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Multiple Solutions on a Ball for a Generalized Lane Emden Equation

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Multiple Solutions on a Ball for a Generalized Lane–Emden Equation

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Abstract

In this work we study the Generalized Lane-Emden equation and the interplay between the exponents involved and their consequences on the existence and non existence of radial solutions on a unit ball in n dimensions. We extend the analysis to the phase plane for a clear understanding of the behavior of solutions and the relationship between their existence and the growth of nonlinear terms, where we investigate the critical exponent p^* and a sub-critical exponent, which we refer to as $\hat{p}$. We discover a structural change of solutions due the existence of this sub-critical exponent which we relate to the same change in behavior of the Lane-Emden equation solutions, for $\alpha, \beta = 0, and p = 2$, due to the same sub-critical exponent. We hypothesize that this sub-critical exponent may be related to a weighted trace embedding.

Key words: p-laplacian, critical Sobolev exponents, weighted Sobolev spaces.

Chapter 1

Introduction and Scope of Thesis

1.1 Introduction

This work examines the behavior and multiplicity of the radial solutions to the Dirichlet problem

$$\nabla \cdot (|x|^{\alpha} |\nabla u|^{p-2} \nabla u) + |x|^{\beta} |u|^{q-1} u = 0, \ x \in \Omega \tag{1.1}$$

$$u|_{\partial\Omega} = 0, \ q > 1, \ 1 < p \leq 2, \alpha, \beta \geq 0$$

where $\Omega = B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ denotes the unit ball in $\mathbb{R}^n$.

We are interested in the interplay between the exponents α, β, p and q and consequences for the existence and nonexistence of radial solutions to equation (1.1).

This equation serves as a generalization to the Lane-Emden equation which has attracted great interest in the literature and has undergone extensive research due to its frequent use in mathematics and astrophysics.

The aim of studying the generalized form of the Lane-Emden equation and its solutions is to develop a broader understanding of the general equations and to develop analytical tools to analyze these equations to cover more applications as they arise in the future.

1.2 Literature Review

The Lane–Emden equation,

$$\Delta u + u^q = 0, x \in \mathbb{R}^n, q > 1, \tag{1.2}$$

has for many years received attention by the scientific and mathematical communities for its frequent appearance in physical and astrophysical applications. It was first introduced in 1869 by Homer Lane, [25], in his attempt to compute both the temperature and the mass density in portions of the sun. In spite of the fact that the results he obtained were incorrect near the surface of the sun, the values for both quantities were reasonable at the interior, [4]. As a result the Lane–Emden equation is still in use today to compute the structure of the interior of polytropic stars.

The study of equation (1.2) came in to use again when it was initiated by R. Emden in 1897 in problems of meteorology, [37]. In the 1920's, it was extensively studied by Thomas Fermi in the theory of atoms and electronic potential for a value of $q = \frac{5}{2}$ and $n = 3$, [37] . Chandrasekhar subsequently introduced the Lane–Emden equation to astrophysics, for star equilibrium problems with $n = 3$, in 1937, [10].

Later, mathematicians including Nirenberg, Ni and Serrin, [21], [22], studied detailed properties of the Lane–Emden equation in $\mathbb{R}^n$, for general n.

Following Emden's solution in $\mathbb{R}^3$ to (1.2) for the value $q = 5$, (his solution reads

$$u = u(r) = \frac{1}{(1 + \frac{1}{3}r^2)^{\frac{1}{2}}}$$

where

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}, x = (x_1, x_2, x_3) \in \mathbb{R}^3)$$

a simple generalization was found for $q = \frac{n+2}{n-2}$ in $\mathbb{R}^n$ and solved in terms of r and n, [37], with

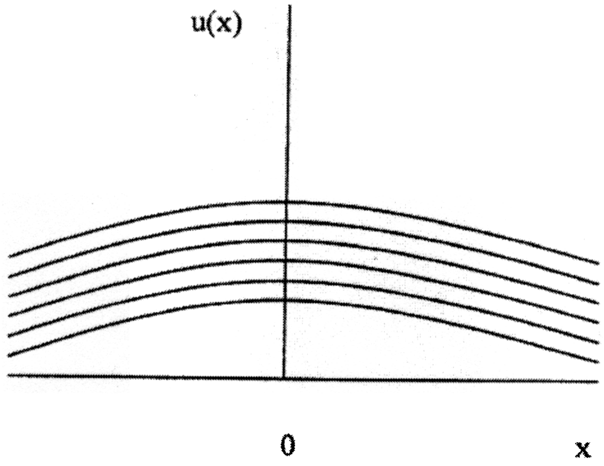
$$u(r) = \frac{1}{[1+Dr^2]^{\frac{n-2}{2}}}$$

where $D = \frac{1}{n(n-2)}$.

In 1986, it was shown by Ni and Serrin,[37], that equation (1.2) exhibits a one-parameter family of ground state solutions, $u > 0$ on $\mathbb{R}^n$, $u(x) \to 0$ as $|x| \to \infty$, consisting of positive, smooth solutions, parametrized by the value $u(0) \in (0, \infty)$. In fact all such solutions are scaled versions of the solution with $u(0) = 1$

This holds true for q greater than or equal to the critical exponent $\frac{n+2}{n-2}$, Figure 1.1, [37].

Figure 1.1



For $q < \frac{n+2}{n-2}$, equation (1.2) no longer has solutions, radial or non radial, while for $q = \frac{n+2}{n-2}$, the equation has been completely solved. Remarkably only radially symmetric, fast-decay solutions can exist, [24], [37].

Serrin and Zou later studied equation (1.2) on the deleted domain $\mathbb{R}^n \backslash \{0\}$. They established nonexistence of solutions for $q < \frac{n}{n-2}$, however they found existence and uniqueness of slowly decaying solutions for $\frac{n}{n-2} < q < \frac{n+2}{n-2}$, [38].

The closely related Dirichlet problem,

$$\Delta u + u^q = 0 \text{ in } \mathbb{B}^n, \tag{1.3}$$

$$u > 0 \text{ in } \mathbb{B}^n, u = 0 \text{ on } \partial\mathbb{B}^n,$$

has also been studied extensively by Serrin, [17]. He was able to prove that the Sobolev critical exponent $\frac{n+2}{n-2}$ "sets up a dividing number" for the existence and nonexistence of positive solutions for (1.3). He showed that only for $q < \frac{n+2}{n-2}$ there exist radial positive solutions to (1.3) while for $q > \frac{n+2}{n-2}$, equation (1.3) has neither radial nor non radial solutions on a ball of radius $r > 0$, [17].

In 1979, [21], Ni, Nirenberg and Gidas studied the Dirichlet problem,

$$\Delta u + f(u) = 0, \tag{1.4}$$

$$|x| < R, u = 0 \text{ on } \partial\Omega,$$

where $\Omega = \{x \in \mathbb{R}^n : |x| < R\}$, and showed that for a finite ball of radius $R > 0$ and $f \in C^1$, if $u > 0$ is a positive solution in $C^2(\Omega)$ of (1.4) then u is radially symmetric and $\frac{\partial u}{\partial r} < 0$ for $0 < r < R$, [21].

Gidas, Ni and Nirenberg followed up their paper on the Dirichlet problem in 1981, by extending the domain to all of R^n to prove radial symmetry of positive solutions in R^n, [22].

Since then there has been considerable interest in positive solutions of the semilinear equation (1.4) in $\mathbb{R}^n$, where the $\mathbb{R}^n$ case has been approached as a limiting case of a finite ball. After Gidas , Ni and Nirenbergs contribution to the symmetry of radial solutions of (1.4) in $\mathbb{R}^n$, it became natural to look at radial solutions of such problems on different types of domains and with differing conditions on u.

There has been considerable amount of literature on the radial solutions of equation (1.4). For example, Atkinson and Peletier, [1], [2] have established the existence of ground state solutions, which tend to zero as the $|x| \to \infty$, using the shooting method for equation (1.4) in $\mathbb{R}^n$. This type of problem arises in phase transition theory, in population genetics and the theory of nucleon cores with various different forms of the nonlinearity f(u). Ground state is a term borrowed from physical context in which this equation arises and by ground states mathematicians mean solutions that tend to zero as x approaches infinity. The shooting method involves varying of u(0), and the problem is to show that if u(0) is chosen sufficiently large then the associated radially symmetric solution has a zero, that is, the Dirichlet problem on some finite ball has a solution. Serrin and Mccloed have also studied the uniqueness of ground states of (1.4) for the case when $f(u) = -u + u^q$, where q is a constant and $q > 1$, [39]. They have shown that for $n = 2$, there is at most one solution for any given q, while when $n = 3$, there is at most one solution for $1 < q < 3$. In both cases the solution is radially symmetric and monotonically decreasing as one moves outward from the center, [39].

In 1973, M. Henon proposed a model to study rotating stellar systems and derived the following generalization of the Lane- Emden equation

$$\Delta u + |x|^l u^q = 0, \quad in \ \ \Omega \tag{1.5}$$

$$u > 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$.

This equation is also referred to as Emden- Fowler equation in astrophysics. Here u represents the density of a single star, [33].

This inspired the mathematicians thereafter to study the following problem

$$\Delta u + f(u, |x|) = 0, \quad in \quad \Omega \tag{1.6}$$

$$u > 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega$$

Where Ω is a smooth bounded domain in $\mathbb{R}^n$ and $f(u, |x|)$ behaves as $(|x|^l u^q)$.

Ni was interested in studying three types of positive radial solutions for equation (1.5) with $q > 1$ and $l > -2$. These solutions were the E-solutions, F-solutions and M solutions as termed by Chandrasekhar, [33].

An E- solution is a classical solution of (1.6) when the domain is a ball. An F-solution is a classical solution of (1.6) when the domain is an annulus and an M-solution is a solution to (1.7) where the domain is a punctured ball at zero.

Equation (1.7) is given as follows

$$\Delta u + f(u, |x|) = 0, \quad in \quad \Omega \backslash \{0\} \tag{1.7}$$

$$u > 0 \text{ in } \Omega \backslash \{0\},\ u = 0 \text{ on } \partial\Omega\ ,\ u \to \infty \text{ as } |x| \to 0$$

The E,F and M solutions are shown in Figure 1.2, [33].

Figure 1.2

i) E Solutions ii) F Solutions iii M- Solutions

In the context of our earlier discussion, the Lane-Emden problem in (1.2) considered on a bounded domain

$$\Delta u + u^q = 0 \quad in \quad \Omega \tag{1.8}$$

$$u > 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega$$

E- solutions exist if and only if $q < \frac{n+2}{n-2}$. It is a different case if the non-linearity depends on $|x|$. The Sobolev "cut off exponent "in this case is changed, in particular, for

$$\Delta u + |x|^l u^q = 0 \ , \ q > 1.$$

A positive E-solution exists if and only if $q < \frac{n+2+2l}{n-2}$.

On the existence of F-solutions for equation (1.8), an F-solution exists for any $q > 1$ and on an annulus. When the domain is a punctured ball at zero, with $l > -2$, and $1 < q < \frac{n+2+2l}{n-2}$ there exists infinitely many M-solutions while for $q > \frac{n+2+2l}{n-2}$, there exists no M-solutions. In general the solvability of such Dirichlet problems depends not only on the nonlinearity, but also on the geometry of the domain, [33].

The following equation arises in geometry and physics and has been studied extensively in recent years:

$$\Delta u + K(x)u^q = 0, \quad x \in \mathbb{R}^n \tag{1.9}$$

There has been a substantial body of research in to the question of existence and non-existence of positive solutions decaying to zero as $|x| \to \infty$ of (1.9). Much of this research focused on the range of q values not including the Sobolev critical exponent $p^* = \frac{n+2}{n-2}$ at which variational methods fail to prove the existence of

positive, decaying solutions. For $q > p^*$, variational methods are sufficient to prove the existence of such positive decaying solutions. Ni and Nirenberg have shown that for K(r) bounded, positive and increasing eventually, equation (1.9) has no positive radial solutions and for K(r) decreasing, equation (1.9) has infinitely many solutions satisfying that $\int_{\mathbb{R}^n} K u^{\frac{2n}{n-2}} dx = +\infty$, [29].

Radially symmetric solutions to equation (1.10)

$$\Delta u + K(x) u^q = 0, \tag{1.10}$$

$$q > 1 \text{ in } \mathbb{R}^n, n > 2$$

Satisfying the following ordinary differential equation corresponding to it

$$u_{rr} + \frac{n-1}{r} u_r + K(r) u^p = 0, r \in (0, \infty) \tag{1.11}$$

$$u(0) = \alpha > 0$$

where $r = |x|$, $u^+ = max\{u, 0\}$.

It is known that this ordinary differential equation under certain conditions on K subject to the initial condition $u(0) = \alpha > 0$ has a unique radial solution u(r). This solution has been classified according to its behavior as r approaches infinity as a crossing, slowly or rapidly decaying solution, [44].

This classification is as follows:

i) $u(r, \alpha)$ is a crossing solution if $u(r, \alpha)$ has a zero in $(0, \infty)$.

ii) $u(r, \alpha)$ is a slowly decaying solution if $u(r, \alpha) > 0$ on $[0, \infty)$ and $r^{n-2} u(r, \alpha) \to \infty$ as $r \to \infty$.

iii) $u(r, \alpha)$ is a rapidly decaying solution if $u(r, \alpha) > 0$ on $[0, \infty)$ and $lim_{r\to\infty} r^{n-2} u(r, \alpha)$ exists and is finite and positive.

Where the conditions on K were taken to be:

1) K(r) is continuous on $(0, \infty)$.

2) $K(r) \geq 0$ and $K(r) \neq 0$ on $(0, \infty)$.

3) $rK(r) \in L^1(0, 1)$

A related problem with $q = \frac{n+2}{n-2}$

$$\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0, \quad in \quad \mathbb{R}^n \tag{1.12}$$

has been studied separately. The main existence result on positive solutions to (1.12), given in [32] by Ni, can be summarized as follows. Let K(x) be bounded, if $|K(x)|$ decays faster than $\frac{C}{|x|^2}$ at infinity for some constant $C > 0$, then (1.12) has infinitely bounded solutions in $\mathbb{R}^n$ with positive lower bound. Also if $K(x)$ is negative and decays slower than $\frac{C}{|x|^2}$ at infinity then (1.12) has no positive solutions in $\mathbb{R}^n$. Ni showed that for K(x) greater than or equal to $\frac{C}{|x|^2}$ at infinity, (1.12) has no positive solutions in $\mathbb{R}^2$.

In [13], Ding and Ni treat the case where K(x) is bounded and nonnegative and obtain existence of infinitely many positive solutions.

As an application to the Lane-Emden equation consider the following astrophysical problem which has been proposed by two astrophysicist, Bertin and Ciotti, as model describing the dynamics of galaxies.

$$-\Delta u(x) = \Phi(r)\,|u|^{q-2}\,u, \quad in \quad \mathbb{R}^3 \tag{1.13}$$

$$u(x) > 0 \text{ in } \mathbb{R}^3$$

$$\int_{\mathbb{R}^3} \Phi(r)u^{q-1}dx < +\infty$$

with $q > 1$ in cylindrical coordinates in $\mathbb{R}^3$, $(x_1, x_2, x_3) \in \mathbb{R}^3$, $r = \sqrt{x_1^2 + x_2^2}$, $z = x_3$ and $u = u(r, z)$. The weight function is a non negative continuous function depending on r only, vanishing both at zero and at infinity, where

$$\Phi(r) = \tfrac{r^{2\alpha}}{(1+r^2)^{\frac{1}{2+\alpha}}} \ (\alpha \geq 0)$$

and the condition $\int_{\mathbb{R}^3} \Phi(r) u^{q-1} dx < +\infty$ guarantees that the solution carries a finite total mass, [3].

Various other equations similar to (1.13) have been proposed to model other phenomena of interest in astrophysics. As a second example we mention the Matukuma equation which is used to model globular clusters of stars.

In 1930, T. Matukuma proposed the mathematical model

$$\Delta u + \frac{u^q}{1 + |x|^2} = 0, \quad x \in \mathbb{R}^3 \tag{1.14}$$

Based on his physical intuition to describe the dynamics of globular clusters of stars, where $u > 0$ is the gravitational potential with

$$\int_{\mathbb{R}_3} \tfrac{u^q}{4\pi[1+|x|^2]} dx$$

In representing the total mass, [26], [27], radial symmetry provides a natural assumption for clusters of stars and for most analysis of the Matukuma equation mathematicians have examined radial solutions, [33].

A more general form of all of the above equations is the nonlinear equation

$$\Delta_p u + f(|x|, u) = 0, \quad x \in \mathbb{R}^n, \quad p > 1 \tag{1.15}$$

where $\Delta_p u$ is the p-laplace operator.

Over many years a significant amount of research has been invested in studying the p-laplace operator. This is due to the enormous richness in applications of this nonlinear operator.

The p-laplacian operator $[\nabla \cdot (|\nabla u|^{p-2} u)]$ and its generalizations arise in the motion of incompressible non-newtonian fluids ($p \neq 2$), pseudoplastic fluids ($p < 2$) and dilatant fluids ($p > 2$). Other applications of the p-laplace operator appear in flow through porous media ($p = 3/2$), Nonlinear Elasticity ($p \geq 2$), Glaciology ($1 < p \leq \frac{4}{3}$), reaction diffusion problems, petroleum extraction and Astronomy, [31]. When $p = 2$ one recovers the Laplace equation and its generalizations, [31], [11].

Serrin, Gazzola and Tang have proved the existence of nonnegative, nontrivial (ground state) radial solutions to a general equation involving the p-laplace operator, [20]

$$\Delta_p u + f(u) = 0, \quad u > 0, \quad in \quad \mathbb{R}^n \tag{1.16}$$

Also they have proved the existence of positive radial solutions of the associated homogeneous Dirichlet- Neumann free boundary problem

$$\Delta_p u + f(u) = 0, \quad u > 0 \;\; in \quad B_R \tag{1.17}$$

$$u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial B_R$$

where B_R is an open ball in $\mathbb{R}^n$ with radius $R > 0$. The nonlinearity $f(u)$ defined for $u > 0$ is required to be Lipschitz continuous on $(0, \infty)$ in L^1 on $(0, 1)$ with $\int_{B_R} f(s)ds < 0$ for small $u > 0$.

Citti proved existence of ground states when $1 < p < n$, $f(0) < 0$, then the ground states connot exist for both cases when $p = 2$ and $p \neq 2$, [12].

The uniqueness of positive radial solutions to the Dirichlet problem

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) + f(u) = 0, \quad in \;\; B_R \tag{1.18}$$

$$u > 0 \text{ in } B_R \text{ , } u = 0 \text{ on } \partial B_R$$

On the finite ball of radius $R > 0$, $n \geq 3$, $1 < p \leq n$, where $f(u)$ is taken to be the function $\mu u^p + u^q$, $\mu > 0$ and $1 \leq p < q \leq \frac{n+2}{n-2}$, has been proved by Erbe and Tang, [17].

The case where the nonlinearity $f(u)$ is $(|u|^{q-1} u)$, has been studied in $\mathbb{R}^n$ and on the unit ball. Saxton and D.Wei have studied the nonlinear p-harmonic (1.18) Dirichlet problem, adapting to the case $p \neq 2$ in [19], and have shown that for p and q satisfying the sub-critical Sobolev embedding condition, (1.18) has infinitely many radially symmetric solutions,[36]. In chapter 2 of this thesis we follow the approach used in [36].

The existence and multiplicity of positive radial solutions to the Dirichlet problem

$$-\Delta_p u = q(|x|)f(u) \tag{1.19}$$

$$x \in B_1, \, u = 0, \text{ for } x \in \partial B_1$$

Where B_1 is the unit ball and the functions $q : (0, 1) \to \mathbb{R}_+$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous, has been established by Ercole and Zumpano in [16].

As an application of the p-laplacian we mention the following example given in [41] which shows how the p-laplacian operator my govern fluid flow in a porous medium. A description used for fluid flow in rock-filled dams is the Missbach's exponential law, which in $\mathbb{R}^2$ states that

$$-\kappa\nabla\Phi = |v|^{m-1}\, v \quad 1 \leq m \leq 2 \tag{1.20}$$

where κ denotes the permeability, Φ the piezometric head and $v = (u, v, 0)$ the average seepage velocity vector.

The component-wise form equation (1.20) is

$$-\kappa(\nabla\Phi)_x = |v|^{m-1}\, u, \quad -\kappa(\nabla\Phi)_y = |v|^{m-1}\, v \tag{1.21}$$

Where $\kappa\nabla\Phi = |v|^m$

Therefore

$$u = -1/\kappa\, |\nabla\Phi|^{\frac{1-m}{m}}\, \Phi_x \quad and \quad v = -1/\kappa\, |\nabla\Phi|^{\frac{1-m}{m}}\, \Phi_y. \tag{1.22}$$

Substituting the last two equations in the continuity equation one obtains the p-harmonic equation

$$\nabla \cdot (|\nabla\Phi|^{p-2}\, \nabla\Phi) = 0 \tag{1.23}$$

where $p = (1 + \frac{1}{m})$, [41].

The p-laplacian also appears in the power-law stokes equation to model the steady flow of a non-Newtonian fluid. The power-law stokes equation reads

$$-k\nabla \cdot (|\nabla(u)|^{p-2}\, \nabla(u)) + \nabla\nu = f \tag{1.24}$$

Where $u(x) = (u_1(x), ..., u_n(x))$ denotes the velocity of a fluid particle at

$x = (x_1, ...x_n) \in \Omega$, $f = (f_1, ...f_n)$ denotes the body force and ν is the scalar pressure. This model of non-Newtonian flow is very popular in chemical engineering, [8], as well as in geophysics, [40], for the design of the extrusion dies, [5], [30], and for the study of lithosphere, [14], [15].

