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Explicit equations for the second Brewster angle of an interface between a transparent and an absorbing medium

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The second Brewster angle $\phi_{B2}$, at which the ratio $R_p/R_s$ of intensity reflectances $R_p$ and $R_s$ for the parallel (p) and the perpendicular (s) polarizations of a dielectric-conductor interface reaches a minimum, is determined by $\text{Im}[(\epsilon - \epsilon_1)/(\epsilon - 2\epsilon)] = 0$, where $\epsilon$ is the complex ratio of dielectric constants of the media of refraction and incidence, $\epsilon = \epsilon_1/(\epsilon_1 + 1)$, and $\epsilon_1 = \sin^2 \phi_{B2}$. An equivalent quartic equation in $u$ is also derived that can be solved exactly and explicitly to determine $\phi_{B2}$ in terms of $\epsilon$.

Equation (12) is, to our knowledge, a new and relatively simple equation that determines the second Brewster angle.
\[
\phi_{B2} = \sin^{-1}u^{1/2}
\]

(13)

for a given complex \(u\) of a given interface.

Equation (12) will always have a solution

\[
0 < u < 1
\]

(14)

irrespective of \(\epsilon\). To prove the existence of the second Brewster angle for any given complex \(\epsilon\), Eq. (12) can be put in the form \(\text{Im}[F(u)] = 0\). As \(u\) increases from 0 to 1, \(F(u)\) follows a trajectory in the complex plane that begins at \(F(0) = -e/4\) and ends at \(F(1) = 1/(1 - e)\). From simple geometrical considerations, it becomes apparent that the end points \(F(0)\) and \(F(1)\) lie on opposite sides of the real axis for any complex \(\epsilon\). This guarantees the intersection of the curve of \(F(u)\) with the real axis; hence the existence of a solution for Eq. (12), at a certain \(u\) in the range of inequality (14).

When \(\epsilon\) is real (i.e., for an interface between two transparent media), the second Brewster angle must coincide with the exact Brewster angle of zero parallel reflectance. This is in that case because Eq. (11) gives \(u = \epsilon/\epsilon + 1\), from which \(\epsilon = u/(1 - u) = \tan^2\phi\), as expected.

To derive from Eq. (12) a more explicit equation for \(u\), we first set

\[
\bar{\epsilon} = \epsilon/(\epsilon + 1)
\]

(15)

and rewrite Eq. (12) in the form

\[
\text{Im}(N/M) = 0,
\]

(16)

where

\[
N = (u - \bar{\epsilon})(u - \bar{\epsilon})^2
= u^2 + \beta_2 u + \beta_1 + \beta_0,
\]

(17a)

\[
M = (u - 2\bar{\epsilon})^2
= u^2 + \gamma_1 u + \gamma_0.
\]

(17b)

and

\[
\beta_0 = -\epsilon^2, \quad \beta_1 = \epsilon^2 + 2\epsilon, \quad \beta_2 = -\epsilon^2.
\]

(18a)

\[
\gamma_0 = \epsilon^2, \quad \gamma_1 = -4\epsilon.
\]

(18b)

Next we write

\[
\beta_k = \beta_{kr} + j\beta_{ki}, \quad \gamma_k = \gamma_{kr} + j\gamma_{ki}, \quad k = 1, 2, 3,
\]

(19)

and

\[
N = N_r + jN_i, \quad M = M_r + jM_i.
\]

(20)

Substitution of Eqs. (20) into Eq. (16) puts the condition for the second Brewster angle in the form

\[
N_r M_i - N_i M_r = 0.
\]

(21)

Finally, substituting from Eqs. (17), (19), and (20) into Eq. (21), we obtain

\[
a_4 u^4 + a_3 u^3 + a_2 u^2 + a_1 u + a_0 = 0,
\]

(22)

which is a quartic equation with coefficients given by

\[
a_0 = \beta_{0r} \gamma_{0i} - \beta_{0i} \gamma_{0r},
\]

\[
a_1 = \beta_{0r} \gamma_{1i} + \beta_{1r} \gamma_{0i} - \beta_{0i} \gamma_{1r} - \beta_{1i} \gamma_{0r},
\]

\[
a_2 = \beta_{0r} \gamma_{2i} + \beta_{2r} \gamma_{1i} - \beta_{0i} \gamma_{2r} - \beta_{2i} \gamma_{1r},
\]

\[
a_3 = \beta_{0r} \gamma_{3i} + \gamma_{0i} - \beta_{1i} \gamma_{2r} - \beta_{2i} \gamma_{1r},
\]

\[
a_4 = \gamma_{1i} - \beta_{2i}.
\]

(23)

For a given complex \(\epsilon\) (of a given interface), the coefficients \(a_k\) can be calculated by use of Eqs. (15), (18), (19), and (23), in that order. Fortunately, a quartic equation can be solved exactly and explicitly\(^2\) (i.e., without numerical iteration), and, from the root \(0 < u < 1\) of Eq. (22), one determines the second Brewster angle by using Eq. (13).

Before concluding, we note that the condition of maximum minimum parallel reflectance at the pseudo-Brewster angle, discussed in Ref. 5, has its analog in terms of the reflectance ratio \(R_p/R_s = |r|^2\) and the second Brewster angle. Thus, if \(r = |r|e^{i\delta}\) denotes the complex normal-incidence interface-reflection coefficient, we find that, for a given \(|r|\), the minimum of \(R_p/R_s\) goes to zero at \(\delta = 0\) and \(\delta = 180^\circ\) and must reach a maximum at some intermediate phase shift \(0 < \delta < 180^\circ\). Such \(\delta\) is expected to differ from that which leads to maximum minimum \(R_p\). A detailed analysis of the condition of maximum minimum \(R_p\) can be pursued along the lines of Ref. 5.

To summarize, we have derived explicit equations, Eqs. (12) and (22), for the second Brewster angle (at which the ratio of parallel to perpendicular reflectance is minimum) of interfaces between isotropic transparent and absorbing media. Whereas the pseudo-Brewster and principal angles are determined by cubic equations, we have found that the second Brewster angle is determined by a quartic equation, Eq. (22), with the variable of all equations being the sine squared of the incidence angle of interest. Fortunately, quartic equations, like cubic equations, are exactly solvable.

Note added in proof: Measurements on an ultrathin gold foil, kindly provided by Gary Reeves of Los Alamos National Laboratory, confirmed the validity of this technique. These results were presented at the Paris Ellipsometry Conference, June 7–10, 1983.

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REFERENCES

6. See, for example, R. M. A. Azzam and N. M. Bashara, Ellipsometry and Polarized Light (North-Holland, Amsterdam, 1977), p. 274.