dielectric–dielectric interface, with associated exact Brewster angles that sum to 90°.

In Fig. 1 all curves appear to pass through a common point, which leads to the interesting conclusion that a pseudo-
Fig. 1. Pseudo-Brewster angle $\phi_{PB}$ as a function of the normal-incidence reflection phase shift $\delta$ for different constant values of the normal-incidence amplitude reflectance, $|r| = 0.1, 0.2, \ldots, 0.9$, as a parameter. Both $\delta$ and $\phi_{PB}$ are in degrees.

Figures 1–3 can be used as nomograms for approximate calculation of complex $\epsilon$ from measured $\phi_{PB}$ and $R_{pm}$. For example, such data locate a point in Fig. 3 from which $|r|$ can be read by interpolation. Next, $|r|$ and $\phi_{PB}$ locate a point in Fig. 1; hence the normal-incidence phase shift $\delta$ is determined. Finally, from $|r|e^{j\delta}$, $\epsilon$ is calculated by using Eq. (25). Of course, nomograms with larger numbers of curves can be computer generated for higher accuracy. Alternatively, the approximate $\epsilon$ can be improved by numerical iteration to minimize the difference between the measured and computed $(\phi_{PB}, R_{pm})$.

Because we are particularly interested in the condition of maximum minimum parallel reflectance, the normal-incidence phase shift required to achieve this condition at a given $|r|$, $\delta_{mm}$, was determined. Figure 4 shows $\delta_{mm}$ as a function of $|r|$. $\delta_{mm}$ decreases from $90^\circ$ to $0$ as $|r|$ increases from $0$ to $1$. The associated maximum minimum parallel reflectance,

Brewster angle of $45^\circ$ corresponds to a normal-incidence phase shift that is restricted to a brief interval $99^\circ \leq \delta \leq 105^\circ$ for $0.1 \leq |r| \leq 0.9$.

After $\phi_{PB}$ is calculated for a given $\epsilon$, the associated minimum parallel reflectance $R_{pm}$ is determined from Eqs. (5), (12), and (13). In Fig. 2 $R_{pm}$ is plotted versus $\delta$ with $|r|$ as a parameter. $R_{pm}$ equals 0 when $\delta = 0, \delta = 180^\circ$ (corresponding to extinction of the reflected wave at exact Brewster angles) and reaches a maximum, $R_{pmm}$, at a certain phase $0 < \delta_{mm} < 180^\circ$. The peak of each $R_{pm}$ versus $\delta$ curve is broad.

From the data of Figs. 1 and 2, $\delta$ can be eliminated, and $R_{pm}$ is related to $\phi_{PB}$ at constant $|r|$. The results appear in Fig. 3.
R_{pmm}, normalized as a fraction of the normal-incidence intensity reflectance, i.e., R_{pmm}/|r|^2, is plotted versus |r| in Fig. 5. We see that such a fraction increases from 0 to 1 as |r| increases from 0 to 1. Finally, Fig. 6 shows \( \phi_{pBmm} \), associated with \( R_{pmm} \), versus |r|. \( \phi_{pBmm} \) decreases monotonically from 45° to 0 as |r| is increased from 0 to 1.

4. SUMMARY
If |r| represents Fresnel’s complex-amplitude normal-incidence reflection coefficient at an interface between a transparent and an absorbing medium, we find that, for a given |r|, the minimum reflectance for the parallel polarization \( R_{pm} \) at the pseudo-Brewster angle \( \phi_{pB} \) reaches a maximum, \( R_{pmm} \), at a certain normal-incidence phase shift \( \delta = \delta_{mm} \). As |r| increases from 0 to 1, \( \delta_{mm} \) decreases from 90° to 0, \( R_{pmm}/|r|^2 \) increases from 0 to 1, and the associated \( \phi_{pBmm} \) decreases from 45° to 0, all monotonically.

These results are obtained after a new form of the equation for the pseudo-Brewster angle [Eq. (20)] is derived. The condition of maximum minimum parallel reflectance is verified through a graphical study of \( R_{pm} \) as a function of \( \delta \) with |r| as a parameter. Furthermore, we plot \( \phi_{pB} \) versus \( \delta \) and \( R_{pm} \) versus \( \phi_{pBm} \) with |r| as a parameter. These graphs can be used as nomograms to determine the complex relative dielectric constant \( \epsilon \) of an interface from measured \( \phi_{pB} \) and \( R_{pm} \).

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6. See, for example, M. Born and E. Wolf, Principles of Optics (Pergamon, New York, 1975), Sec. 1.5.2.
7. The other solution of Eq. (16), \( u = \epsilon \), is unacceptable when \( \epsilon > 1 \) or \( \epsilon < 0 \). When \( 0 < \epsilon < 1 \), it yields the critical angle of total internal reflection, \( \sin^{-1}\sqrt{\epsilon} \).
9. This is Method F of Ref. 3.