1.3 The Generalized Lane-Emden equation as a general model

The generalized Lane-Emden equation

$$\nabla \cdot (|x|^{\alpha} |\nabla u|^{p-2} \nabla u) + |x|^{\beta} |u|^{q-1} u = 0, \tag{1.1}$$
$$u|_{\partial\Omega} = 0 \ , \ q > 1, \ 1 < p \leq 2, \alpha, \beta \geq 0.$$

Generalizes the above equations and their applications for appropriate choices of α, β, p and q.

Equations (1.2) , (1.3) and (1.8) can be obtained from equation (1.1) by choosing $\alpha, \beta = 0$ and $p = 2$.

Equation (1.5) can be obtained from (1.1) by setting $\alpha = 0$, $\beta = l > 0$, and $p = 2$.

Equation (1.4) can be obtained from (1.1) be choosing $\alpha, \beta = 0$ and $p = 2$, where $f(u)$ may involve the non-linearity $(|u|^{q-1} u)$ or other functions of u. When $f(u) = \lambda u + |u|^{q-1} u$, where $\lambda \geq 0$ is a constant, the equation is referred to as the eigenvalue problem.

Equations (1.6) and (1.7) are obtained from the equation (1.1) by setting $\alpha = 0$, $p = 2$, and $\beta = 0$ or not depending on the nature of the function $f(|x|, u)$.

Equations (1.9), (1.10) and (1.12) are obtained from the equation (1.1) by choosing $\alpha = 0$, $p = 2$, and $\beta = 0$ or not depending on the nature of the function $K(x)$. The function $K(x)$ takes the form of $\Phi(r)$ in the model (1.13) or the form $(1+|x|^2)^{-1}$ in the model (1.14) representing the Matukuma equation.

Equations (1.16), (1.17) and (1.18) are obtained from (1.1) by setting $\alpha, \beta = 0$ and $p \neq 2$.

Equations (1.15) and (1.19) are obtained from (1.1) by setting $\alpha = 0$, $p \neq 2$, and $\beta = 0$ or not depending on the nature of the function $f(|x|, u)$.

1.4 The Generalized Lane-Emden equation and applications

In this thesis we will consider two application models of the Generalized Lane-Emden equation from two different fields of science, Astronomy and Engineering.

The first application is given by the equation

$$\Delta u + |x|^{\beta} |u|^{q-1} u = 0 \tag{1.25}$$

$$u|_{\partial\Omega} = 0$$

$$q > 1, \beta \geq 0, n > 2$$

The second application is given by the equation

$$-\nabla \cdot \left(|x|^{-ap} |\nabla u|^{p-2} \nabla u\right) = |x|^{-(a+2)p+c} |u|^{q-1} u \tag{1.26}$$

$$u|_{\partial\Omega} = 0$$

$$1 < p \leq 2, q > 1, a < -1, c > 0$$

Where Ω is the unit ball in $\mathbb{R}^n$.

1.4.1 The first application and literature

The equation

$$\Delta u + K(|x|)f(u) = 0 \tag{1.27}$$

was proposed as a model describing the dynamics of Galaxies by G.Bertin, [6]. Based on the assumption that galaxies are axial symmetric, a cylindrical symmetry of the problem was derived where $f(u) = u^q$, $u = u(r, z)$, $K(|x|) = \Psi(r)$, where $\Psi(r) = \frac{r^{2\alpha}}{(1+r^2)^{1/2+\alpha}}$, $\alpha \geq 0$ and $r = \sqrt{x_1^2 + x_2^2}$, $z = x_3$, [3].

Various equations similar to equation (1.27) have been proposed to model several phenomena of interest in astrophysics, for example the Matukuma equation (1.14).

These models were used to model globular clusters of stars and so radial symmetry was the natural symmetry assumption, [3].

Equation (1.27) has it's roots from many mathematical and physical fields, e.g, the scalar curvature equation (1.12), as well as the study of Riemannian Geometry and scalar field equation for the standing wave of nonlinear Schrodinger and Klein-Gordan equations. For $K(|x|) = 1$ we have the Lane-Emden equation that plays an important role in astrophysics, [28].

Extensive research was done on (1.27) and it's generalizations by many scientists in these fields. For example, in [29], [28], the nonlinearity $f(u)$ is taken to equal $|u|^{q-1} u$ in $\mathbb{R}^n$ and $K(r)$ is a smooth and positive function on $(0, \infty)$ where $K(r) \in L^1(0, 1)$ for $q > 1$, $n > 2$,

In our application we consider the following equation on a unit ball centered at zero with $K(x) = |x|^\beta$

$$\Delta u + |x|^\beta |u|^{q-1} u = 0, \quad x \in B^n \tag{1.28}$$

$$u|_{\partial B^n} = 0, \beta \geq 0, q > 1, n > 2$$

1.4.2 The second application and literature

Degenerate nonlinear elliptic equations of the type

$$-\nabla \cdot (a(x)\nabla u) = g(\lambda, x, u), \quad x \in \Omega. \tag{1.29}$$

Where λ is a real parameter, Ω is a (bounded or unbounded) domain in $\mathbb{R}^n$ $(n \geq 2)$ and $a(x)$ is a nonnegative measurable weight function that is allowed to have zeros at some points, have long history and come from the consideration of standing waves in anisotropic Schrödinger equation, [35]. Such problems of anisotropic media can be regarded as equilibrium solutions of the evolution equations

$$u_t = F(\lambda, u, \nabla u) \quad in \quad \Omega \times (0, T) \tag{1.30}$$

Where $u = u(x, t)$ is the state of a certain system. The study of nontrivial solutions of the problem $F(\lambda, u, \nabla u) = 0$ in the given domain is motivated by important phenomena such as the irrotational flow of a fluid along a flat-bottomed canal with $F(\lambda, 0, 0) = 0$. Other problems of this type also appear in reaction diffusion processes, [35],[9].

F may be taken to involve the quasilinear differential operator

$$\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u), \quad 1 < p < \infty \tag{1.31}$$

The generalized Lane-Emden equation appears as an application in fluid mechanics as a physical phenomena related to equilibrium of anisotropic media which possible are perfect insulators. For instance, if $\check{T}$ is the sheer stress and $\nabla_p u$ is the velocity gradient then these quantities obey a relation of the form $T\check{(}x) = a(x)\nabla_p u(x)$ where $\nabla_p u = |\nabla u|^{p-2} \nabla u$ is the p-laplacian operator, $p > 1$ is an arbitrary number. The case $p = 2$ (respectively $p < 2$, $p > 2$) corresponds to a Newtonian (respectively pseudo plastic, dilatant) fluid. The resulting equations of motion then involve

the nonlinear, inhomogeneous expression $\nabla \cdot (a\nabla_p u)$, which reduces to $a\nabla \cdot (\nabla_p u)$ for a being a constant, [7], [31].

For example, several existence results for the following eigenvalue problem involving a p-laplacian and a nonlinear boundary condition on unbounded domains appears in [31],

$$-\nabla \cdot (a(x) \left|\nabla u\right|^{p-2} \nabla u) = \lambda f(x) \left|u\right|^{p-2} u + q(x) \left|u\right|^{q-2} u \quad x \in \Omega \tag{1.32}$$

$$a(x) \left|\nabla u\right|^{p-2} \nabla u \cdot \nu + b(x) \left|u\right|^{p-2} u = f(x, u) \ , \ x \in \partial\Omega$$

Where ν denotes the unit outward normal on the smooth boundary $\partial\Omega$, $\Omega \subset \mathbb{R}^n$ is an unbounded domain, $\lambda \succ 0$, the functions a,b and h are positive while f and g are subcritical nonlinearities, $0 < a_0 < a \in L^\infty(\Omega)$, while $b : \partial\Omega \to \mathbb{R}$ is a continuous function satisfying

$$\frac{c}{(1+|x|)^{p-1}} \leq b(x) \leq \frac{C}{(1+|x|)^{p-1}} \tag{1.33}$$

for some constants $0 < c \leq C$, $p, q, n \in \mathbb{R}, 1 < p < q < p^* = \frac{np}{n-p}$. For further examples see [11].

Quasilinear problems with variable coefficients also appear in the mathematical model of the torsional creep (elastic for $p = 2$, plastic as p approaches infinity). This study is based on the observation that a prismatic material rod subject to torsional moment at sufficiently high temperatures and for an extended period of time exhibits a permanent deformation called creep, [31].

A specific example of the equation above that has been under study by many scientists is the degenerate problem

$$-\nabla \cdot (|x|^{-ap} \left|\nabla u\right|^{p-2} \nabla u) = |x|^{-(a+1)p+c} f(u), \quad in \quad \Omega \tag{1.34}$$

$$u = 0 \text{ on } \partial\Omega$$

Where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with C^1 boundary and $0 \in \Omega$,

$1 \ < p < \ n$, $0 \leq a < \frac{n-p}{p}$, $c > 0$, [42].

The existence of multiple solutions to equation (1.34) with asymptotically linear term at infinity has been studied by Xuan, B in [42], where the asymptotically linear term is $lim_{|t|\to\infty} \frac{f(t)}{t} = l < \infty$.

In our application (1.26) we take Ω to be the unit ball in n dimensions and $f(u) = |u|^{q-1} u$, $a < -1$ and $c > 0$. This equation appears in physical phenomena related to equilibrium of anisotropic media which possibly are perfect insulators or perfect conductors.

1.5 Organization of the thesis

In this thesis we prove the existence and non-uniqueness of an infinite number of solutions to the Generalized Lane-Emden equation in relation to a critical Sobolev exponent using variational analysis as well as a complete phase plane analysis where there appears another sub-critical exponent influencing the behavior of solutions. A connection between both settings will be made subsequently.

We start with a weighted Sobolev embedding theorem in chapter 2. Then we use the Pohozaev's identity to prove the non-existence of both general and radial solutions of the Generalized Lane-Emden equation for $q+1 \geq p^* = \frac{p(n+\beta)}{n+\alpha-p}$,

$1 < p \leq 2, q > 1, \alpha, \beta \geq 0$. Next we use the shooting argument to prove the existence and non-uniqueness locally and globally of an infinite number of radial solutions to the corresponding ordinary differential equation for $p < q+1 < p^*$,

$1 < p \leq 2$, $\alpha, \beta \geq 0$. We end chapter 2 with few properties of these radial solutions.

In chapter 3 we move to phase plane analysis. We start by transforming the Generalized Lane-Emden equation and its associated boundary condition in to an autonomous system in phase plane. We perform complete phase plane analysis which includes finding the finite critical points of the system, local behavioral analysis of trajectories, global portraits and critical exponent analysis. In this chapter we obtain the critical exponents that influence the existence and behavior of solutions of the Generalized Lane-Emden equation.

We conclude with chapter 4 where we introduce the Lane-Emden equation in to phase plane to obtain the sub-critical and critical exponents in phase plane with a complete description of the behavior of solutions for $p < q+1 < p^*, p^* = \frac{2n}{n-2}$. We also include similar results to two applications from the fields of Astronomy and Engineering.

We start by providing definitions and notations that are a basis to our work in this thesis. The symbol Ω will refer to a general domain subset of $\mathbb{R}^n$ and $\partial\Omega$ will represent the boundary of the domain.

Definition 1.5.1:

i) A function $\omega(x)$ defined on a domain $\Omega \subset \mathbb{R}^n$ is said to be a weight function if it is strictly positive, finite measurable almost everywhere on Ω.

ii) Let $\omega(x)$ be a weight function and $1 \leq p < \infty$, the weighted Lebesgue space $L^p(\Omega, \omega)$ is the set of all measurable functions u defined on Ω such that $\int_\Omega \omega(x) |u(x)|^p dx < \infty$. The space $L^p(\Omega, \omega)$ is a Banach space with the norm $\|u\|_{L^p(\Omega,\omega)} = (\int_\Omega |u(x)|^p \omega(x) dx)^{1/p}$.

iii) Let $1 \leq p < \infty$, ω is a weight function as defined in i), then the weighted Sobolev space $W^{1,p}(\Omega, \omega)$ is the set of all functions $u \in L^p(\Omega, \omega)$ such that $\frac{\partial u}{\partial x_i} \in L^p(\Omega, \omega)$ for all $i = 1, 2, ..., n$. The space $W^{1,p}(\Omega, \omega)$ is a Banach space with the norm

$\|u\|_{W^{k,p}(\Omega,\omega)} = [\|u\|^p_{L^p(\Omega,\omega)} + \sum \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega,\omega)}]^{\frac{1}{p}}$, (Appendix A2).

We will be using the symbol $\hookrightarrow$ for continuous embedding and $\hookrightarrow\hookrightarrow$ for compact embedding.

Taking $\omega(x)$ to be the function $|x|^\alpha$ or $|x|^\beta$ and $\Omega = B^n$, we define the following norms and function spaces.

Definition 1.5.2:

i) The weighted $q+1$ norm is:

$\|u\|_{L^{q+1}(B^n,|x|^\beta)} = (\int_{B^n} |x|^\beta |u(x)|^{q+1} dx)^{\frac{1}{q+1}}$ for $1 \leq q+1 < \infty$.

and

ii) The weighted p norm:

$\|\nabla u\|_{L^p(B^n,|x|^\alpha)} = (\int_{B^n} |x|^\alpha \, |\nabla u(x)|^p \, dx)^{\frac{1}{p}}$ for $1 \leq p < \infty$.

Definition 1.5.3:

$$W^{1,p}(B^n, |x|^\alpha) \equiv \left\{ u \in W^{1,p}_{loc}(B^n) : \|\nabla u\|_{L^p(B^n,|x|^\alpha)} = \int_{B^n} |x|^\alpha \left(|u|^p + |\nabla u|^p\right) dx)^{\frac{1}{p}} < \infty \right\}$$

and

$$L^{q+1}(B^n, |x|^\beta) \equiv \left\{ u \in W^{q+1}_{loc}(B^n) : \|u\|_{L^{q+1}(B^n,|x|^\beta)} = \int_{B^n} |x|^\beta |u|^{q+1} \, dx)^{\frac{1}{q+1}} < \infty \right\}$$

Chapter 2

Existence Of Solutions Using Variational Methods.

In this chapter we use a shooting argument to prove the existence and nonuniqueness of radial solutions to (1.1) in $W^{1,p}(B^n, |x|^\alpha) \cap L^{q+1}(B^n, |x|^\beta)$, where for $q+1 < p^*$, $p^* = \frac{p(n+\beta)}{n+\alpha-p}$ the following inequality $\|u\|_{L^{q+1}(B^n,|x|^\beta)} \leq C \|\nabla u\|_{L^p(B^n,|x|^\alpha)}$ holds and the embedding $W^{1,p}(B^n, |x|^\alpha) \hookrightarrow L^{q+1}(B^n, |x|^\beta)$ is compact, [42]. Then we prove the non-existence of radial and general solutions on a star shaped domain to (1.1) using Pohozaev's identity for $q+1 \geq p^*$.

In this chapter we also prove that the class of all nontrivial solutions are bounded below for $p < q+1 < p^* = \frac{p(n+\beta)}{n+\alpha-p}$, $1 < p \leq 2, q > 1, \alpha, \beta \geq 0$ by a constant dependent on p, n, α, β. We then prove that the maximum amplitude of radial solutions to the Generalized Lane-Emden equation occurs at the origin and establish the existence of a finite number of zeros.

The Generalized Lane-Emden equation (1.1) may be considered as the Euler-Lagrange equation for the functional

$$F(u) = \int_\Omega (\frac{1}{p} |x|^\alpha |\nabla u|^p - \frac{1}{q+1} |x|^\beta |u|^{q+1}) dx \tag{2.1}$$

Upon taking the first variation of the functional $F(u)$ in (2.1) we have

$$\delta F(u) = \delta \int_\Omega (\frac{1}{p} |x|^\alpha |\nabla u|^p - \frac{1}{q+1} |x|^\beta |u|^{q+1}) dx \tag{2.2}$$

where $\delta F(u) = \frac{d}{d\epsilon} F(u+\epsilon w)|_{\epsilon=0}, \forall w \in C_0^\infty(\Omega)$ and $\epsilon w = \delta u$. Since $\delta \ |\nabla u|^p \ = \ p \ |\nabla u|^{p-2} \nabla u \ \cdot \ \nabla \ \delta u$ and $\delta |u|^{q+1} = (q+1) |u|^{q-1} u \delta u$, we have

$$\delta F(u) = \int_\Omega (|x|^\alpha |\nabla u|^{p-2} \nabla u \cdot \nabla \delta u - |x|^\beta |u|^{q-1} u \delta u) dx \tag{2.3}$$

Then

$$\delta F(u) = -\int_\Omega [\nabla \cdot (|x|^\alpha |\nabla u|^{p-2} \nabla u) + |x|^\beta |u|^{q-1} u] \delta u dx \tag{2.4}$$

We consider radial solutions regular at the origin satisfying

$$< F'(u), u >= 0 \tag{2.5}$$

That is

$$\int_\Omega u(\nabla \cdot (|x|^\alpha |\nabla u|^{p-2} \nabla u) + |x|^\beta |u|^{q-1} u)dx = 0 \tag{2.6}$$

This simplifies in to

$$\int_\Omega (-\nabla u \cdot (|x|^\alpha |\nabla u|^{p-2} \nabla u) + |x|^\beta |u|^{q+1})dx = 0 \tag{2.7}$$

Hence

$$\int_{B^n} |x|^\alpha |\nabla u|^p \, dx = \int_{B^n} |x|^\beta |u|^{q+1} \, dx \tag{2.8}$$

In this thesis we examine solutions for which the terms in this relation are finite.

Solutions to (1.1) satisfying Dirichlet boundary conditions on a domain Ω are characterized as critical points of the energy functional (2.1). Studying the existence and qualitative properties of these critical points corresponds to studying these properties for the solutions of the equation (1.1). This has been known to be done using variational methods.

We consider the radial equation corresponding to the Generalized Lane-Emden equation.

$$r^{1-n}(r^{n+\alpha-1} |u_r|^{p-2} u_r)_r + r^\beta |u|^{q-1} u = 0 \tag{2.9}$$

Which after simplification and collecting like terms becomes

$$(p-1)r^\alpha |u_r|^{p-2} u_{rr} + (\alpha + n - 1)r^{\alpha-1} |u_r|^{p-2} u_r + r^\beta |u|^{q-1} u = 0 \tag{2.10}$$

2.1 Weighted Sobolev embeddings

We start by stating two well known results (Theorems 2.1.1 and 2.1.2) and a Conjecture. Theorem 2.1.1 and Conjecture 2.1.1 provide us with embedding results needed for weighted Sobolev spaces and Theorem 2.1.2 is a trace embedding result for unweighted spaces.

Before stating the theorem and the conjecture we define $\hat{p}$ as the sub-critical exponent mentioned in the literature for the Lane-Emden equation ($\hat{p} = \frac{2(n-1)}{n-2}$), [17]. In chapter 3 we obtain a sub-critical exponent through phase plane analysis, again labeled $\hat{p}$, which corresponds to a change in the structure of solutions to the Generalized Lane-Emden equation, here $\hat{p} = \frac{p(n+\beta)-(p+\beta-\alpha)}{n+\alpha-p}$. We also define p^*, the critical exponent for the Lane-Emden equation ($p^* = \frac{2n}{n-2}$), [17], where $p^* = \frac{p(n+\beta)}{\alpha+n-p}$ for the Generalized Lane-Emden equation.

Theorem 2.1.1. *Weighted Sobolev Embedding Theorem.*

Let $\alpha, \beta \geq 0$, $1 < p \leq 2$ *and* $p < q+1 \leq p^*, p^* = \frac{p(n+\beta)}{\alpha+n-p}$. *Then* $\|u\|_{L^{q+1}(B^n,|x|^\beta)} \leq C \|\nabla u\|_{L^p(B^n,|x|^\alpha)}$ *and the embedding* $W^{1,p}(B^n, |x|^\alpha) \hookrightarrow L^{q+1}(B^n, |x|^\beta)$ *is continuous. If the upper bound for* $q+1$ *is strict then the embedding is compact, [42], (see appendix A1 for a proof).*

Theorem 2.1.2. *Trace Embedding Theorem*

Consider a unit ball in n *dimensions,* B^n. *Let* $1 \leq q+1 \leq \hat{p}, \hat{p} = \frac{p(n-1)}{n-p}$, *then* $W^{1,p}(B^n) \hookrightarrow L^{q+1}(\partial B^n)$ *is continuous and the inequality* $\|u\|_{L^{q+1}(\partial B^n)} \leq C \|u\|_{W^{1,p}(B^n)}$ *holds, [9].*

Conjecture 2.1.1. *Weighted Trace Embedding Theorem*

Let $\alpha, \beta \geq 0$, $1 < p \leq 2$ *and* $p < q+1 \leq \hat{p}, \hat{p} = \frac{p(n+\beta)-(p+\beta-\alpha)}{n+\alpha-p}$ *then*

$W^{1,p}(B^n,|x|^{\alpha}) \hookrightarrow L^{q+1}(\partial B^n,|x|^{\beta})$ *is continuous and the inequality* $\|u\|_{L^{q+1}(\partial B^n,|x|^{\beta})} \leq C\,\|u\|_{L^p(B^n,|x|^{\alpha})}$ *holds. If the upper bound of* $q+1$ *is strict then the trace embedding is compact.*

2.2 Non-existence of solutions for $q+1 \geq p^*$

In this section we prove the non existence of solutions for the generalized Lane-Emden equation, radial and general, for $q+1 \geq p^* = \frac{p(n+\beta)}{n+\alpha-p}$ using Pohozaev's identity.

2.2.1 Non-existence of radial solutions for $q+1 \geq p^*$

Before we prove the nonexistence of solutions for $q+1 \geq p^* = \frac{p(n+\beta)}{n+\alpha-p}$, consider the radial form of the Generalized Lane-Emden equation

$$(p-1)r^\alpha |u_r|^{p-2} u_{rr} + (\alpha+n-1)r^{\alpha-1} |u_r|^{p-2} u_r + r^\beta |u|^{q-1} u = 0 \qquad (2.10)$$

Upon multiplying (2.10) by u_r we have:

$$(p-1)r^\alpha |u_r|^{p-2} u_{rr}u_r + (\alpha+n-1)r^{\alpha-1} |u_r|^{p-2} u_r u_r + r^\beta |u|^{q-1} uu_r = 0. \quad (2.11)$$

Since

$$(p-1)r^\alpha |u_r|^{p-2} u_r u_{rr} = \frac{d}{dr}[\frac{p-1}{p} r^\alpha |u_r|^p] - \alpha(\frac{p-1}{p}) r^{\alpha-1} |u_r|^p, \qquad (2.12)$$

and

$$r^\beta |u|^{q-1} uu_r = \frac{d}{dr}[\frac{1}{q+1} r^\beta |u|^{q+1}] - \frac{\beta}{q+1} r^{\beta-1} |u|^{q+1}, \qquad (2.13)$$

Equation (2.11) becomes

$$\frac{d}{dr}[\frac{p-1}{p} r^\alpha |u_r|^p + \frac{1}{q+1} r^\beta |u|^{q+1}] = -(\frac{\alpha}{p} + n - 1) r^{\alpha-1} |u_r|^p + \frac{\beta}{q+1} r^{\beta-1} |u|^{q+1}. \qquad (2.14)$$

Integrating both sides of equation (2.14) with respect to r from 0 to r we obtain $\int_0^r \frac{d}{ds}[\frac{p-1}{p} s^\alpha |u_s|^p + \frac{1}{q+1} s^\beta |u|^{q+1}]ds$

$$= \int_0^r [-(\frac{\alpha}{p} + n - 1) s^{\alpha-1} |u_s|^p + \frac{\beta}{q+1} s^{\beta-1} |u|^{q+1}]ds \qquad (2.15)$$

and letting $E(r) = \frac{p-1}{p} r^\alpha |u_r|^p + \frac{1}{q+1} r^\beta |u|^{q+1}$, we then have

$$E(r) = \int_0^r [-(\frac{\alpha}{p} + n - 1) s^{\alpha-1} |u_s|^p + \frac{\beta}{q+1} s^{\beta-1} |u|^{q+1}]ds + E(0) \qquad (2.16)$$

We examine bounded solutions such that $u(0)$ and $u_r(0)$ are both finite, with $\alpha, \beta \geq 0, 1 < p \leq 2, q > 1$. In particular, let $u(0) = \sigma, |\sigma| < \infty, u_r(0) = 0, u(1) = 0,$

$u_r(1) = -\delta, -\delta \in (-\infty, \infty)$, consequently $E(0) = 0$ for $\alpha, \beta > 0$. For $\alpha, \beta = 0$, $E(r) = \frac{p-1}{p} |u_r|^p + \frac{1}{q+1} |u|^{q+1}$ and $E(0) = \frac{|\sigma|^{q+1}}{q+1}$. For reference on the case $\alpha, \beta = 0$ see [36].

Therefore

$$E(r, \sigma) = \int_0^r [-(\frac{\alpha}{p} + n - 1)s^{\alpha-1} |u_s|^p + \frac{\beta}{q+1} s^{\beta-1} |u|^{q+1}]ds \tag{2.17}$$

Consider (2.14) again, integrating both sides from ϵ to 1 with respect to r we have

$$\frac{p-1}{p} r^{\alpha} |u_r|^p |_{\epsilon}^{1} + \frac{1}{q+1} r^{\beta} |u|^{q+1} |_{\epsilon}^{1} = \int_{\epsilon}^{1} [-(\frac{\alpha}{p} + n - 1) r^{\alpha-1} |u_r|^p + \frac{\beta}{q+1} r^{\beta-1} |u|^{q+1}]dr \tag{2.18}$$

Letting $\epsilon \to 0$, for $1 < p \leq 2, q > 1, u(0) = |\sigma| < \infty, u_r(0) = 0, u(1) = 0,$

$u_r(1) = -\delta, -\delta \in (-\infty, \infty)$, the integrand $\int_0^1 -(\frac{\alpha}{p} + n - 1) r^{\alpha-1} |u_r|^p \, dr$ is finite for $\alpha \geq 0$ and the integrand $\int_0^1 \frac{\beta}{q+1} r^{\beta-1} |u|^{q+1} \, dr$ is finite for $\beta \geq 0$.

Let $E_1(r, \sigma) = r^{-\beta} E(r, \sigma) = \frac{p-1}{p} r^{\alpha-\beta} |u_r|^p + \frac{1}{q+1} |u|^{q+1}$, next we prove that $E_1(r, \sigma)$ is a monotone decreasing function for a fixed value of σ.

Lemma 2.2.1. *$\frac{d}{dr} E_1(r, \sigma) \leq 0$ for $\alpha, \beta \geq 0, 1 < p \leq 2, q > 1, u(0) = |\sigma| < \infty, u_r(0) = 0,$ $r \geq 0$.*

Proof:

Consider multiplying (2.10) by $r^{-\beta} u_r$, we then have

$$(p-1) r^{\alpha-\beta} |u_r|^{p-2} u_r u_{rr} + (\alpha + n - 1) r^{\alpha-\beta-1} |u_r|^p + |u|^{q-1} u u_r = 0 \tag{2.19}$$

After simplification and collecting like terms equation (2.19) gives

$$\frac{d}{dr} [\frac{p-1}{p} r^{\alpha-\beta} |u_r|^p + \frac{1}{q+1} |u|^{q+1}] = -(n + \beta - 1 + \frac{\alpha - \beta}{p}) r^{\alpha-\beta-1} |u_r|^p \tag{2.20}$$

Observe that $n-1+\frac{\alpha}{p}+\frac{(p-1)\beta}{p}>0$ for $\alpha,\beta\geq 0$ and $1<p\leq 2$, hence

$$\frac{dE_1}{dr}\leq 0 \quad for \quad r\geq 0 \tag{2.21}$$

Integrating (2.21) from 0 to r gives

$$E_1(r)\leq E_1(0) \quad for \quad r\geq 0 \tag{2.22}$$

Observe that the function, $E_1(r)$, is bounded provided that $\alpha,\beta\geq 0, 1<p\leq 2, q>1$, $u_r(0)=0$ and $u(0)=|\sigma|$ are finite, which in return implies that $u(r)$ and $u_r(r)$ are bounded. We therefore have

$$E_1(r)\leq \frac{1}{q+1}|\sigma|^{q+1} \tag{2.23}$$

Integrating (2.21) with respect to r from $r=0$ to $r=1$ gives the following estimate for $u_r(1)=-\delta$

$$E_1(1)\leq E_1(0) \tag{2.24}$$

Hence,

$$|\delta|^p\leq \frac{p}{(p-1)(q+1)}|\sigma|^{q+1} \tag{2.25}$$

Lemma 2.2.2. *Let $u(r)$ satisfy (2.9), $u(0)=\sigma$, $u_r(0)=0, \alpha,\beta\geq 0, 1<p\leq 2$, $p^*=\frac{p(n+\beta)}{n+\alpha-p}$, then*

$$\frac{1}{n+\beta}r^nE(r,\sigma)=\lambda\int_0^r s^{n+\beta-1}|u|^{q+1}ds-\frac{1}{p^*}r^{n+\alpha-1}|u_r|^{p-2}u_ru \tag{2.26}$$

for $r\in[0,1]$ and $\lambda=\frac{1}{q+1}-\frac{1}{p^*}$

Proof:

Consider

$(r^nE)_r=r^{n-1}(rE_r+nE)$

Where

$$E_r=-(\frac{\alpha}{p}+n-1)r^{\alpha-1}|u_r|^p+\frac{\beta}{q+1}r^{\beta-1}|u|^{q+1} \tag{2.27}$$

And

$$rE_r = -(\frac{\alpha}{p} + n - 1)r^{\alpha} |u_r|^p + \frac{\beta}{q+1} r^{\beta} |u|^{q+1} \tag{2.28}$$

Hence

$$rE_r + nE = [-(\frac{\alpha}{p} + n - 1) + \frac{n(p-1)}{p}] r^{\alpha} |u_r|^p + \frac{n+\beta}{q+1} r^{\beta} |u|^{q+1} \tag{2.29}$$

Which simplifies in to

$$rE_r + nE = -(\frac{n+\alpha-p}{p}) r^{\alpha} |u_r|^p + \frac{n+\beta}{q+1} r^{\beta} |u|^{q+1} \tag{2.30}$$

Then

$$(r^n E)_r = -(\frac{n+\alpha-p}{p}) r^{\alpha+n-1} |u_r|^p + \frac{n+\beta}{q+1} r^{\beta+n-1} |u|^{q+1} \tag{2.31}$$

Now consider equation (2.10), upon multiplying by r^{n-1} we have

$$(p-1)r^{n+\alpha-1} |u_r|^{p-2} u_{rr} + (\alpha+n-1)r^{\alpha+n-2} |u_r|^{p-2} u_r = -r^{\beta+n-1} |u|^{q-1} u \tag{2.32}$$

Note also that

$$(r^{n+\alpha-1} |u_r|^{p-2} u_r)_r = (p-1)r^{n+\alpha-1} |u_r|^{p-2} u_{rr} + (n+\alpha-1)r^{n+\alpha-2} |u_r|^{p-2} u_r \tag{2.33}$$

Hence equation (2.32) can be written as

$$(r^{n+\alpha-1} |u_r|^{p-2} u_r)_r = -r^{\beta+n-1} |u|^{q-1} u. \tag{2.34}$$

And so

$$(r^{n+\alpha-1} |u_r|^{p-2} u_r)_r u = -r^{\beta+n-1} |u|^{q+1} \tag{2.35}$$

Next

$$(r^{n-1+\alpha} |u_r|^{p-2} u_r u)_r = (r^{\alpha+n-1} |u_r|^{p-2} u_r)_r u + (r^{\alpha+n-1} |u_r|^{p-2} u_r) u_r \tag{2.36}$$

And so

$$(r^{n-1+\alpha} |u_r|^{p-2} u_r u)_r = (r^{n-1+\alpha} |u_r|^{p-2} u_r)_r u + r^{\alpha+n-1} |u_r|^p \tag{2.37}$$

Equations (2.35) and (2.37) together give

$$(r^{n-1+\alpha}\left|u_r\right|^{p-2}u_r u)_r = -r^{\beta+n-1}\left|u\right|^{q+1} + r^{n+\alpha-1}\left|u_r\right|^p \tag{2.38}$$

Now

$$(\frac{1}{n+\beta}r^n E + \frac{1}{p^*}r^{n+\alpha-1}\left|u_r\right|^{p-2}u_r u)_r = \frac{1}{n+\beta}(r^n E)_r + \frac{1}{p^*}(r^{n-1+\alpha}\left|u_r\right|^{p-2}u_r u)_r \tag{2.39}$$

$$= \tfrac{1}{n+\beta}[-\tfrac{n+\alpha-p}{p}r^{n+\alpha-1}\left|u_r\right|^p + \tfrac{n+\beta}{q+1}r^{n+\beta-1}\left|u\right|^{q+1}]$$

$$+\frac{1}{p^*}[r^{n+\alpha-1}\left|u_r\right|^p - r^{n+\beta-1}\left|u\right|^{q+1}] \tag{2.40}$$

Collecting like terms we have

$$(\tfrac{1}{n+\beta}r^n E + \tfrac{1}{p^*}r^{n+\alpha-1}\left|u_r\right|^{p-2}u_r u)_r =$$

$$[\frac{1}{p^*} - \frac{n+\alpha-p}{p(n+\beta)}]r^{n+\alpha-1}\left|u_r\right|^p + [\frac{1}{q+1} - \frac{1}{p^*}]r^{n+\beta-1}\left|u\right|^{q+1} \tag{2.41}$$

Since $p^* = \frac{p(n+\beta)}{\alpha+n-p}$ then $[\frac{1}{p^*} - \frac{\alpha+n-p}{p(n+\beta)}] = 0$. Hence we have

$$(\frac{1}{n+\beta}r^n E + \frac{1}{p^*}r^{n-1+\alpha}\left|u_r\right|^{p-2}u_r u)_r = [\frac{1}{q+1} - \frac{1}{p^*}]r^{n+\beta-1}\left|u\right|^{q+1} \tag{2.42}$$

Integrating both sides of equation (2.42) from 0 to r with respect to r gives

$$\frac{1}{n+\beta}r^n E + \frac{1}{p^*}r^{n+\alpha-1}\left|u_r\right|^{p-2}u_r u = \int_0^r (\frac{1}{q+1} - \frac{1}{p^*})s^{n+\beta-1}\left|u(s)\right|^{q+1}ds \tag{2.43}$$

rearranging equation (2.43) gives the desired result

$$\frac{1}{n+\beta}r^n E = \lambda\int_0^r s^{n+\beta-1}\left|u(s)\right|^{q+1}ds - \frac{1}{p^*}r^{n+\alpha-1}\left|u_r\right|^{p-2}u_r u \tag{2.44}$$

Where $\lambda = (\frac{1}{q+1} - \frac{1}{p^*})$ and $r \in [0,1]$.

Theorem 2.2.1. *For* $\alpha, \beta \geq 0, 1 < p \leq 2, q > 1, u(0) = |\sigma| < \infty, u_r(0) = 0,$ $q+1 \geq p^*, p^* = \frac{p(n+\beta)}{\alpha+n-p}$*, no radial solutions exist for the generalized Lane-Emden equation.*

Proof

Let $r = r_0 > 0$ denote the value of r at which $u(r)$ has its first zero such that $u(r_0) = 0$. Substituting this value in (2.44) gives the equation

$$\frac{1}{n+\beta} r_0^n E(r_0) = (\frac{1}{q+1} - \frac{1}{p^*}) \int_0^{r_0} r^{n+\beta-1} |u(r)|^{q+1} \, dr. \tag{2.45}$$

If $q+1 = p^*$, $p^* = \frac{p(n+\beta)}{n+\alpha-p}$, then $E(r_0) = \frac{p-1}{p} r_0^\alpha |u_r(r_0)|^p + \frac{1}{q+1} r_0^\beta |u(r_0)|^{q+1} = 0$ and consequently $u_r(r_0) = 0$. Since $r_0 > 0$, then $u(r_0) = 0$ on $(r_0 - \epsilon, r_0 + \epsilon)$ for some $\epsilon > 0$, which violates the assumption on r_0 being the first zero of $u(r)$.

If $q + 1 > p^*$, then $E(r_0) < 0$ for nontrivial solutions, which is a contradiction since $E(r)$ is positive for all $r > 0$ and $1 < p \leq 2$, $q > 1, \alpha, \beta \geq 0$. Therefore either $u(r) = 0$, which implies that all possible solutions are trivial solutions, or no radial solutions exist for $q + 1 \geq p^*$. This leaves us with the case $q + 1 < p^*$ to examine existence of solutions, [33].

2.2.2 Non-existence of general solutions for $q+1 \geq p^*$

We also establish the nonexistence of more general solutions for $q+1 \geq p^*$ by applying Pohozaev's identity to equation (1.1) .

Lemma 2.2.3. *For $\alpha, \beta \geq 0, 1 < p \leq 2, q > 1, q+1 \geq p^*$, $p^* = \frac{p(n+\beta)}{\alpha+n-p}$, there exist no solutions to the generalized Lane-Emden equation over any domain which is smooth and star-shaped with respect to the origin.*

Proof:

Multiplying (1.1) by $x \cdot \nabla u$ on both sides of the equation and integrating by parts gives (appendix A3)

$$\frac{p-1}{p}\int_{\partial B^n} |x|^\alpha |\nabla u|^p x{\cdot}ndS + \frac{\alpha+n-p}{p}\int_{B^n} |x|^\alpha |\nabla u|^p dx = \int_{B^n} \frac{n+\beta}{q+1} |x|^\beta |u|^{q+1} dx \tag{2.46}$$

By (2.8)

$$\textstyle\int_{B^n} |x|^\alpha |\nabla u|^p dx = \int_{B^n} |x|^\beta |u|^{q+1} dx \tag{2.8}$$

Combining the two identities (2.46) and (2.8), we obtain

$$\frac{p-1}{p}\int_{\partial B^n} |x|^\alpha |\nabla u|^p x{\cdot}\nu dS + \frac{n+\alpha-p}{p}\int_{B^n} |x|^\beta |u|^{q+1} dx = \int_{B^n} \frac{n+\beta}{q+1} |x|^\beta |u|^{q+1} dx \tag{2.47}$$

Hence we have

$$\frac{p-1}{p}\int_{\partial B^n} |x|^\alpha |\nabla u|^p x \cdot \nu dS = -[\frac{n+\alpha-p}{p} - \frac{n+\beta}{q+1}]\int_{B^n} |x|^\beta |u|^{q+1} dx \tag{2.48}$$

It is clear that for $\frac{n+\beta}{q+1} = \frac{n+\alpha-p}{p}$, that is for $q+1 = p^* = \frac{p(n+\beta)}{n+\alpha-p}$, $q > 1$, $1 < p \leq 2$, $n > 2$, $\alpha, \beta \geq 0$, only trivial solutions exist for (1.1).

For $(\frac{n+\alpha-p}{p} - \frac{n+\beta}{q+1}) > 0$, $q > 1$, $1 < p \leq 2$, $\alpha, \beta \geq 0$, we have a contradiction since the right hand side of the equation is positive and so is the left hand side for the

absolute values. This implies that for $\frac{n+\alpha-p}{p} > \frac{n+\beta}{q+1}$, that is for $q+1 > \frac{p(n+\beta)}{n+\alpha-p} = p^*$, provided that $p < n+\alpha$, no nontrivial solutions exist.

2.3 Existence and nonuniqueness of infinitely many radially symmetric solutions to the Generalized Lane-Emden equation

Proving the existence of infinitely many radially symmetric solutions to the boundary value problem

$$(p-1)r^{\alpha}\left|u_r\right|^{p-2}u_{rr}+(\alpha+n-1)r^{\alpha-1}\left|u_r\right|^{p-2}u_r+\left|r\right|^{\beta}\left|u\right|^{q-1}u=0\ , \tag{2.10}$$

$$u(1)=u_r(0)=0 \tag{2.49}$$

is done using the shooting argument which relates (2.10), (2.49) to the initial value problem (2.10) with initial conditions (2.50),

$$u(0)=\sigma, u_r(0)=0 \tag{2.50}$$

The value of σ is chosen in a manner that the solution to (2.10), (2.50) also satisfies $u(1)=0$. This solution exists for $1<p\leq 2$, $\alpha,\beta\geq 0$, $p<q+1<p^*$ and is unique for $r\in[0,1]$(see section 2.4). Using the shooting argument we will be able to show that there exists a sequence $\{\sigma_n\}$, $n=1,2,3,...$ of values of σ, each of which gives rise to a corresponding solution of (2.10), (2.49). Thus the σ_n parametrize an infinite sequence of solutions, $\{u_n(r)\}$, n = 1,2,3, ... to the boundary value problem (2.10), (2.49), with $\sigma_n=u_n(0)\to\infty$ as $n\to\infty$.

We will also introduce a quantity measure we refer to as

$$\chi(r,\sigma)=(r^{2\alpha}\left|u_r\right|^{2(p-1)}+r^{2\beta}u^2)^{1/2} \tag{2.51}$$

We will use the property that

$$\chi(r,\sigma)\to\infty\ as\left|\sigma\right|\to\infty \tag{2.52}$$

uniformly for $r\in[0,1]$. It is clear from the definitions of (2.51) and the Energy function $E(r,\sigma)$ that $\chi(r,\sigma)\to\infty$ if and only if $E(r,\sigma)\to\infty$, hence we may

establish (2.52) by showing

$$E(r,\sigma) \to \infty \; as \, |\sigma| \to \infty \tag{2.53}$$

uniformly in $r \geq 0$.

Suppose that $0 \leq \rho \leq 1$ and $r = r_\rho(\sigma)$ are such that, for a finite σ, we have

$$u(r_\rho(\sigma)) = \rho\sigma, |u(r)| \geq |\sigma| \, \rho, \forall 0 \leq r \leq r_\rho(\sigma) \tag{2.54}$$

$$u(r_0(\sigma)) = 0, r_1(\sigma) = 0 \tag{2.55}$$

Definition 2.3.1:

Let $\Phi_p(x) = |x|^{p-2} x$, for $x \in \Re$, $p > 1$ and denote its inverse by $\Phi_{p'}(x)$, where $\frac{1}{p'} + \frac{1}{p} = 1$.

Then we have the following estimate for $r_\rho(\sigma)$.

Lemma 2.3.1. *Let $r_\rho(\sigma)$ be as defined above. Then for some positive constant $C = C(p,n,\alpha,\beta)$, $r_\rho(\sigma)$ satisfies*

$$|\sigma|^{-\frac{1}{\theta}} (1-\rho)^{\frac{p-1}{\beta+p-\alpha}} \leq C r_\rho(\sigma) \leq |\sigma|^{-\frac{1}{\theta}} \rho^{-\frac{q}{p+\beta-\alpha}} (1-\rho)^{\frac{p-1}{\beta+p-\alpha}} \tag{2.56}$$

Where $\theta = \frac{\beta+p-\alpha}{q+1-p}$ and $C = (\frac{p-1}{p+\beta-\alpha})^{\frac{p-1}{\beta+p-\alpha}} (\frac{1}{n+\beta})^{\frac{1}{\beta+p-\alpha}}$.

Proof:

Consider multiplying (2.9) by r^{n-1} and integrating with respect to r from 0 to r, we then have

$$|u_r|^{p-2} u_r = -\frac{1}{r^{n+\alpha-1}} \int_0^r s^{n+\beta-1} |u(s)|^{q-1} u ds \tag{2.57}$$

Note that $\Phi_p(u_r) = |u_r|^{p-2} u_r$ and $\Phi_{q+1}(u) = |u|^{q-1} u$, hence applying $\Phi_{p'}$ to both sides of equation (2.57) gives

$$u_r(r) = \Phi_{p'}(-\frac{1}{r^{n+\alpha-1}} \int_0^r s^{n+\beta-1} \Phi_{q+1}(u(s)) ds) \tag{2.58}$$

Since

$$\Phi_{p'}(-\frac{1}{r^{n+\alpha-1}}) = \left|-\frac{1}{r^{n+\alpha-1}}\right|^{p'-2}(-\frac{1}{r^{n+\alpha-1}}) = -(\frac{1}{r^{\frac{n+\alpha-1}{p-1}}}) \tag{2.59}$$

Then (2.58) gives

$$u_r(r) = -(\frac{1}{r^{\frac{n+\alpha-1}{p-1}}})\Phi_{p'}(\int_0^r s^{n+\beta-1}\Phi_{q+1}(u(s))ds) \tag{2.60}$$

Integrating with respect to r both sides of equation (2.60) from 0 to $r_\rho(\sigma)$ we have

$$\int_0^{r_\rho(\sigma)} u_r(r)dr = -\int_0^{r_\rho(\sigma)} \frac{1}{r^{\frac{n+\alpha-1}{p-1}}}\Phi_{p'}(\int_0^r s^{n+\beta-1}\Phi_{q+1}(u(s))ds)dr \tag{2.61}$$

Where

$$\int_0^{r_\rho(\sigma)} u_r(r)dr = u(r_\rho(\sigma)) - u(0) = \rho\sigma - \sigma = -\sigma(1-\rho) \tag{2.62}$$

Hence we have

$$\sigma(1-\rho) = \int_0^{r_\rho(\sigma)} \frac{1}{r^{\frac{n+\alpha-1}{p-1}}}\Phi_{p'}(\int_0^r s^{n+\beta-1}\Phi_{q+1}(u(s))ds)dr \tag{2.63}$$

Since $|u(r)| \geq |\sigma|\,\rho$ for all $0 \leq r \leq r_\rho(\sigma)$, then $|u(0)| \geq |u(r)| \geq |u(r_\rho(\sigma))|$ and therefore $|\sigma|\,\rho \leq |u(r)| \leq |\sigma|$ for $r \in [0, r_\rho(\sigma)]$. Hence we have

$$\int_0^{r_\rho(\sigma)} \frac{1}{r^{\frac{n+\alpha-1}{p-1}}}\Phi_{p'}(\int_0^r s^{n+\beta-1}(|\sigma|\,\rho)^q ds)dr \leq |\sigma|\,(1-\rho)$$

$$\leq \int_0^{r_\rho(\sigma)} \frac{1}{r^{\frac{n+\alpha-1}{p-1}}}\Phi_{p'}(\int_0^r s^{n+\beta-1}(|\sigma|^q)ds)dr \tag{2.64}$$

Where

$\Phi_{q+1}(|\sigma|\,\rho) = |\sigma\rho|^{q-1}\,|\sigma|\,\rho = (|\sigma|\,\rho)^q$, similarly $\Phi_{q+1}(|\sigma|) = |\sigma|^q$.

Evaluating the inside integrals of equation (2.64) gives

$$\int_0^{r_\rho(\sigma)} \frac{1}{r^{\frac{n+\alpha-1}{p-1}}}\Phi_{p'}[\frac{(|\sigma|\rho)^q}{n+\beta}s^{n+\beta}|_0^r]dr \leq |\sigma|\,(1-\rho)$$

$$\leq \int_0^{r_\rho(\sigma)} \frac{1}{r^{\frac{n+\alpha-1}{p-1}}}\Phi_{p'}[\frac{|\sigma|^q}{n+\beta}s^{n+\beta}|_0^r]dr \tag{2.65}$$

Which for $n+\beta>0$ gives

$$\int_0^{r_\rho(\sigma)} \frac{1}{r^{\frac{n+\alpha-1}{p-1}}}\Phi_{p'}[\frac{(|\sigma|\,\rho)^q}{n+\beta}r^{n+\beta}]dr \leq |\sigma|\,(1-\rho) \leq \int_0^{r_\rho(\sigma)} \frac{1}{r^{\frac{n+\alpha-1}{p-1}}}\Phi_{p'}[\frac{|\sigma|^q}{n+\beta}r^{n+\beta}]dr \tag{2.66}$$

Which in turn simplifies to

$$\int_0^{r_\rho(\sigma)} \frac{1}{r^{\frac{n+\alpha-1}{p-1}}}[\frac{(|\sigma|\,\rho)^q}{n+\beta}r^{n+\beta}]^{\frac{1}{p-1}}dr \leq |\sigma|\,(1-\rho) \leq \int_0^{r_\rho(\sigma)} \frac{1}{r^{\frac{n+\alpha-1}{p-1}}}[\frac{|\sigma|^q}{n+\beta}r^{n+\beta}]^{\frac{1}{p-1}}dr \tag{2.67}$$

Where

$$\Phi_{p'}[\frac{(|\sigma|\,\rho)^q}{n+\beta}r^{n+\beta}] = \left|\frac{(|\sigma|\,\rho)^q}{n+\beta}r^{n+\beta}]\right|^{\frac{p}{p-1}-2}(\frac{(|\sigma|\,\rho)^q}{n+\beta}r^{n+\beta}) = [\frac{(|\sigma|\,\rho)^q}{n+\beta}r^{n+\beta}]^{\frac{1}{p-1}} \tag{2.68}$$

And

$$\Phi_{p'}[\frac{|\sigma|^q}{n+\beta}r^{n+\beta}] = [\frac{|\sigma|^q}{n+\beta}r^{n+\beta}]^{\frac{1}{p-1}} \tag{2.69}$$

Hence we have

$$(\frac{(|\sigma|\,\rho)^q}{n+\beta})^{\frac{1}{p-1}}\int_0^{r_\rho(\sigma)} r^{\frac{(\beta-\alpha+1)}{(p-1)}}dr \leq |\sigma|\,(1-\rho) \leq (\frac{|\sigma|^q}{n+\beta})^{\frac{1}{p-1}}\int_0^{r_\rho(\sigma)} r^{\frac{\beta-\alpha+1}{p-1}}dr \tag{2.70}$$

For $p>\alpha-\beta, p>1$, we have

$$(\frac{p-1}{p+\beta-\alpha})(\frac{(|\sigma|\,\rho)^q}{n+\beta})^{\frac{1}{p-1}}(r_\rho(\sigma))^{\frac{(p+\beta-\alpha)}{p-1}} \leq |\sigma|\,(1-\rho) \leq (\frac{p-1}{p+\beta-\alpha})(\frac{|\sigma|^q}{n+\beta})^{\frac{1}{p-1}}(r_\rho(\sigma))^{\frac{(p+\beta-\alpha)}{p-1}} \tag{2.71}$$

Hence

$$|\sigma|^{-\frac{1}{\theta}}\,(1-\rho)^{\frac{(p-1)}{(\beta+p-\alpha)}} \leq Cr_\rho(\sigma) \leq |\sigma|^{-\frac{1}{\theta}}\,\rho^{-\frac{q}{(\beta+p-\alpha)}}(1-\rho)^{\frac{(p-1)}{(\beta+p-\alpha)}} \tag{2.72}$$

Where $C=(\frac{1}{n+\beta})^{\frac{1}{p+\beta-\alpha}}(\frac{p-1}{p+\beta-\alpha})^{\frac{p-1}{p+\beta-\alpha}}$ and $\theta=\frac{p+\beta-\alpha}{q+1-p}$.

It is evident now that as $\sigma\to\infty$, $|\sigma|^{-1/\theta}\to 0$ for $\theta>0$ or $|\sigma|^{-1/\theta}\to\infty$ for $\theta<0$. Hence $r_\rho(\sigma)\to\infty$ or 0 as $|\sigma|\to\infty$, for $\rho\in[0,1]$, depending on the sign of θ. We will prove that the sign plays an important role in the existence of solutions to the PDE in the phase plane setting in chapter 3. In particular, we prove that

solutions satisfying the associated boundary condition exist for $\theta > 0$. In section 2.4.2, theorem 2.4.1, we also prove that radial bounded solutions approach the origin with zero slope for $\beta - \alpha + 1 > 0$, which in turn implies that $\theta = \frac{p+\beta-\alpha}{q+1-p}$ is positive for $1 < p \leq 2$, $q > 1$, $\alpha, \beta \geq 0$.

Using Lemmas 2.2.2 and 2.3.1 we now prove (2.53)

Corollary 2.3.1. *Let $u(r)$ satisfy (2.9), (2.49), (2.50) and $\alpha, \beta \geq 0, 1 < p \leq 2$, $q+1 \neq p, \theta = \frac{p+\beta-\alpha}{q+1-p} < 0$, then*

$$E(r,\sigma) = \tfrac{p-1}{p} r^{\alpha} |u_r|^p + \tfrac{1}{q+1} r^{\beta} |u|^{q+1} \to \infty \text{ as } |\sigma| \to \infty$$

uniformly for $r \in [0,1]$

Proof:

By (2.26), for all $r \in [0,1]$

$$(n+\beta)\lambda \int_0^r s^{\beta+n-1} |u|^{q+1}\, ds = \frac{n+\beta}{p^*} r^{n+\alpha-1} |u_r|^{p-2} u_r u + r^n E(r,\sigma) \tag{2.73}$$

Therefore for $r \in [0,1]$ and $\alpha \geq 0$ then $|r^{n+\alpha-1}| \leq |r^{\alpha}|$, and we have

$$(n+\beta)\lambda \int_0^r s^{\beta+n-1} |u|^{q+1}\, ds \leq \frac{n+\beta}{p^*} r^{\alpha} |u_r|^{p-1} |u| + E(r,\sigma) \tag{2.74}$$

Using the arithmetic-geometric inequality $a^c b^d \leq ca + db$, $c+d = 1$, $a, b \geq 0$, taking $a = |u_r|^p$, $b = |u|^p$, $c = 1 - \frac{1}{p}$, and $d = \frac{1}{p}$ we have,

$$(n+\beta)\lambda \int_0^r s^{\beta+n-1} |u|^{q+1}\, ds \leq \frac{n+\beta}{p^*} r^{\alpha} [\frac{p-1}{p} |u_r|^p + \frac{1}{p} |u|^p] + E(r,\sigma) \tag{2.75}$$

Multiplying by $\frac{1}{n+\beta}$ and using the arithmetic-geometric inequality again taking $a = |u|^{q+1}$, $c = \frac{p}{q+1}$,$b = 1$, $d = 1 - \frac{p}{q+1}$ and for $\alpha \geq \beta, r \in [0,1]$ we have

$$\lambda \int_0^r s^{\beta+n-1} |u|^{q+1}\, ds \leq \frac{1}{p^*} [\frac{p-1}{p} r^{\alpha} |u_r|^p + \frac{1}{q+1} r^{\beta} |u|^{q+1} + \frac{1}{p} r^{\beta} (1 - \frac{p}{q+1})] + \frac{1}{n+\beta} E(r,\sigma) \tag{2.76}$$

Equation (2.76) implies that for any $r \in [0,1]$ and arbitrary $r_\rho(\sigma) \leq r$

$$\lambda \int_0^r s^{\beta+n-1} |u|^{q+1}\, ds \leq [\frac{n+\alpha}{(n+\beta)p}]E(r,\sigma) + C' \tag{2.77}$$

With $C' = \frac{1}{p^*}[\frac{1}{p} - \frac{1}{q+1}]$

By (2.54), then for all $r \geq r_\rho(\sigma)$, and $n+\beta > 0$ we have

$$\frac{\lambda}{n+\beta}(r_\rho(\sigma))^{n+\beta}(|\sigma|\,\rho)^{q+1} \leq [\frac{n+\alpha}{(n+\beta)p}]E(r,\sigma) + C' \tag{2.78}$$

Using the inequality (2.56), for all $r \geq r_\rho(\sigma)$

$$\frac{\lambda}{n+\beta} |\sigma|^{q+1} |\rho|^{q+1} (1-\rho)^{\frac{(\beta+n)(p-1)}{\beta+p-\alpha}} (|\sigma|^{-\frac{1}{\theta}})^{\beta+n} \leq C([\frac{n+\alpha}{(n+\beta)p}]E(r,\sigma) + C') \tag{2.79}$$

It is clear that for $\theta < 0, q+1 > 0, \beta+n > 0$, the energy function of solutions $E(r,\sigma) \to \infty$ as $|\sigma| \to \infty$ for any $r_\rho(\sigma) \leq r$ since $\rho \in [0,1]$.

Integrating equation (2.60) on the interval (0,r) leads to defining the mapping T given by

$$Tu(r) = \sigma - \int_0^r \frac{1}{t^{\frac{n+\alpha-1}{p-1}}} \Phi_{p'}(\int_0^r s^{n+\beta-1}\Phi_{q+1}(u)ds)dt \tag{2.80}$$

The fixed point $u(r,\sigma)$ of T is a continuous function of the initial data σ and r since the conditions $1 < p \leq 2 < n, \beta \geq 0$ and $p < q+1 < p^*$ guarantee that the functional $\Phi_{p'}(\int_0^r s^{n+\beta-1}\Phi_{q+1}(u)ds)$ is locally lipschitz (section 2.4). This and equation (2.60) imply that $u_r(r,\sigma)$ is also a continuous function of σ and r.

Let $\chi = (r^{2\alpha} |u_r|^{2(p-1)} + r^{2\beta}u^2)^{1/2}$. We define $\Theta(r,\sigma)$ such that

$$r^\beta u(r,\sigma) = \chi cos\Theta(r,\sigma) \tag{2.81}$$

And

$$-r^\alpha |u_r(r,\sigma)|^{p-2} u_r(r,\sigma) = \chi sin\Theta(r,\sigma) \tag{2.82}$$

Lemma 2.3.2. $\Theta(r,\sigma)$ *is a uniformly continuous function of* σ *for* $0 \leq r \leq 1$.

Proof:

A natural consequence of the uniform continuity of $u(r,\sigma)$ and $u_r(r,\sigma)$ on $r \in [0,1]$.

Lemma 2.3.3. *$\Theta(r,\sigma) \geq 0$ for all $r \in [0,1]$ and $\sigma > 0$*

Proof:

Suppose that there exists an $r \in (0,1]$ such that $\Theta(r,\sigma) < 0$ (note that $\Theta(0,\sigma) = \frac{\pi}{2}$). Since $u(r,\sigma)$ is continuous in r and $u(0,\sigma) = \sigma > 0$ then by (2.57) there exists $\epsilon > 0$ such that $\Theta(r,\sigma) > 0$ for $0 < r \leq \epsilon$. Then for a fixed $\sigma > 0$, by the continuity of $\Theta(r,\sigma)$ and the Mean Value Theorem, there exists $r^* > \epsilon > 0$ satisfying $\Theta(r^*,\sigma) = 0$ and thus some $\delta > 0$ such that

i) in $(r^* - \delta, r^*)$, $u(r,\sigma) > 0$ and $u_r(r,\sigma) < 0$.

ii) in $(r^*, r^* + \delta)$, $u(r,\sigma) > 0$ and $u_r(r,\sigma) > 0$.

iii) $u(r^*,\sigma) > 0$, $u_r(r^*,\sigma) = 0$

If $u(r) > 0$ for all $r \in (0, r^*)$, then let $\hat{r} = 0$. If $u(r,\sigma) \leq 0$ for some $r \in (0, r^*)$, then i) implies that u has a local maximum at some $s \in (0, r^*)$, where $u_r(s,\sigma) = 0$, and we let $\hat{r}$ be the largest of possible values of s. Therefore there exists an $\hat{r}$ in $(0, r^*)$ such that $u_r(\hat{r},\sigma) = 0$ and $u(r,\sigma) > 0$ in $[\hat{r}, r^*]$.

Using (2.57) we have

$$\Phi_p(u_r(r^*,\sigma)) = -\frac{1}{r^{*(n+\alpha-1)}} \int_0^{r^*} s^{n+\beta-1} \Phi_{q+1}(u(s))ds \tag{2.83}$$

$$= -\frac{1}{r^{*(n+\alpha-1)}} \int_0^{\hat{r}} s^{n+\beta-1} \Phi_{q+1}(u)ds - \frac{1}{r^{*(n+\alpha-1)}} \int_{\hat{r}}^{r*} s^{n+\beta-1} \Phi_{q+1}(u)ds \tag{2.84}$$

And so

$$\Phi_p(u_r(r^*,\sigma)) = \frac{\hat{r}^{n+\alpha-1}}{r^{*(n+\alpha-1)}} \Phi_p(u_r(\hat{r},\sigma)) - \frac{1}{r^{*(n+\alpha-1)}} \int_{\hat{r}}^{r*} s^{n+\beta-1} \Phi_{q+1}(u)ds \tag{2.85}$$

Since both $u_r(\hat{r},\sigma)$ and $u_r(r^*,\sigma) = 0$ we have,

$$0 = -\frac{1}{r^{*(n+\alpha-1)}} \int_{\hat{r}}^{r^*} s^{n+\beta-1} \Phi_{q+1}(u)ds \tag{2.86}$$

Which implies $u(r) = 0$. Since $u(r) > 0$ by assumption on $[\hat{r}, r^*]$, this leads to $\Theta(r, \sigma) \geq 0$ by contradiction.

By differentiating $\Theta(r, \sigma)$ with respect to r and using (2.9) we obtain:

$$\Theta_r(r, \sigma) = \frac{(n + \beta - 1)r^{\alpha+\beta-1} |u_r|^{p-2} uu_r + r^{2\beta} |u|^{q+1} + r^{\alpha+\beta} |u_r|^p}{\chi^2} \tag{2.87}$$

Theorem 2.3.1. $\Theta(1, \sigma) \to \infty$ *as* $\sigma \to \infty$

Proof:

Write $[s_0, 1]$ as $U_{k=1}^m [r_k, r_{k+1}]$, where m is a positive integer, $0 < s_0 < 1$, $r_k < r_{k+1}$, $s_0 = r_1, r_{m+1} = 1$ and the set $\{r_k\}_{k=1}^m$ contains all the zeros of u in $[s_0, 1]$. Since as a function of r, Θ is continuous in $[r_k, r_{k+1}]$ and differentiable in (r_k, r_{k+1}).

We have,

$$\Theta(1, \sigma) - \Theta(s_0, \sigma) = \sum_{k=1}^{m} (\Theta(r_{k+1}, \sigma) - \Theta(r_k, \sigma)) \tag{2.88}$$

$$= lim_{\delta \to 0} \sum_{k=1}^{m} (\Theta(r_{k+1} - \delta, \sigma) - \Theta(r_k + \delta, \sigma)) \tag{2.89}$$

$$= lim_{\delta \to 0} \sum_{k=1}^{m} \int_{r_k+\delta}^{r_{k+1}-\delta} \frac{(n + \beta - 1)r^{\alpha+\beta-1} |u_r|^{p-2} uu_r + r^{2\beta} |u|^{q+1} + r^{\alpha+\beta} |u_r|^p}{\chi^2} dr \tag{2.90}$$

$$= \int_{s_0}^{1} \frac{(n + \beta - 1)r^{\alpha+\beta-1} |u_r|^{p-2} uu_r + r^{2\beta} |u|^{q+1} + r^{\alpha+\beta} |u_r|^p}{\chi^2} dr \tag{2.91}$$

$$\geq \int_{s_0}^{1} \frac{r^{\alpha+\beta} |u_r|^p + r^{2\beta} |u|^{q+1} - [\frac{n+\beta-1}{2s_0}][r^{2\alpha} |u_r|^{2(p-1)} + r^{2\beta} u^2]}{\chi^2} dr \tag{2.92}$$

$$= \int_{s_0}^{1} \frac{r^{\alpha+\beta} |u_r|^p + r^{2\beta} |u|^{q+1}}{\chi^2} dr - [\frac{n + \beta - 1}{2}](\frac{1}{s_0} - 1) \tag{2.93}$$

which can be rewritten as

$$\Theta(1, \sigma) + \frac{n + \beta - 1}{2}(\frac{1}{s_0} - 1) \geq \int_{s_0}^{1} \frac{r^{\alpha+\beta} |u_r|^p + r^{2\beta} |u|^{q+1}}{\chi^2} dr \tag{2.94}$$

The integrand in (2.94) equals

$$I = \frac{r^{\alpha+\beta} |u_r|^p + r^{2\beta} |u|^{q+1}}{r^{2\alpha} |u_r|^{2(p-1)} + r^{2\beta} u^2} \tag{2.95}$$

by definition of χ^2.

Since $p > 2(p-1)$ and $|u_r(r)|, |u(r)| \to \infty$ if and only if $E(r,\sigma) \to \infty$ we conclude from corollary 2.3.1 and lemma 2.3.2 the integrand above approaches infinity uniformly on $[s_0, 1]$ as $\sigma \to \infty$ for $q > 1$ and $\alpha, \beta \geq 0$.

Therefore

$$\Theta(1,\sigma) \to \infty \quad as \quad \sigma \to \infty. \tag{2.96}$$

By the continuity of $\Theta(1,\sigma)$ in σ and by (2.96) there exists a sequence of pairs $\{\sigma_k, k\}$ such that $\Theta(1,\sigma_k) = \frac{(2k-1)\pi}{2}$ as a result we have this theorem.

Remark: $\Theta(1,\sigma) \to \infty$ as $\sigma \to -\infty$ can be shown in the same manner.

Theorem 2.3.2. *Let $\alpha, \beta \geq 0, 1 < p \leq 2, p < q+1 < p^* = \frac{p(n+\beta)}{\alpha+n-p}$. Then (2.9),(2.49) has infinitely many radially symmetric solutions lying in*

$W^{1,p}(B^n, |x|^\alpha) \cap L^{q+1}(B^n, |x|^\beta)$.

2.4 Existence of a radial solution to the initial value problem.

In this section we prove the existence of a fixed point of $T(u(r))$ of (2.80), for the initial value problem (2.10), (2.50), and show that this fixed point is continuously dependent on the initial data uniformly on $[0, 1]$.

Let $R > 0$, σ be a real number and $B_R^\epsilon(\sigma) = \{u : u \in C[0, \epsilon], \|u - \sigma\|_\infty \leq R\}$ where $\epsilon < \sqrt[p+\beta-\alpha]{\frac{(n+\beta)(p+\beta-\alpha)^{p-1}}{R^{q+1-p}}} \sqrt[p+\beta-\alpha]{\frac{1}{q^{p-1}}}$, where for $\beta - \alpha + 1 > 0$ we have $u_r(0) = 0$. It is clear that $B_R^\epsilon(\sigma)$ is a closed subset of $C[0, \epsilon]$. We show that T leaves $B_R^\epsilon(\sigma)$ invariant and is a contraction on $B_R^\epsilon(\sigma)$ with respect to the sup-norm. Suppose that $u \in B_R^\epsilon(\sigma)$ then by (2.80)

$$|Tu(r) - \sigma| \leq \left| \int_0^r \frac{1}{t^{\frac{n+\alpha-1}{p-1}}} \Phi_{p'}(\int_0^t s^{n+\beta-1} \Phi_{q+1} u(s) ds) dt \right| \tag{2.97}$$

$$\leq \frac{1}{(n+\beta)^{\frac{1}{p-1}}} (\frac{p-1}{p+\beta-\alpha}) r^{\frac{\beta+p-\alpha}{p-1}} (\|u\|_\infty)^{\frac{q}{p-1}} \tag{2.98}$$

$$\leq \frac{1}{(n+\beta)^{\frac{1}{p-1}}} (\frac{p-1}{p+\beta-\alpha}) \epsilon^{\frac{\beta+p-\alpha}{p-1}} R^{\frac{q}{p-1}} \leq R \tag{2.99}$$

This inequality holds if and only if $\epsilon \leq \sqrt[p+\beta-\alpha]{\frac{(n+\beta)(p+\beta-\alpha)^{p-1}}{R^{q+1-p}}} \sqrt[p+\beta-\alpha]{\frac{1}{(p-1)^{p-1}}}$ which follows since $\sqrt[p+\beta-\alpha]{\frac{1}{(p-1)^{p-1}}} > \sqrt[p+\beta-\alpha]{\frac{1}{q^{p-1}}}$ and $\epsilon < \sqrt[p+\beta-\alpha]{\frac{(n+\beta)(p+\beta-\alpha)^{p-1}}{R^{q+1-p}}} \sqrt[p+\beta-\alpha]{\frac{1}{q^{p-1}}}$ therefore T leaves $B_R^\epsilon(\sigma)$ invariant.

Let $u, v \in B_R^\epsilon(\sigma)$, then

$$Tu(r) - Tv(r) = \int_0^r \frac{1}{t^{\frac{n+\alpha-1}{p-1}}} [\Phi_{p'}(\int_0^t s^{n+\beta-1} \Phi_{q+1}(u(s)) ds) - \Phi_{p'}(\int_0^t s^{n+\beta-1} \Phi_{q+1}(v(s)) ds)] dt \tag{2.100}$$

And so

$$|Tu(r) - Tv(r)| \leq \left| \int_0^r \frac{1}{t^{\frac{n+\alpha-1}{p-1}}} (\chi[u](t) - \chi[v](t)) dt \right| \tag{2.101}$$

Where $\chi[u](t) = \Phi_{p'}(\int_0^t s^{n+\beta-1} \Phi_{q+1}(u(s)) ds)$ and $\chi[v](t)$ similarly.

Let $G(\lambda; t) = \chi[\lambda u + (1 - \lambda) v](t)$ then $\chi[u](t) - \chi[v](t) = G(1; t) - G(0; t)$

Now

$$G(\lambda;t)=\Phi_{p'}(\int_0^t s^{n+\beta-1}\Phi_{q+1}(\lambda u(s)+(1-\lambda)v(s))ds) \tag{2.102}$$

since $\frac{d}{dz}\Phi_{p'}(z)=(p'-1)\left|z\right|^{p'-2}$ and $p^{'}=\frac{p}{p-1}$ then

$G_\lambda(\lambda;t)=\frac{1}{p-1}\left|\int_0^t s^{n+\beta-1}\Phi_{q+1}(\lambda u(s)+(1-\lambda)v(s))ds\right|^{\frac{2-p}{p-1}}$

$$\cdot\int_0^t s^{n+\beta-1}q\left|\lambda u(s)+(1-\lambda)v(s)\right|^{q-1}(u(s)-v(s))ds \tag{2.103}$$

By the mean value theorem

$$\chi[u](t)-\chi[v](t)=G_\lambda(\lambda;t)\quad for\ some\quad 0<\lambda<1 \tag{2.104}$$

Therefore by (2.101), (2.103), (2.104) we have

$\left|Tu(r)-Tv(r)\right|\le\frac{q}{p-1}\int_0^r\frac{1}{t^{\frac{n+\alpha-1}{p-1}}}[\left|\int_0^t s^{n+\beta-1}\Phi_{q+1}(\lambda u+(1-\lambda)v)ds\right|^{\frac{2-p}{p-1}}$

$$\cdot\int_0^t s^{n+\beta-1}\left|\lambda u(s)+(1-\lambda)v(s)\right|^{q-1}\left|u(s)-v(s)\right|ds]dt \tag{2.105}$$

Therefore

$$\left|Tu(r)-Tv(r)\right|\le\frac{q}{p+\beta-\alpha}\frac{1}{(n+\beta)^{\frac{1}{p-1}}}R^{\frac{q+1-p}{p-1}}r^{\frac{p+\beta-\alpha}{p-1}}\left\|u-v\right\|_\infty \tag{2.106}$$

For $r\in[0,\epsilon]$. Hence,

$$\left\|Tu(r)-Tv(r)\right\|_\infty\le C\left\|u-v\right\|_\infty \tag{2.107}$$

Where $C=\frac{q}{p+\beta-\alpha}\frac{1}{(n+\beta)^{\frac{1}{p-1}}}R^{\frac{q+1-p}{p-1}}\epsilon^{\frac{p+\beta-\alpha}{p-1}}<1$ since $\epsilon<\sqrt[p+\beta-\alpha]{\frac{(n+\beta)(p+\beta-\alpha)^{p-1}}{R^{q+1-p}}}\sqrt[p+\beta-\alpha]{\frac{1}{q^{p-1}}}$.

So T is a contraction on $B_R^\epsilon(\sigma)$ and has a unique fixed point in $B_R^\epsilon(\sigma)$. It can be verified that a fixed point of (2.80) is a solution of (2.9), (2.49). Using the monotonic property of the energy inequality, $E_{1r}(r,\alpha)\le 0$, in lemma 2.2.1 we observe that this solution is uniformly bounded on $[0,r]$ for any r and therefore can be extended to $[0,1]$.

Suppose that $u(r)$, $v(r)$ satisfy (2.9), then

$$u(r) - v(r) =$$

$$u(0)-v(0)+\int_0^r \frac{1}{t^{\frac{n+\alpha-1}{p-1}}}[\Phi_{p'}(\int_0^t s^{n+\beta-1}\Phi_{q+1}(u(s))ds)-\Phi_{p'}(\int_0^t s^{n+\beta-1}\Phi_{q+1}v(s)ds)]dt \tag{2.108}$$

And so

$$|u(r) - v(r)| \leq |u(0) - v(0)| + \int_0^r \frac{1}{t^{\frac{n+\alpha-1}{p-1}}} |\chi[u](t) - \chi[v](t)| \, dt \tag{2.109}$$

Hence

$$|u(r) - v(r)| \leq |u(0) - v(0)| + \tfrac{q}{p-1} \int_0^r \tfrac{1}{t^{\frac{n+\alpha-1}{p-1}}}[\left|\int_0^t s^{n+\beta-1}\Phi_{q+1}(\lambda u(s) + (1-\lambda)v(s))ds\right|^{\frac{2-p}{p-1}}$$

$$\cdot \int_0^t s^{n+\beta-1} |\lambda u(s) + (1-\lambda)v(s)|^{q-1} |u(s) - v(s)| \, ds]dt \tag{2.110}$$

Using Lemma 2.2.1, we may bound $u(s)$ and $v(s)$ uniformly on $[0, 1]$ by a constant C and so by (2.110) we have

$$|u(r) - v(r)|$$

$$\leq |u(0) - v(0)| + C\int_0^r \frac{1}{t^{\frac{n+\alpha-1}{p-1}}}[\left|\int_0^t s^{n+\beta-1}ds\right|^{\frac{2-p}{p-1}} \cdot \int_0^t s^{n+\beta-1} |u(s) - v(s)| \, ds]dt \tag{2.111}$$

$$\leq |u(0) - v(0)| + C\int_0^r t^{-n-\frac{(\alpha-1)-\beta(p-2)}{p-1}}[\int_0^t s^{n+\beta-1} |u(s) - v(s)| \, ds]dt \tag{2.112}$$

$$\leq |u(0) - v(0)| + C\int_0^r t^{\frac{1+\beta-\alpha}{p-1}} \int_0^t |u(s) - v(s)| \, dsdt \tag{2.113}$$

$$\leq |u(0) - v(0)| + Cr^{\frac{p+\beta-\alpha}{p-1}} \int_0^r |u(t) - v(t)| \, dt \tag{2.114}$$

Where C is a generic constant. By Gronwall's inequality and (2.114) we have

$$|u(r) - v(r)| \leq C\, |u(0) - v(0)| \tag{2.115}$$

for all $r \in [0, 1]$.

2.4.1 Solutions to the Generalized Lane-Emden equation are bounded

In this section we prove that individual, general and radial, solutions of (1.1) are bounded below for $\alpha, \beta \geq 0, 1 < p \leq 2, p < q+1 < p^*$.

Lemma 2.4.1. *General solutions for the Generalized Lane-Emden equation (1.1) are bounded below by a constant* $C = C(p, n, \alpha, \beta)$ *for* $\alpha, \beta \geq 0, 1 < p \leq 2$, $p < q+1 < p^*$.

Proof:

Consider (2.8)

$$\textstyle\int_{B^n}(|x|^\alpha |\nabla u|^p)dx = \int_{B^n}(|x|^\beta |u|^{q+1})dx \qquad (2.8)$$

When both sides of equation (2.8) are raised to the power $\frac{1}{q+1}$ we have

$$(\int_{B^n}(|x|^\alpha |\nabla u|^p)dx)^{\frac{1}{q+1}} = \|u\|_{L^{q+1}(B^n,|x|^\beta)} \tag{2.116}$$

The weighted Sobolev inequality $\|u\|_{L^{q+1}(B^n,|x|^\beta)} \leq C\,\|\nabla u\|_{L^p(B^n,|x|^\alpha)}$ for $p < q+1 < p^*$, $p^* = \frac{p(n+\beta)}{n+\alpha-p}, \alpha, \beta \geq 0, 1 < p \leq 2 < n$, where C is a constant dependent on p, n, α and β, is in fact the inequality

$$(\int_{B^n} |x|^\beta |u|^{q+1}\, dx) \leq C(\int_{B^n} |x|^\alpha |\nabla u|^p\, dx)^{\frac{q+1}{p}} \tag{2.117}$$

Putting the equations (2.8) and (2.117) together we have,

$$(\int_{B^n} |x|^\alpha |\nabla u|^p\, dx) \leq C(\int_{B^n} |x|^\alpha |\nabla u|^p\, dx)^{\frac{q+1}{p}} \tag{2.118}$$

Therefore for u not equal to zero

$$(\int_{B^n} |x|^\alpha |\nabla u|^p)dx)^{1-(\frac{q+1}{p})} \leq C. \tag{2.119}$$

And

$$(\int_{B^n} |x|^\beta |u|^{q+1}\, dx)^{1-(\frac{q+1}{p})} \leq C \tag{2.120}$$

The exponent simplifies to $-(\frac{q+1-p}{p})$. We observe that for $q+1>p$ the exponent is negative and hence the class of all non trivial solutions are bounded below by a constant.

We can also prove in a similar manner the same property for radial solutions of (2.9) for $\alpha, \beta \geq 0, 1 < p \leq 2, p < q+1 < p^*$.

Corollary 2.4.1. *Radial solutions for the Generalized Lane-Emden equation (2.9) are bounded below for $\alpha, \beta \geq 0, 1 < p \leq 2, p < q+1 < p^*$ by a constant $C(n, p, \alpha, \beta)$.*

Proof:

The weighted radial norm is defined as

$$\|u\|_{L^{q+1}(B^n, |x|^\beta)} = (\int_0^1 r^{\beta+n-1} |u|^{q+1} \, dr)^{\frac{1}{q+1}} \tag{2.121}$$

Where $u = u(r)$, and

$$\|\nabla u\|_{L^p(B^n, |x|^\alpha)} = (\int_0^1 r^{\alpha+n-1} |u_r|^p \, dr)^{\frac{1}{p}} \tag{2.122}$$

The weighted Sobolev inequality (2.117) in radial form for $p < q+1 < p^*$ and C, a constant dependent on p, n, α and β, is in fact the inequality

$$(\int_0^1 r^{\beta+n-1} |u|^{q+1} \, dr)^{\frac{1}{q+1}} \leq C(\int_0^1 r^{\alpha+n-1} |u_r|^p \, dr)^{\frac{1}{p}} \tag{2.123}$$

Applying the same analysis used for the proof of lemma 2.4.1, we Observe that the class of all nontrivial radial solutions in $L^{q+1}(B^n, |x|^\beta)$ is bounded below by a constant dependent on p, n, α and β.

2.4.2 Decay estimates for radial solutions

Consider equation (2.9)

$$(r^{n+\alpha-1}\left|u_r\right|^{p-2}u_r)_r = -r^{\beta+n-1}\left|u\right|^{q-1}u \tag{2.9}$$

For simplicity we use the convention $\left|a\right|^{\gamma-1}a = a^{\gamma}$, we then have

$$(r^{n+\alpha-1}u_r^{p-1})_r = -r^{n+\beta-1}u^q \tag{2.124}$$

Integrating both sides of equation (2.124) from ϵ to r with respect to r gives

$$r^{n+\alpha-1}u_r^{p-1}(r) - \epsilon^{n+\alpha-1}u_r^{p-1}(\epsilon) = -\int_{\epsilon}^{r} t^{n+\beta-1}u^q(t)dt \tag{2.125}$$

We are interested in bounded radial solutions in $C^1[0,1]$, in particular, bounded at the origin with $u_r(0) = 0$. Letting $lim_{\epsilon\to 0}\epsilon^{n+\alpha-1}u_r^{p-1}(\epsilon) = 0$ as $\epsilon \to 0$, we then have

$$r^{n+\alpha-1}u_r^{p-1}(r) = -\int_0^r t^{n+\beta-1}u^q(t)dt \tag{2.126}$$

Adding $\int_0^r t^{n+\beta-1}u^q(0)dt$ and applying the absolute value to both sides of equation (2.126) gives

$$\left|r^{n+\alpha-1}u_r^{p-1}(r) + \int_0^r t^{n+\beta-1}u^q(0)dt\right| \leq \left|\int_0^r t^{n+\beta-1}(u^q(t) - u^q(0))dt\right| \tag{2.127}$$

Which for $n+\beta > 0$ and near zero gives

$$\left|r^{n+\alpha-1}u_r^{p-1}(r) + \frac{\sigma^q}{n+\beta}r^{n+\beta}\right| \leq Cr^{n+\beta} \tag{2.128}$$

Therefore

$$\left|u_r^{p-1}(r) + \frac{\sigma^q}{n+\beta}r^{\beta-\alpha+1}\right| \leq Cr^{\beta-\alpha+1} \tag{2.129}$$

The condition $u_r(0) = 0$ requires $u_r^{p-1}(r)$ to decay at least as fast as $\frac{\sigma^q}{n+\beta}r^{\beta-\alpha+1}$ and $\beta-\alpha+1 > 0$, hence

$$\left|u_r(r)\right| \leq O(r^{\frac{\beta-\alpha+1}{p-1}}) \tag{2.130}$$

Theorem 2.4.1. *For $\beta - \alpha + 1 > 0, 1 < p \leq 2, q > 1$, radial solutions of (2.9) satisfying the boundary condition in (2.49) approach the origin with zero slope such that $u_r(0) = 0$.*

2.5 Some properties of radial solutions.

Consider multiplying both sides of equation (2.9) by $\frac{1}{r^\beta}u_r$ and integrating from r_1 to r_2, for $0 \leq r_1 < r_2 \leq 1$. We then have,

$$\int_{r_1}^{r_2} \frac{u_r}{r^{n+\beta-1}}(r^{n+\alpha-1}\left|u_r\right|^{p-2}u_r)_r dr = -\int_{r_1}^{r_2}\left|u\right|^{q-1}uu_r dr \tag{2.131}$$

Integrating by parts on the left hand side of equation (2.131) we have

$$r^{\alpha-\beta}\left|u_r(r_2)\right|^p - r^{\alpha-\beta}\left|u_r(r_1)\right|^p - \textstyle\int_{r_1}^{r_2} r^{\alpha-\beta}\left|u_r\right|^{p-2}u_r u_{rr}dr + \int_{r_1}^{r_2}(n+\beta-1)r^{\alpha-\beta-1}\left|u_r\right|^p dr$$

$$= -\int_{u(r_1)}^{u(r_2)}\left|\omega\right|^{q-1}\omega d\omega \tag{2.132}$$

Expanding equation (2.9), multiplying by $\frac{u_r}{p-1}$ and $r^{-\beta-n+1}$ we have

$$r^{\alpha-\beta}\left|u_r\right|^{p-2}u_r u_{rr} = -\frac{n+\alpha-1}{p-1}r^{\alpha-\beta-1}\left|u_r\right|^p - \frac{1}{p-1}\left|u\right|^{q-1}uu_r \tag{2.133}$$

Substituting (2.133) in (2.132) and collecting like terms we have,

$$r^{\alpha-\beta}\left|u_r(r_2)\right|^p - r^{\alpha-\beta}\left|u_r(r_1)\right|^p + \tfrac{p(n+\beta-1)+(\alpha-\beta)}{p-1}\textstyle\int_{r_1}^{r_2} r^{\alpha-\beta-1}\left|u_r\right|^p dr$$

$$= -\frac{p}{p-1}\int_{u(r_1)}^{u(r_2)}\left|\omega\right|^{q-1}\omega d\omega \tag{2.134}$$

Where $n-1+\frac{\alpha}{p}+\frac{(p-1)\beta}{p} > 0$ for $\alpha, \beta \geq 0, 1 < p \leq 2, q > 1$. Equation (2.134) can be rewritten as follows

$$\tfrac{p-1}{p}r^{\alpha-\beta}[\left|u_r(r_2)\right|^p - \left|u_r(r_1)\right|^p] + (n+\beta-1+\tfrac{\alpha-\beta}{p})\textstyle\int_{r_1}^{r_2} r^{\alpha-\beta-1}\left|u_r\right|^p dr$$

$$+\frac{1}{q+1}[\left|u(r_2)\right|^{q+1} - \left|u(r_1)\right|^{q+1}] = 0 \tag{2.135}$$

From (2.135) we obtain the following theorem.

Theorem 2.5.1. *Let $u(r)$ be a solution of (2.9), (2.49), suppose that*

$0 < r_1 < r_2 \leq 1$.

i) If $u(r_1) = u(r_2) = 0$, then $|u_r(r_1)| \geq |u_r(r_2)|$.

ii) If $u_r(r_1) = u_r(r_2) = 0$, then $|u(r_1)| \geq |u(r_2)|$.

Property ii) leads to the final property,

iii) $max_{0 \leq r \leq 1} |u(r)| = |u(0)| = \sigma$.

Hence, the maximum amplitude of solutions occurs at the center of the ball.

2.5.1 Maximum amplitude of radial solutions

Theorem 2.5.2. *Let $u(r)$ be a nontrivial solution of (2.9), (2.49), then*

$max_{0\leq r\leq 1}|u(r)| \geq [(n+\beta)(\frac{p+\beta-\alpha}{p-1})^{p-1}]^{\frac{1}{q+1-p}}$.

That is the maximum amplitude of any solution is bounded below uniformly by $[(n+\beta)(\frac{p+\beta-\alpha}{p-1})^{p-1}]^{\frac{1}{q+1-p}}$. Consequently by property iii), if $|u(0)| < [(n+\beta)(\frac{p+\beta-\alpha}{p-1})^{p-1}]^{\frac{1}{q+1-p}}$ then $u(r)$ can not be a solution of (2.9), (2.49).

Proof:

Using the first inequality in (2.56) for $r_\rho(\sigma)$ and letting $\rho = 0$ we have

$$|\sigma|^{-\frac{q+1-p}{\beta+p-\alpha}} \leq (\frac{p-1}{p+\beta-\alpha})^{\frac{p-1}{\beta+p-\alpha}}(\frac{1}{n+\beta})^{\frac{1}{p+\beta-\alpha}} r_0(\sigma) \tag{2.136}$$

Hence,

$$|\sigma| \geq [(\frac{1}{r_0(\sigma)})^{\beta+p-\alpha}(\frac{p+\beta-\alpha}{p-1})^{p-1}(\frac{1}{n+\beta})]^{\frac{1}{q+1-p}} \tag{2.137}$$

Recall that $u(0) = \sigma$, $r_0(\sigma) = 1$ if u has no zero in $[0,1]$, and $r_0(\sigma) \leq 1$ in general, then by property iii) of theorem 2.5.1 and equation (2.137) we have

$$max_{0\leq r\leq 1}|u(r)| \geq [(\frac{p+\beta-\alpha}{p-1})^{p-1}(\frac{1}{n+\beta})]^{\frac{1}{q+1-p}} \tag{2.138}$$

It is clear that the closer $r_0(\sigma)$ to $r = 0$ the larger the amplitude of the radial solution at the origin.

2.5.2 Existence of a finite number of zeros for a solution

Theorem 2.5.3. *Let $u(r)$ be a nontrivial bounded solution of (2.9), (2.49), then u has only finitely many zeros in $[0,1]$ and the total number of zeros of u in $[0,1]$ is no more than the greatest integer less than or equal to $1+\frac{1}{2}(\frac{p-1}{p})^{\frac{p-1}{p}}(|u(0)|)^{\frac{q+1-p}{p}}$.*

Proof:

Suppose that the number of zeros of u in $[0,1]$ is infinite. Then we can choose a sequence $\{r_k\}$ of such zeros which converges to some $r_0 \in [0,1]$. Since $u \in C^1[r_k, r_{k+1}]$, by Rolle's Theorem there exists $\eta_k \in (r_k, r_{k+1})$ satisfying $u_r(\eta_k) = 0$ and $\eta_k \to r_0$ as $k \to \infty$. Therefore $u(r_0) = u_r(r_0) = 0$ for some $r_0 \in [0,1]$, and we have

$$u(r) = -\int_{r_0}^{r} \Phi_{p'}(\frac{1}{t^{n+\alpha-1}}\int_{r_0}^{t} s^{n+\beta-1}\Phi_{q+1}(u(s))ds)dt \quad for \quad r \in [0,1] \tag{2.139}$$

Let $\delta > 0$. Then by using equation (2.139) we have,

$$|u(r)| \leq \left|\int_{r_0}^{r} \Phi_{p'}(\frac{1}{t^{n+\alpha-1}}\int_{r_0}^{t} s^{n+\beta-1}\Phi_{q+1}(u(s))ds)dt\right| \tag{2.140}$$

$$\leq \left|\int_{r_0}^{r} \Phi_{p'}(\frac{t^{\beta-\alpha+1}}{n+\beta})dt\right| Sup_{\xi\in B_\delta(r_0)}|u(\xi)|^{\frac{q}{p-1}} \tag{2.141}$$

$$\leq \frac{1}{(n+\beta)^{p'-1}}\left|\int_{r_0}^{r} t^{\frac{\beta-\alpha+1}{p-1}}dt\right| Sup_{\xi\in B_\delta(r_0)}|u(\xi)|^{\frac{q}{p-1}} \tag{2.142}$$

And therefore

$$|u(r)| \quad \leq \frac{p-1}{p+\beta-\alpha}\frac{1}{(n+\beta)^{p'-1}}\delta^{\frac{p+\beta-\alpha}{p-1}} Sup_{\xi\in B_\delta(r_0)}|u(\xi)|^{\frac{q}{p-1}} \tag{2.143}$$

For $r \in B_\delta(r_0)$, where $B_\delta(r_0) = (r_0-\delta, r_0+\delta)\cap[0,1]$ and $p' = \frac{p}{p-1}$.

Since $p' > 0$ and $q > p-1$ using the equation (2.143) and letting $\delta > 0$ be sufficiently small, then for a non trivial solution where $\|u\|_{\infty,B_\delta(r_0)} \to \infty$ as $\delta \to 0$ we have

$$\|u\|_{\infty,B_\delta(r_0)}^{1-\frac{q}{p-1}} \leq \frac{p-1}{p+\beta-\alpha}\frac{\delta^{\frac{p+\beta-\alpha}{p-1}}}{(n+\beta)^{p'-1}} \to 0 \quad as \quad \delta \to 0 \tag{2.144}$$

This contradicts the assumption that u has infinitely many zeros converging to r_0. Therefore u has finitely many zeros in [0,1].

Assume now that $r_1 < r_2 < .... < r_k < r_{k+1} < ....r_m$ are the zeros of u in $[0,1]$. Let $\eta_k \in (r_k, r_{k+1})$ be such that $u_r(\eta_k) = 0$ and $u(\eta_k) = max_{r_k \le r \le r_{k+1}} |u(r)|$. We then have,

$$u(r) = u(\eta_k) - \int_{\eta_k}^{r} \Phi_{p'}(\frac{1}{t^{n+\alpha-1}} \int_{\eta_k}^{t} s^{n+\beta-1}\Phi_{q+1}(u(s))ds))dt \tag{2.145}$$

for $r > \eta_k$, and therefore

$$0 = u(\eta_k) - \int_{\eta_k}^{r_{k+1}} \Phi_{p'}(\frac{1}{t^{n+\alpha-1}} \int_{\eta_k}^{t} s^{n+\beta-1}\Phi_{q+1}(u(s))ds))dt \tag{2.146}$$

Let us assume that $u(r) > 0$ for $r \in (r_k, r_{k+1})$ using equation (2.146) and letting $u(\eta_k) = M_k$

$$0 \ge M_k - \int_{\eta_k}^{r_{k+1}} \frac{1}{t^{\frac{n+\alpha-1}{p-1}}} (\int_{\eta_k}^{t} s^{n+\beta-1}(M_k)^q ds)^{\frac{1}{p-1}} dt \tag{2.147}$$

Therefore

$$\int_{\eta_k}^{r_{k+1}} \frac{1}{t^{\frac{n+\alpha-1}{p-1}}} (\int_{\eta_k}^{t} s^{n+\beta-1} ds)^{\frac{1}{p-1}} dt \ge (M_k)^{-\frac{q+1-p}{p-1}} \tag{2.148}$$

Since $s \le t \le r_{k+1}$ and for $\beta > \alpha$ we have

$$\int_{\eta_k}^{r_{k+1}} \frac{1}{t^{\frac{n+\alpha-1}{p-1}}} (\int_{\eta_k}^{r_{k+1}} t^{n+\alpha-1} dt)^{\frac{1}{p-1}} dt \ge (M_k)^{-\frac{q+1-p}{p-1}} \tag{2.149}$$

Evaluating (2.149) gives

$$\frac{p-1}{p}(r_{k+1} - \eta_k)^{\frac{p}{p-1}} \ge (M_k)^{-\frac{q+1-p}{p-1}} \tag{2.150}$$

This simplifies to

$$r_{k+1} - \eta_k \ge (\frac{p}{p-1})^{\frac{p-1}{p}} (|u(0)|)^{-\frac{q+1-p}{p}} \tag{2.151}$$

In a similar manner we can show

$$\eta_k - r_k \ge (\frac{p}{p-1})^{\frac{p-1}{p}} (|u(0)|)^{-\frac{q+1-p}{p}} \tag{2.152}$$

Adding the equations (2.151) and (2.152) together we have,

$$r_{k+1} - r_k \geq 2(\frac{p}{p-1})^{\frac{p-1}{p}}(|u(0)|)^{-\frac{q+1-p}{p}} \tag{2.153}$$

Hence

$$1 \geq \sum_{1}^{m-1}(r_{k+1} - r_k) \geq 2(m-1)(\frac{p}{p-1})^{\frac{p-1}{p}}(|u(0)|)^{-\frac{q+1-p}{p}} \tag{2.154}$$

From which we obtain

$$m \leq 1 + \frac{1}{2}(\frac{p-1}{p})^{\frac{p-1}{p}}(|u(0)|)^{\frac{q+1-p}{p}} \tag{2.155}$$

and the theorem is proved.

Chapter 3

The Generalized Lane-Emden Equation and phase plane

.

3.1 The Generalized Lane-Emden equation as an autonomous system

The Generalized Lane-Emden equation (1.1) with the associated boundary condition for radial solutions reduces to the ordinary differential equation (ODE)

$$(p-1)r^{\alpha}\left|u_r\right|^{p-2}u_{rr}+(n+\alpha-1)r^{\alpha-1}\left|u_r\right|^{p-2}u_r+r^{\beta}\left|u\right|^{q-1}u=0 \qquad (2.10)$$

$$r=|x|,\ 0\le r\le 1,\ u=u(r) \text{ and } u(1)=0.$$

This radial version of the equation can be further transformed into an autonomous system for examination in the phase plane using the following change of variables: $r=e^{-t}$, $u_r=-e^{t}u_t$ and $u_{rr}=e^{2t}[u_{tt}+u_t]$. This transformation will remove the radial variable r and replace it with a variable, t. We then have

$$(p-1)e^{-(\alpha-p)t}\left|u_t\right|^{p-2}u_{tt}-(n+\alpha-p)e^{-(\alpha-p)t}\left|u_t\right|^{p-2}u_t+e^{-\beta t}\left|u\right|^{q-1}u=0 \quad (3.1)$$

Multiplying by $e^{(\alpha-p)t}$ and collecting like terms gives

$$(p-1)\left|u_t\right|^{p-2}u_{tt}-(n+\alpha-p)\left|u_t\right|^{p-2}u_t+e^{-(\beta+p-\alpha)t}\left|u\right|^{q-1}u=0. \qquad (3.2)$$

We now use a second transformation to obtain an autonomous system by introducing the variables v and v_t. Let $u=e^{\theta t}v$, then $u_t=e^{\theta t}(\theta v+v_t)=e^{\theta t}w$ and $u_{tt}=e^{\theta t}(\theta w+w_t)$, then substituting these values in (3.2) gives

$$(p-1)\left|e^{\theta t}w\right|^{p-2}e^{\theta t}(w_t+\theta w)-(n+\alpha-p)\left|e^{\theta t}w\right|^{p-2}e^{\theta t}w+e^{-(\beta+p-\alpha)t}\left|e^{\theta t}v\right|^{q-1}e^{\theta t}v=0 \qquad (3.3)$$

After simplifying and collecting like terms in (3.3) we have

$$(p-1)e^{\theta(p-1)t}\left|w\right|^{p-2}w_t+(p-n-\alpha+\theta(p-1))e^{\theta(p-1)t}\left|w\right|^{p-2}w+e^{(\alpha-p-\beta+\theta q)t}\left|v\right|^{q-1}v=0 \qquad (3.4)$$

To eliminate the exponential term in (3.4) we set $\theta(p-1) = \alpha - \beta - p + \theta q$. This results in $\theta = \frac{\beta+p-\alpha}{q+1-p}$.

Equation (3.4) can be rearranged as the autonomous system

$$(p-1)\left|w\right|^{p-2} w_t + \phi \left|w\right|^{p-2} w + \left|v\right|^{q-1} v = 0 \tag{3.5}$$

$$v_t + \theta v - w = 0$$

Where $\phi = p - n - \alpha + \theta(p-1)$.

Finally, letting $\eta = \left|w\right|^{p-2} w$, $\frac{d\eta}{dt} = (p-1)\left|w\right|^{p-2} w_t$ and $w = \eta \left|\eta\right|^{\frac{2-p}{p-1}}$ sets up a simpler autonomous system

$$\eta_t = -\phi\eta - \left|v\right|^{q-1} v = P(\eta, v) \tag{3.6}$$

$$v_t = -\theta v + \eta \left|\eta\right|^{\frac{2-p}{p-1}} = Q(\eta, v)$$

Next, we introduce the associated boundary condition to the phase plane. The Generalized Lane-Emden equation is defined on a unit ball centered at the origin. In radial coordinates the boundary of this ball is at $r = 1$ and its center is located at $r = 0$. Let us denote the slope of the solution at the boundary by $-\delta$ such that $u_r(1) = -\delta$ for $-\delta \in (-\infty, \infty)$.

The dirichlet boundary condition $u|_{\partial\Omega} = 0$ in radial form is $u(1) = 0$. Using the change of variables introduced earlier where $r = e^{-t}$, the boundary $r = 1$ becomes $t = 0$ in the phase plane. When $r = 0$ the transformation implies that t approaches infinity. Solutions starting at $r = 1$ and ending at $r = 0$ in the radial setting are equivalent to solutions starting at $t = 0$ and approaching infinity in the phase plane. We use the transformations $u = e^{\theta t} v$, $u_r = -e^t u_t$, and $u_t = e^{\theta t}(\theta v + v_t)$ to find the initial and final points in the $v - v_t$ plane. The approach used in this chapter is in essence a reverse of the shooting argument used in chapter 2.

The starting point at $t = 0$ in the phase plane.

Since $u = 0$ at $t = 0$,($r = 1$), the transformation $u = e^{\theta t}v$ gives $v(0) = 0$. The transformation $u_r = -e^t u_t$ for these values of r and t gives $u_t = \delta$, since $u_r(1) = -\delta$. Whereas the transformation $u_t = e^{\theta t}(\theta v + v_t)$ results in $v_t = \delta$ for $t = 0$, $v = 0$ and $u_t = \delta$. Hence solutions that satisfy the boundary condition for the partial differential equation, $u(r = 1) = 0$, start on the v_t axis at $t = 0$ at some point $(0, \delta)$.

End point in the phase plane as $t \to \infty$

To locate the required asymptotic behavior of solutions for the PDE starting at $(0, \delta)$ in the phase plane, we deduce the values of v and v_t as $t \to \infty$ using the transformation $u = e^{\theta t}v$. Substituting the values $u(r = 0) = \sigma$ and $t = \infty$, we find that $v \approx \sigma e^{-\theta t}$ as $t \to \infty$. Further, the transformations $u_r = -e^t u_t$,$u_t = e^{\theta t}(\theta v + v_t)$ and the estimate in (2.130) of u_r, we have $u_t \leq O(r^{\frac{\beta-\alpha+p}{p-1}}) = e^{\theta t}(\theta\sigma e^{-\theta t} + v_t)$ which in turn implies that $v_t \approx -\theta\sigma e^{-\theta t}$ as $t \to \infty$ for $\beta - \alpha + 1 > 0$.

Observe that solutions satisfying the dirichlet boundary condition and approaching the origin with a zero slope begin at time zero at the point $(0, \delta)$ in the $v - v_t$ plane and approach the origin as $t \to \infty$ for $\theta > 0$.

3.2 Existence of solutions in phase plane

Using the phase plane transformations we show that solutions to the partial differential equation satisfying the given boundary condition exist for $1 < p < q+1 < p^*$ and $\theta > 0$ as is stated in the following lemma.

Lemma 3.2.1. *Let u be a solution of the Generalized Lane-Emden equation in $L^{q+1}(B^n, |x|^\beta)$, then v and w tend to zero as $t \to \infty$ for $p < q+1 < p^*, 1 < p \leq 2$, $n + \alpha > p$ and $\theta > 0$, where $\theta = \frac{p+\beta-\alpha}{q+1-p}$ and*
$|v|^{q+1}$, $|w|^p \to o(exp[(n+\beta) - (\frac{p+\beta-\alpha}{q+1-p})(q+1)]t)$ as $t \to \infty$.

Proof:

Equation (2.18) implies

$$r^{n+\beta-1} |u|^{q+1} \approx o(r^{-1}) \tag{3.7}$$

Therefore

$$|u|^{q+1} \approx o(r^{-(n+\beta)}) \tag{3.8}$$

Using the phase plane transformations where $u = e^{\theta t} v$ and $r = e^{-t}$ we have

$$|v|^{q+1} \approx o(e^{[(n+\beta)-\theta(q+1)]t}) \tag{3.9}$$

The exponent $(n+\beta) - \theta(q+1)$ in (3.9) must be less than zero to ensure decay of solutions. Therefore

$$(\frac{p+\beta-\alpha}{q+1-p})(q+1) > n+\beta \tag{3.10}$$

and so

$$(p+\beta-\alpha)(q+1) > (n+\beta)(q+1-p) \tag{3.11}$$

Hence

$$(q+1)[n+\beta-p-\beta+\alpha] < p(n+\beta) \tag{3.12}$$

Therefore for $n + \alpha > p$

$$q + 1 < \frac{p(n+\beta)}{n+\alpha-p} = p^* \tag{3.13}$$

Similarly

$$r^{n+\alpha-1} |u_r|^p \approx o(r^{-1}) \tag{3.14}$$

this implies that

$$|u_r|^p \approx o(r^{-(n+\alpha)}) \tag{3.15}$$

Using the transformations $r = e^{-t}$, $u_r = -e^t u_t$ we have

$$\left|-e^t u_t\right|^p \approx o(e^{(n+\alpha)t}) \tag{3.16}$$

Therefore

$$|u_t|^p \approx o(e^{(n+\alpha-p)t}) \tag{3.17}$$

Using the phase plane transformation $u_t = e^{\theta t} w$ gives

$$|w|^p \approx o(e^{(n+\alpha-p(\theta+1)t}) \tag{3.18}$$

To ensure decay of solutions we set $n + \alpha - p - \theta p < 0$. Therefore

$$-\theta p < -(n+\alpha) + p \tag{3.19}$$

And so

$$\theta > \frac{n+\alpha-p}{p} \tag{3.20}$$

Inequality (3.20) for $p + \beta - \alpha > 0$ implies that

$$q + 1 < \frac{p(n+\beta)}{n+\alpha-p} = p^*. \tag{3.21}$$

The inequality $q + 1 > p$ was needed to avoid the singularity at the origin and the inequality (3.20) for $p + \beta - \alpha > 0$ restricts θ to be greater than zero for solutions to satisfy the boundary condition for $1 < p < q + 1 < p^*$.

3.3 Local analysis of the autonomous system

Consider the case where $q > 1$, and $1 < p \leq 2$ to avoid the singularity at the origin of the autonomous system in (3.6). The equilibrium points of the nonlinear system

$$\eta_t = -\phi\eta - |v|^{q-1} v = P(\eta, v)$$

$$v_t = -\theta v + \eta |\eta|^{\frac{2-p}{p-1}} = Q(\eta, v)$$

are found by setting the functions $P(\eta, v)$ and $Q(\eta, v)$ equal to zero and solving for η and v. Setting $Q(\eta, v) = 0$ gives $v = \frac{\eta}{\theta} |\eta|^{\frac{2-p}{p-1}}$. Substituting this value in $P(\eta, v) = 0$ gives,

$$-\phi\eta - \frac{\eta}{\theta} \left|\frac{\eta}{\theta}\right|^{q-1} |\eta|^{(\frac{2-p}{p-1})(q-1)} |\eta|^{\frac{2-p}{p-1}} = 0 \tag{3.22}$$

Hence

$$-\phi\eta - |\theta|^{1-q} \theta^{-1} \eta |\eta|^{\frac{q+1-p}{p-1}} = 0 \tag{3.23}$$

Therefore

$$\eta[\phi + |\theta|^{1-q} \theta^{-1} |\eta|^{\frac{q+1-p}{p-1}}] = 0 \tag{3.24}$$

which in turn leads to $\eta = 0$ or $\eta = \pm(-\phi\theta |\theta|^{q-1})^{\frac{p-1}{q+1-p}}$. When $\phi\theta > 0$ the only critical point is $(0, 0)$. For $\phi\theta < 0$, where $q + 1 \neq p$, there are three finite critical points. These three points are $(\eta, v) = ((-\theta\phi |\theta|^{q-1})^{\frac{p-1}{q+1-p}}, \frac{1}{\theta}(-\phi\theta |\theta|^{q-1})^{\frac{1}{q+1-p}})$, $(-(-\theta\phi |\theta|^{q-1})^{\frac{p-1}{q+1-p}}, -\frac{1}{\theta}(-\phi\theta |\theta|^{q-1})^{\frac{1}{q+1-p}})$ and $(0, 0)$.

To find these three finite critical points observe that when we set $Q(v, \eta) = 0$ we obtain $v = \frac{\eta}{\theta} |\eta|^{\frac{2-p}{p-1}}$ which for $\theta > 0$ implies that v is positive when η is positive and v is negative when η is negative. Since $\eta_{\pm} = \pm \left(-\theta\phi |\theta|^{q-1}\right)^{\frac{p-1}{q+1-p}}$, there exist three finite critical points for the case $\phi\theta < 0$: $(0, 0)$, (v_+, η_+) and (v_-, η_-).

Taking η_+ gives $v_+ = \frac{\eta_+}{\theta} |\eta_+|^{\frac{2-p}{p-1}}$

We therefore have

$$v_+ = \frac{1}{\theta}\left(-\theta\phi\,|\theta|^{q-1}\right)^{\frac{p-1}{q+1-p}}\left|\left(-\theta\phi\,|\theta|^{q-1}\right)^{\frac{p-1}{q+1-p}}\right|^{\frac{2-p}{p-1}} \tag{3.25}$$

Hence for $\phi\theta < 0$

$$v_+ = \frac{1}{\theta}\left(-\theta\phi\,|\theta|^{q-1}\right)^{\frac{1}{q+1-p}} \tag{3.26}$$

Similarly, substituting $\eta_- = -(-\theta\phi\,|\theta|^{q-1})^{\frac{p-1}{q+1-p}}$ in $v_- = \frac{\eta_-}{\theta}\,|\eta_-|^{\frac{2-p}{p-1}}$ gives

$$v_- = -\frac{1}{\theta}\left(-\theta\phi\,|\theta|^{q-1}\right)^{\frac{p-1}{q+1-p}}\left|-(-\theta\phi\,|\theta|^{q-1})^{\frac{p-1}{q+1-p}}\right|^{\frac{2-p}{p-1}} \tag{3.27}$$

Therefore for $\phi\theta < 0$

$$v_- = -\frac{1}{\theta}\left(-\theta\phi\,|\theta|^{q-1}\right)^{\frac{1}{q+1-p}} \tag{3.28}$$

So far we only considered the case $q > 1, 1 < p \leq 2$ to avoid the singularity at the origin. The autonomous system in (3.6) is singular for $p < 1$, $p > 2$, $q < 1$, $\alpha, \beta < 0$ at the point $(\eta, v) = (0, 0)$ driving the critical point to infinity creating what is commonly referred to as " critical points at infinity ". In solving $|\eta|^{\frac{q+1-p}{p-1}} = (-\phi\theta\,|\theta|^{q-1})$ for η in (3.24) we notice the presence of other singularities. For example, for $q + 1 < p$, or $p < 1$, η evaluated at zero is not defined which results in mapping η and v to infinity. A similar situation is created when $\theta = 0$ and $q < 1$ where θ is exactly equal to zero when $p = \alpha - \beta$. Another case is obvious when $q + 1 = p$ in the exponent of (3.24). The case $q + 1 = p$ yielding critical points at infinity can also be seen when considering the denominator of θ, where $\theta = \frac{\beta+p-\alpha}{q+1-p}$ approaches infinity when $q+1 = p$ which in turn will move ϕ to infinity as well, resulting in critical points at infinity. The last case is when $\theta < 0$ or $\phi < 0$ and $p < 1$ or $q + 1 < p$.

These singular cases represent case iii), figure 1.2, of the radial setting as unbounded solutions near the origin where $u(r)$ and $u_r(r)$ approach infinity as $r \to 0^+$.

Next we use the method of linearization and the Hartman-Grobman theorem to determine the local behavior of trajectories of the autonomous system in (3.6) around the finite critical points in phase plane. In simple words, the Hartman-Grobman theorem states that near a hyperbolic equilibrium point x_0, the nonlinear system $x' = f(x)$ has the same qualitative structure as the linear system $x' = Ax$, where $A = Df(x_0)$ represents the Jacobian matrix evaluated at the critical point x_0.

Theorem 3.3.1. *: The Hartman-Grobman Theorem.*

If $Df(x_0)$ has no zero or purely imaginary eigenvalues, then there is a homeomorphism h defined on some neighborhood U of x_0 in $\Re^n$ locally taking orbits of the nonlinear flow ϕ_t of the nonlinear system, $x' = f(x)$, to those of the linear flow $x' = Ax$, where $A = Df(x_0)$. The homeomorphism preserves the sense of orbits and can also be chosen to preserve parametrization by time, [23].

i) The case $\phi\theta > 0$.

The Jacobian matrix of the autonomous system in (3.6) is

$$\begin{pmatrix} -\phi & -q\,|v|^{q-1} \\ \frac{1}{p-1}\,|\eta|^{\frac{2-p}{p-1}} & -\theta \end{pmatrix}$$

The Jacobian matrix when evaluated at the critical point $(0,0)$ results in two eigenvalues. The eigenvalue $-\phi$, with corresponding eigenvector $[1 \quad 0]$, and the eigenvalue $-\theta$, with corresponding eigenvector $[0 \quad 1]$. The local solution is therefore

$$[\eta \quad v] = c_1[0 \quad 1]e^{-\theta t} + c_2[1 \quad 0]e^{-\phi t} \tag{3.29}$$

The eigenvalues for the critical point $(0, 0)$ can be positive or negative influencing the behavior of trajectories about $(0, 0)$ depending on the signs of θ and ϕ, which in turn depends on the parameters p, q, n , β and α. Recall that for the nonlinear autonomous system to be nonsingular we restricted $q+1$ to be greater than p, and for solutions to satisfy the boundary condition we observed that $\theta > 0$. $\theta = \frac{p+\beta-\alpha}{q+1-p}$ is positive when $p > \alpha - \beta$.

The sign of $\phi = p - n - \alpha + \theta(p-1)$ is positive when $\theta > \frac{n+\alpha-p}{p-1}$. Since $\theta > 0$, then $\phi > 0$ when $q + 1 < \frac{p(\beta+n-1)+(\alpha-\beta)}{n+\alpha-p} = \hat{p}$. On the other hand, ϕ is negative when $q + 1 > \frac{p(\beta+n-1)+(\alpha-\beta)}{n+\alpha-p} = \hat{p}$.

When $\phi\theta > 0$, $\theta > 0$, $\phi > 0$, the eigenvalues are both negative and the critical point $(0, 0)$ is a sink and therefore asymptotically stable. For $1 < p \leq 2$, $\alpha + n > p$, we have $\theta(p-1) < \theta$ and $\phi = \theta(p-1) - (\alpha + n - p) < \theta$ which in turn implies that $-\phi > -\theta$. Therefore $e^{-\phi t}[1 \quad 0]$ is the strong attractor.

Solutions to the partial differential equation satisfying the asymptotic condition, $e^{\theta t}w \to 0$ as $t \to \infty$, eventually lie along the weak attractor $e^{-\theta t}[0 \quad 1]$ as can be seen from the following calculation.

$$e^{\theta t}w = e^{\theta t}\,|\eta|^{\frac{2-p}{p-1}}\,\eta = e^{\theta t}\left|c_1(0)e^{-\theta t}\right|^{\frac{2-p}{p-1}}\left(c_1(0)e^{-\theta t}\right) = 0. \tag{3.30}$$

Along the strong attractor $e^{-\phi t}[1 \quad 0]$ we have

$$e^{\theta t}w = e^{\theta t}\,|\eta|^{\frac{2-p}{p-1}}\,\eta = e^{\theta t}\left|c_2 e^{-\phi t}\right|^{\frac{2-p}{p-1}}\left(c_2 e^{-\phi t}\right) = ce^{(\theta - \frac{1}{p-1}\phi)t} \tag{3.31}$$

Since $\frac{\phi}{p-1} < \theta$, solutions satisfying the asymptotic condition do not lie along the strong attractor.

ii) The case $\phi\theta < 0$

For a closer look and deeper analysis of the behavior of solutions in the phase plane we transform the system further in to one dependent on the variables v and v_t.

Consider the first equation of the autonomous system in (3.5), substituting the value $(v_t + \theta v)$ for w, $(\theta v_t + v_{tt})$ for w_t, γ for v_t and collecting like terms, we have

$$(p-1)\left|\gamma+\theta v\right|^{p-2}\gamma_t + [\phi+\theta(p-1)]\left|\gamma+\theta v\right|^{p-2}\gamma + [\phi\theta\left|\gamma+\theta v\right|^{p-2} + |v|^{q-1}]v = 0 \tag{3.32}$$

$$v_t = \gamma$$

as the corresponding autonomous system dependent on v and v_t. This system can be arranged as follows

$$\gamma_t = -(\frac{\phi}{p-1}+\theta)\gamma - \frac{\phi\theta}{p-1}v - \frac{|v|^{q-1}v}{p-1}\left|\gamma+\theta v\right|^{2-p} = P(\gamma, v) \tag{3.33}$$

$$v_t = \gamma = Q(\gamma, v)$$

Setting both equations of the system in (3.33) equal to zero, we find the finite critical points to be the point $(0,0)$ for the case $\phi\theta > 0$ and $(\pm(-\phi\theta\left|\theta\right|^{p-2})^{\frac{1}{q+1-p}}, 0)$, $(0,0)$ for the case $\phi\theta < 0$.

We obtain these critical points by setting $Q(\gamma, v) = 0$ and substituting $\gamma = 0$ in $P(\gamma, v)$ of the system (3.33)

$$0 = -\frac{[\phi\theta + |\theta|^{2-p}|v|^{q+1-p}]v}{p-1} \tag{3.34}$$

Hence $v = 0$ or $|v|^{q+1-p} = -\theta\phi\left|\theta\right|^{p-2}$. Observe that θ must not equal zero to avoid a singularity of the system for $1 < p \leq 2$.

Next we linearize about the three finite critical points to understand local behavior of trajectories there.

The Jacobian matrix for the system in (3.33) is:

$$\begin{pmatrix} \frac{\partial P}{\partial \gamma} & \frac{\partial P}{\partial v} \\ 1 & 0 \end{pmatrix}$$

Where

$\frac{\partial P}{\partial \gamma} = [\frac{-\phi}{p-1} - \theta] - \frac{2-p}{p-1} v(\gamma + \theta v) \left|v\right|^{q-1} \left|\gamma + \theta v\right|^{-p}$

And

$\frac{\partial P}{\partial v} = -\frac{\phi\theta}{p-1} - \frac{2-p}{p-1}\theta v(\gamma + \theta v) \left|v\right|^{q-1} \left|\gamma + \theta v\right|^{-p} - \frac{q}{p-1} \left|\gamma + \theta v\right|^{2-p} \left|v\right|^{q-1}$.

When the Jacobian matrix is evaluated at $(0, 0)$ the matrix gives

$$\begin{pmatrix} -\frac{\phi}{p-1} - \theta & -\frac{\phi\theta}{p-1} \\ 1 & 0 \end{pmatrix}$$

(see appendix A4).

Setting the determinant of the matrix $(A - \lambda I)$ equal to zero, we solve for the eigenvalues

$$\lambda_{1,2} = \frac{-(\frac{\phi}{p-1} + \theta) \pm \sqrt{(\frac{\phi}{p-1} + \theta)^2 - \frac{4\phi\theta}{p-1}}}{2} \tag{3.35}$$

which simplifies to

$$\lambda_{1,2} = \frac{-(\frac{\phi}{p-1} + \theta) \pm (\frac{\phi}{p-1} - \theta)}{2} \tag{3.36}$$

Equation (3.36) results in two eigenvalues, $\lambda_1 = -\frac{\phi}{p-1}$ and $\lambda_2 = -\theta$. The corresponding eigenvectors are found by solving the system $[A - \lambda I]X = 0$ where the matrix $(A - \lambda I)$ is

$$\begin{pmatrix} -\frac{\phi}{p-1} - \theta - \lambda & -\frac{\phi\theta}{p-1} \\ 1 & 0 - \lambda \end{pmatrix}$$

The eigenvalue $\lambda_1 = -\frac{\phi}{p-1}$ has the corresponding eigenvector $[\frac{\phi}{p-1} \quad -1]$ and the eigenvalue $\lambda_2 = -\theta$ has the corresponding eigenvector $[\theta \quad -1]$. The local solution is therefore written as

$$[v_t \quad v] = c_1[\frac{\phi}{p-1} \quad -1]e^{-\frac{\phi}{p-1}t} + c_2[\theta \quad -1]e^{-\theta t} \tag{3.37}$$

The vector $[\frac{\phi}{p-1} \quad -1]e^{-\frac{\phi}{p-1}t}$ is the strong attractor in the case of $\phi\theta > 0$ since $\frac{\phi}{p-1} = \theta - \frac{n+\alpha-p}{p-1}$ which for $n+\alpha > p$, $1 < p \leq 2$ is in fact smaller than θ hence $-\frac{\phi}{p-1} > -\theta$.

For $\theta > 0, \phi < 0$ and $\phi\theta < 0$ the origin is a saddle where solutions satisfying the asymptotic condition, $e^{\theta t}w \to 0$, approach zero along the weak attractor. Other trajectories leave the origin along the local flow component $[\frac{\phi}{p-1} \quad -1]e^{-\frac{\phi}{p-1}t}$ as can be seen from the following calculations.

Along the local flow component $[\frac{\phi}{p-1} \quad -1]e^{\frac{-\phi}{p-1}t}$*:*

$$e^{\theta t}w = e^{\theta t}(v_t + \theta v) = e^{\theta t}[-c_1\theta e^{-\frac{\phi}{p-1}t} + c_1\frac{\phi}{p-1}e^{-\frac{\phi}{p-1}t}] = c_1 e^{(\theta - \frac{\phi}{p-1})t}[-\theta + \frac{\phi}{p-1}] \tag{3.38}$$

It is clear that the exponent $(\theta - \frac{\phi}{p-1})$ is positive for $\phi < 0$ and the constant $[-\theta + \frac{\phi}{p-1}]$ does not equal to zero.

Along the weak attractor $[\theta \quad -1]e^{-\theta t}$*:*

$$e^{\theta t}w = e^{\theta t}(v_t + \theta v) = e^{\theta t}[c_2\theta e^{-\theta t} - c_2\theta e^{-\theta t}] = c_2 e^{(\theta-\theta)t}[\theta - \theta] \tag{3.39}$$

Now consider the critical point $((-\phi\theta\,|\theta|^{p-2})^{\frac{1}{q+1-p}}, 0)$. Linearization about this point yields the following matrix:

$$\begin{pmatrix} -(\phi+\theta) & \frac{q+1-p}{p-1}\theta\phi \\ 1 & 0 \end{pmatrix}$$

The above matrix is obtained by substituting the critical point $((-\phi\theta\,|\theta|^{p-2})^{\frac{1}{q+1-p}}, 0)$ in the Jacobian matrix of the autonomous system (3.33), where

$$\frac{\partial P}{\partial \gamma} = -(\frac{\phi}{p-1} + \theta) - (\frac{2-p}{p-1})\theta\,|\theta|^{-1}\,|\phi| \tag{3.40}$$

For $\phi < 0$ and $\theta > 0$ we have

$$\frac{\partial P}{\partial \gamma} = \frac{-\phi}{p-1} - \theta + \frac{2-p}{p-1}\phi = -(\phi+\theta) \tag{3.41}$$

And,

$$\frac{\partial P}{\partial v} = \frac{-\phi\theta}{p-1} + \frac{q+2-p}{p-1}\theta^{2-p}\,|\theta|^{p-2}\,\theta\phi \tag{3.42}$$

Which simplifies to

$$\frac{\partial P}{\partial v} = \frac{q+1-p}{p-1}\theta\phi \tag{3.43}$$

The eigenvalues for the Jacobian matrix linearized about the point $((-\phi\theta\,|\theta|^{p-2})^{\frac{1}{q+1-p}}, 0)$ are obtained by setting the determinant of the matrix $(A - \lambda I) = 0$.

$$\lambda_{1,2} = \frac{-(\phi+\theta) \pm \sqrt{(\phi+\theta)^2 - 4(-\frac{q+1-p}{p-1}\theta\phi)}}{2} \tag{3.44}$$

For $1 < p \leq 2$, $q+1 > p$, $\phi < 0$ and $\theta > 0$, the term $(-\frac{q+1-p}{p-1})\phi\theta$ in (3.44) is positive. The value under the square root is negative creating complex eigenvalues. When $|\phi| > |\theta|$ the term $-(\phi+\theta)$ is positive creating an unstable spiral point with trajectories spiraling out of the critical point. For $|\phi| < |\theta|$ the term $-(\phi+\theta)$ is negative creating an unstable spiral point with trajectories spiraling in to the critical point.

The same results apply to the critical point $(-(-\phi\theta\left|\theta\right|^{p-2})^{\frac{1}{q+1-p}}, 0)$ since the absolute value eliminates the negative sign in the v coordinate of the point resulting in similar results.

3.4 Critical exponent analysis

In the previous section, we obtained from linearization about the finite critical points of the autonomous system (3.33) two different cases, the case $\phi\theta > 0$ and the case $\phi\theta < 0$. In each case the behavior of trajectories about the finite critical points is determined by the sign of ϕ and the value of ϕ relative to the value of θ. In this section we prove that such behavior of trajectories is influenced by two critical exponents dependent on the parameters p, q, β, α and n. These two critical exponents are what we referred to as p^* and $\hat{p}$ in the weighted Sobolev embedding theorem and weighted trace embedding conjuncture respectively in section 2.1.

Lemma 3.4.1. *For $p < q+1 < \hat{p}, \hat{p} = \frac{p(\beta+n-1)+(\alpha-\beta)}{n+\alpha-p}, 1 < p \leq 2, \phi\theta > 0$, $0 < \phi < \theta$, the origin is a sink of the autonomous system (3.6). Solutions satisfying the asymptotic condition, $e^{\theta t}w \to 0$, approach the origin along the weak attractor as $t \to \infty$.*

Proof:

Local linearization about the origin for this case indicated the presence of a sink at the origin with all trajectories approaching $(0,0)$ as $t \to \infty$. In particular we found that solutions to the partial differential equation satisfying the asymptotic condition, $e^{\theta t}w \to 0$, approach the origin along the weak attractor $[\theta \quad -1]e^{-\theta t}$.

Since $\phi = p - n - \alpha + \theta(p-1) > 0$ then $\theta = \frac{\beta+p-\alpha}{q+1-p} > \frac{n+\alpha-p}{p-1}$ which for $(\beta + p - \alpha) > 0$, $(\theta > 0)$, implies that $q + 1 < \hat{p}$, where $\hat{p} = \frac{p(\beta+n-1)+(\alpha-\beta)}{n+\alpha-p}$.

So far we considered two cases for the values of ϕ and θ, namely $|\phi| > |\theta|$ and $|\phi| < |\theta|$. For these two cases, linearization indicated the presence of spiral points at $(\pm(-\phi\theta\,|\theta|^{p-2})^{\frac{1}{q+1-p}}, 0)$ and a saddle at the origin. We now consider the case when $\phi = -\theta$.

Lemma 3.4.2. *The autonomous system in the phase plane (3.33) corresponding to the Generalized Lane-Emden equation (1.1) for* $q+1=p^*, p^*=\frac{p(\beta+n)}{\alpha+n-p}$, $1<p\leq 2$, $q>1$, $\phi\theta<0$, $\phi=-\theta$ *has three finite critical points, a saddle origin and centers at* $(\pm(-\phi\theta\,|\theta|^{p-2})^{\frac{1}{q+1-p}},0)$.

Proof:

When $\phi=-\theta$, $q+1=p^*$, where $p^*=\frac{p(\beta+n)}{\alpha+n-p}$. This case marks the appearance of the critical exponent p^*.

Substituting $\phi=-\theta$ in (3.37) gives

$$[v_t \quad v]=c_1[\frac{-\theta}{p-1} \quad -1]e^{\frac{\theta}{p-1}t}+c_2[-\theta \quad 1]e^{-\theta t} \tag{3.45}$$

Which indicates the presence of a saddle about $(0,0)$.

Now consider the point $((-\phi\theta\,|\theta|^{p-2})^{\frac{1}{q+1-p}},0)$, the eigenvalues for the Jacobian matrix evaluated at this point are

$$\lambda_{1,2}=\frac{-(\phi+\theta)\pm\sqrt{(\phi+\theta)^2-4(-(\frac{q+1-p}{p-1})\phi\theta)}}{2} \tag{3.46}$$

When $\phi=-\theta$ is substituted in (3.46) we have

$$\lambda_{1,2}=\frac{\pm\sqrt{-4(\frac{q+1-p}{p-1})\theta^2}}{2} \tag{3.47}$$

It is clear that the term under the square root is negative for $q+1>p$ and $1<p\leq 2$. Hence the point $((-\phi\theta\,|\theta|^{p-2})^{\frac{1}{q+1-p}},0)$ has complex eigenvalues with zero real parts. This implies the presence of a center about this critical point. The same applies to the critical point $(-(-\phi\theta\,|\theta|^{p-2})^{\frac{1}{q+1-p}},0)$.

Next we show that the critical exponent, $p^*=\frac{p(\beta+n)}{\alpha+n-p}$, marks the end of the behavior of trajectories creating a saddle about $(0,0)$ and other trajectories spiraling in to the critical points $(\pm(-\phi\theta\,|\theta|^{p-2})^{\frac{1}{q+1-p}},0)$ and marks the start of a new phase

where orbits are created about the critical points $(\pm(-\phi\theta\,|\theta|^{p-2})^{\frac{1}{q+1-p}}, 0)$. The critical exponent p^* also marks the end of a phase where trajectories create centers about the critical points $(\pm(-\phi\theta\,|\theta|^{p-2})^{\frac{1}{q+1-p}}, 0)$ and sets the start of another phase where other trajectories spiral out of the these two critical points. We refer to the case where $\phi = -\theta$ as the bifurcation case, such that $\phi + \theta$ is the bifurcation term. A change in the bifurcation value changes the qualitative behavior of trajectories in the phase plane as stated in lemmas 3.4.3 and 3.4.4 for $\phi > -\theta$ and $\phi < -\theta$ respectively.

Lemma 3.4.3. *The autonomous system (3.33) corresponding to the Generalized Lane-Emden equation has three finite critical points,* $(0, 0)$ *a saddle and* $(\pm(-\phi\theta\,|\theta|^{p-2})^{\frac{1}{q+1-p}}, 0)$ *spirals with trajectories spiraling in to the critical points, for* $\hat{p} < q + 1 < p^*$, $1 < p \leq 2$, $\phi\theta < 0$, $\theta > 0$, $\phi < 0$, *and* $\phi > -\theta$.

Proof:

Linearization about the finite critical points in this case indicated the presence of a saddle about the origin and spirals with trajectories spiraling in to the critical points $(\pm(-\phi\theta\,|\theta|^{p-2})^{\frac{1}{q+1-p}}, 0)$. When $\phi < 0$ and $\theta > 0$, the inequality $q + 1 > \hat{p}$ holds, where $\hat{p} = \frac{p(\beta+n-1)+(\alpha-\beta)}{n+\alpha-p}$.

For $\phi > -\theta$, $\phi < 0, \theta > 0$, we obtain the inequality $\theta = \frac{\beta+p-\alpha}{q+1-p} > \frac{n+\alpha-p}{p}$. This implies that $q + 1 < p^*$, where $p^* = \frac{p(n+\beta)}{\alpha+n-p}$.

Lemma 3.4.4. *The critical points* $(\pm(-\phi\theta\,|\theta|^{p-2})^{\frac{1}{q+1-p}}, 0)$ *are spirals, with trajectories spiraling out, of the autonomous system (3.33) corresponding to the Generalized Lane-Emden equation (1.1) for* $q + 1 > p^*$, $1 < p \leq 2$, $\phi\theta < 0, \phi < 0, \theta > 0$, *and* $\phi < -\theta$.

Proof:

For $\phi\theta < 0$, $\phi < 0$, $\theta > 0$, $\phi < -\theta$, linearization pointed to a saddle about the origin and spirals at the critical points $(\pm(-\phi\theta\,|\theta|^{p-2})^{\frac{1}{q+1-p}}, 0)$ with trajectories spiraling out.

When $\phi < 0$, $q + 1 > \hat{p}, \hat{p} = \frac{p(\beta+n-1)+(\alpha-\beta)}{n+\alpha-p}$. On the other hand $\phi < -\theta$ implies that $\theta = \frac{\beta+p-\alpha}{q+1-p} < \frac{n+\alpha-p}{p}$ and since $\theta > 0$ it is then clear that $q + 1 > p^*$, where $p^* = \frac{p(\beta+n)}{\alpha+n-p}$.

3.5 Global phase plane analysis of solutions

In order to understand the behavior of trajectories in the phase plane one needs to understand the local behavior and the global picture. A method that helps sketch this picture is the index of an equilibrium point that provides information on the nature and complexity of the equilibrium point especially in the nonlinear cases, [23]. The index of a critical point is defined as $k = \frac{1}{2\pi}\int_C d\left\{arctan\left(\frac{dy}{dx}\right)\right\} = \frac{1}{2\pi}\int_C d\left\{arctan\left(\frac{g(x,y)}{f(x,y)}\right)\right\} = \frac{1}{2\pi}\int_C \frac{fdg-gdf}{f^2+g^2}$, Where C is a circle enclosing the critical point of the autonomous system

$$y' = f(x,y) \quad x' = g(x,y).$$

Index theory and Bendixon's criteria can help in understanding the global picture by indicating the possibility of the existence of a limit cycle. Follows are the index theory and Bendixon's criteria

Theorem 3.5.1. *Index theory.*

i) The index of a sink, a source or a center is +1.

ii) The index of a hyperbolic saddle point is -1.

iii) The index of a closed orbit is +1.

iv) The index of a closed curve containing any fixed points is 0.

v) The index of a closed curve is equal to the sum of the indices of the fixed points within it, [23].

Theorem 3.5.2. *Bendixon's criteria.*

Let $f \in C^1(E)$, where E is a simply connected region in $\Re^2$. If the divergence of the vector field f, $\nabla \cdot f$, is not identically zero and does not change sign in E, then the autonomous system $x' = f(x)$ has no closed orbit lying entirely in E, [34].

Orbits created in the phase plane are unique and not intersecting except possibly at $t = \pm\infty$. For our Generalized Lane-Emden equation this is true for $q+1 > p, 1 < p \leq 2, \alpha, \beta \geq 0$. This fact will be useful in sketching the local phase portraits and attempting to locate periodic orbits or limit cycles.

In this section we consider the global behavior of trajectories for the different cases mentioned in section 3.4 and sketch global trajectories for each case. We will also specify the only two cases for which solutions to our PDE, that satisfy the associated asymptotic condition, exist and sketch the solutions in the phase plane for each of these two cases.

Case 1) $p < q + 1 < \hat{p}$ **(** $\phi\theta > 0, 0 < \phi < \theta$**)** **.**

There are no critical points at infinity in this case and $(0, 0)$ is the only finite critical point. Solutions satisfying the PDE and its associated asymptotic condition, $e^{\theta t}w \to 0$, approach the origin along the weak attractor $[-\theta \quad 1]e^{-\theta t}$ as $t \to \infty$. For a solution in the radial setting starting at $r = 1$ satisfying $u_r(1) = -\delta$, $-\delta \in (-\infty, \infty)$, to reach $r = 0$ with $u_r(0) = 0$, it will begin at the point $(0, \delta)$ in the phase plane and end at the origin in the phase plane along the weak attractor $[-\theta \quad 1]e^{-\theta t}$ as t approaches infinity. As it travels in the radial setting it intersects the r axis a number of times creating a number of zeros all of which translate to intersections in the phase plane with the η (or correspondingly v_t) axis, see Figure 3.1. Hence, a single Orbit is created in the phase plane that corresponds to the infinite number of solutions in the radial setting. Each radial solution starts at a different $-\delta$ value corresponding to the different slopes at $r = 1$ that the infinite radial solutions may exhibit, see Figure 3.2.

There are no limit cycles in the phase plane for this case as can be shown using Bendixon's criterion

$$\frac{\partial P}{\partial \eta} + \frac{\partial Q}{\partial v} = -(\phi + \theta) \tag{3.48}$$

For $\theta\phi > 0, \theta > 0, \phi > 0, 0 < \phi < \theta$ the term $-(\phi + \theta)$ is negative, see Figure 3.3.

Case 2) $\hat{p} < q + 1 < p^*$ **(**$\phi\theta < 0, \phi < 0, \phi > -\theta$ **)**

There are no critical points at infinity in this case as well. It remains to check if there are any limit cycles in the phase plane. Using the index theory, $(0, 0)$ is a saddle with index $= -1$ and the other two finite critical points $(\pm(-\phi\theta\,|\theta|^{p-2})^{\frac{1}{q+1-p}}, 0)$ are spirals with an index of $+1$ each. The sum of these index values is $+1$ which implies the possibility of the existence of a limit cycle surrounding all three finite

critical points. Using Bendixon's criterion we have

$$\frac{\partial P}{\partial \eta} + \frac{\partial Q}{\partial v} = -(\phi + \theta) \tag{3.49}$$

For $\theta\phi < 0, \theta > 0, \phi < 0, \phi > -\theta$ the term $-(\phi + \theta)$ is negative.

Since there are no critical points at infinity, no limit cycles in the phase plane and trajectories are unique and not intersecting, we connect the orbits in the phase plane and complete the global picture. Trajectories leaving $(0, 0)$ along the local flow component $[\frac{\phi}{p-1} \quad - 1]e^{-\frac{\phi}{p-1}t}$ will be connected with trajectories spiraling in to the other two finite critical points, $(\pm(-\phi\theta\,|\theta|^{p-2})^{\frac{1}{q+1-p}}, 0)$. Solutions to our PDE are solutions that start at $(0, \delta)$ and approach $(0, 0)$ along the weak attractor as $t \to \infty$. Trajectories spiraling in to the other two critical points do not satisfy the PDE and its associated asymptotic condition, see Figure 3.4.

Case 3) $q + 1 = p^*$ $(\phi\theta < 0, \phi = -\theta\ )$

Local analysis about the finite critical points in this case indicated that $(0, 0)$ is a saddle with index of -1 and the other two critical points, $(\pm(-\phi\theta\,|\theta|^{p-2})^{\frac{1}{q+1-p}}, 0)$, are centers with an index of $+1$ each. Hence a closed curve containing all three points will have an index of $+1$, representing the sum of the indices of the three finite critical points within it.

Using Bendixon's criterion we find that

$$\frac{\partial P}{\partial \eta} + \frac{\partial Q}{\partial v} = -(\phi + \theta) = 0 \tag{3.50}$$

For $\phi = -\theta$.

Using the fact that trajectories are not intersecting, we connect the trajectories starting at $(0, \delta)$ and approaching the origin along the weak attractor with trajectories leaving the origin along the local flow component. This situation creates

closed orbits in the phase plane circulating around the centers created about the critical points $(\pm(-\phi\theta\,|\theta|^{p-2})^{\frac{1}{q+1-p}}, 0)$. There are no solutions to the PDE satisfying the associated asymptotic condition in this case, see Figure 3.5.

Case 4) $q+1 > p^*$ $(\phi\theta < 0, \phi < 0, \phi < -\theta)$

Using index theory, $(0,0)$ is a saddle with index -1 and the other two critical points $(\pm(-\phi\theta\,|\theta|^{p-2})^{\frac{1}{q+1-p}}, 0)$ are spirals with an index of $+1$ each, the sum of these index values is $+1$ which implies the possibility of the existence of a limit cycle surrounding all three points. Using Bendixon's criterion we have

$$\frac{\partial P}{\partial \eta} + \frac{\partial Q}{\partial \eta} = -(\phi + \theta) \tag{3.51}$$

For $\phi\theta < 0, \phi < 0, \phi < -\theta$ the term $-(\phi+\theta)$ is positive. We connect trajectories spiraling out of the two critical points $(\pm(-\phi\theta\,|\theta|^{p-2})^{\frac{1}{q+1-p}}, 0)$ with trajectories entering the origin along the weak attractor. Trajectories leaving the origin along the local flow component will circulate about the orbits spiraling out of the spiral points and leave to infinity. Solutions to our PDE are trajectories that start at $(0,\delta)$ and approach $(0,0)$ as $t \to \infty$ along the weak attractor, hence there are no solutions to the PDE satisfying the associated asymptotic condition. This is consistent with the results obtained earlier using Pohozaev's identity in chapter 2; solutions to the Generalized Lane-Emden equation (1.1) satisfying the associated boundary condition do not exist for $q+1 \geq p^*$, see Figure 3.6.

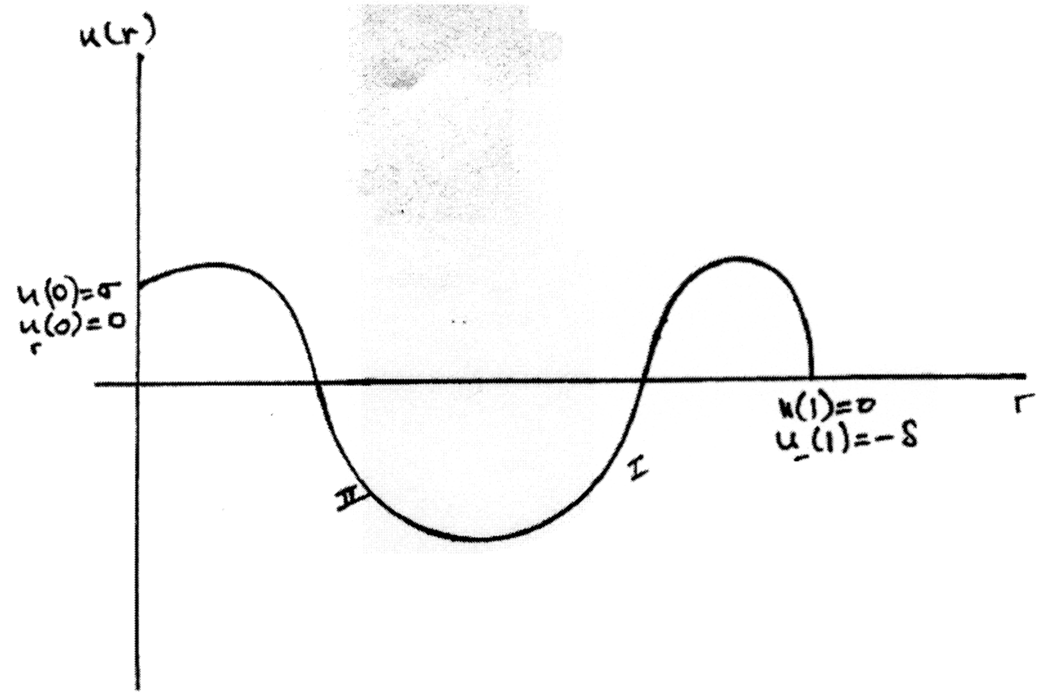


Figure 3.1

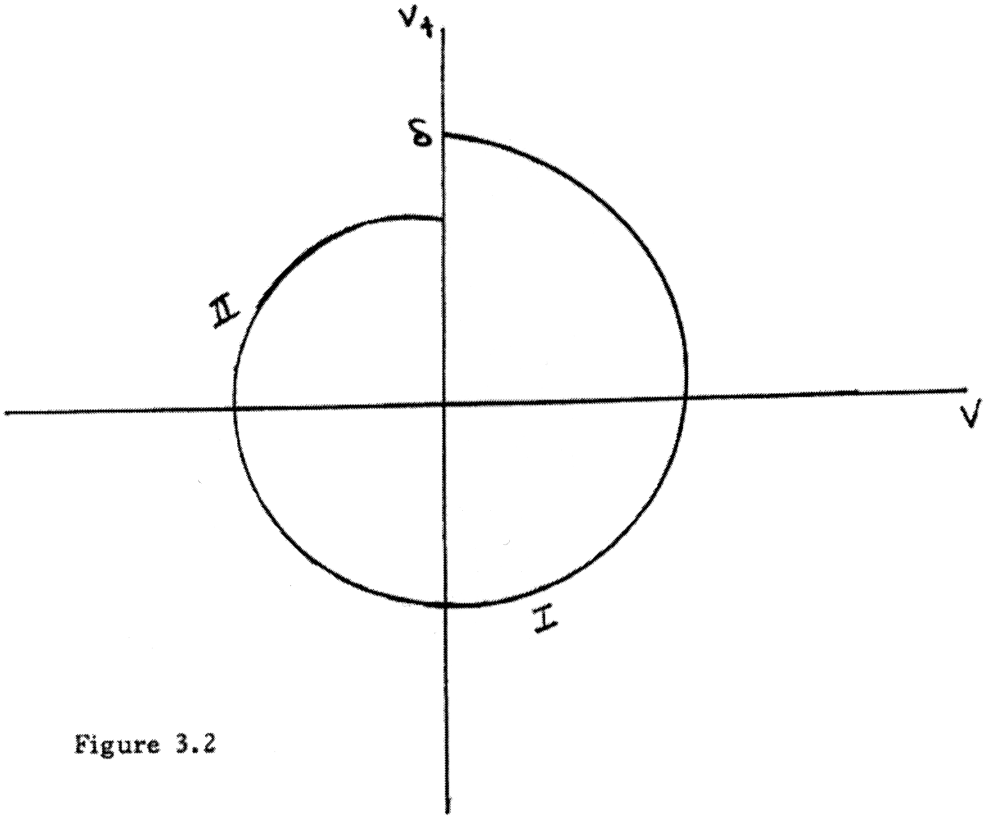


Figure 3.2

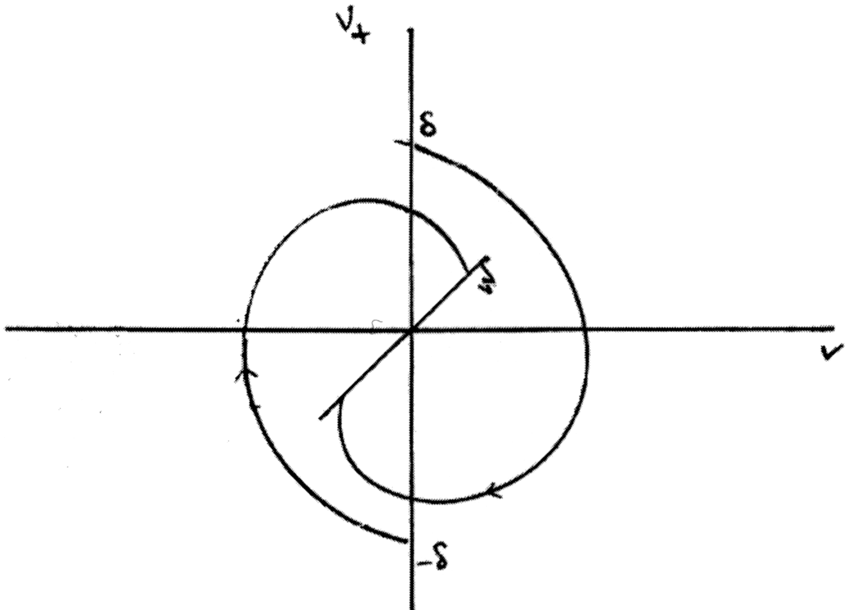


Figure 3.3 The case $1 < p < q + 1 < p^\wedge$.

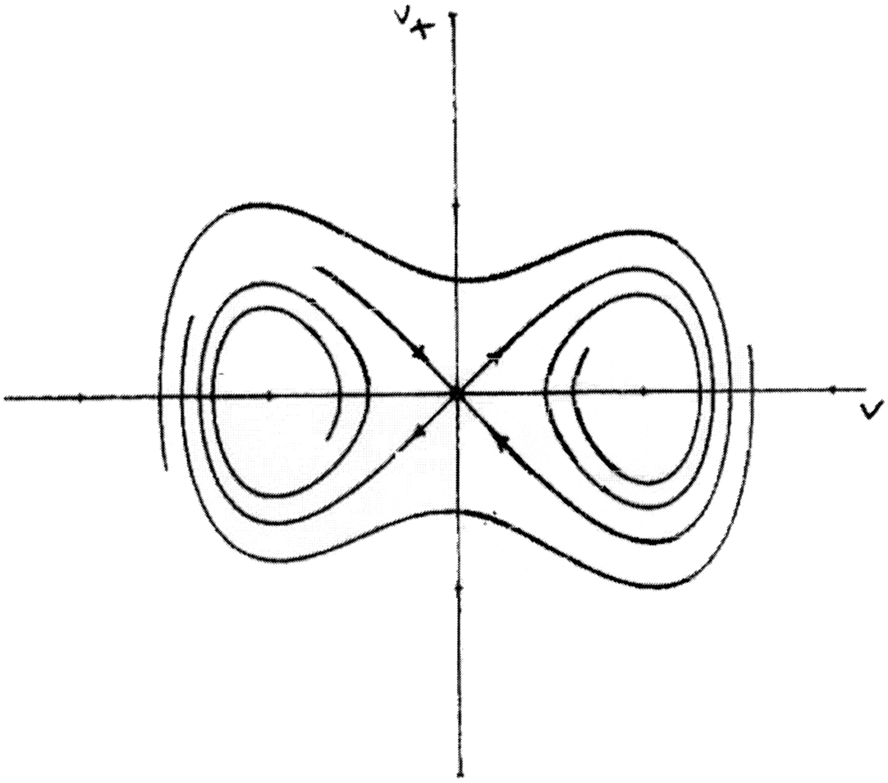


Figure 3.4 The case $p^\wedge < q +\quad < p^*$

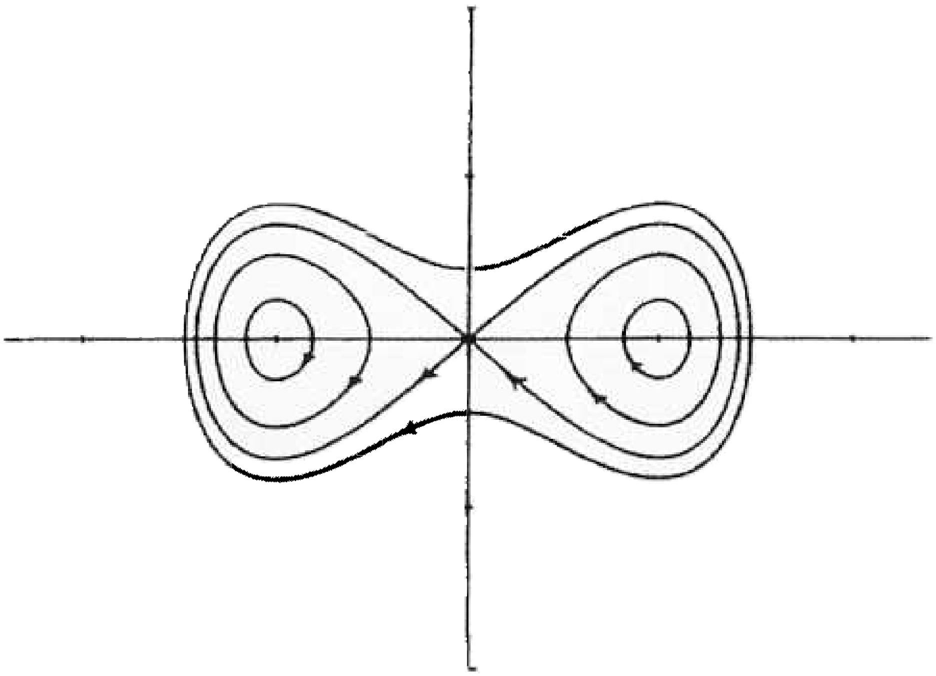


Figure 3.5 The case q+1 = p*.

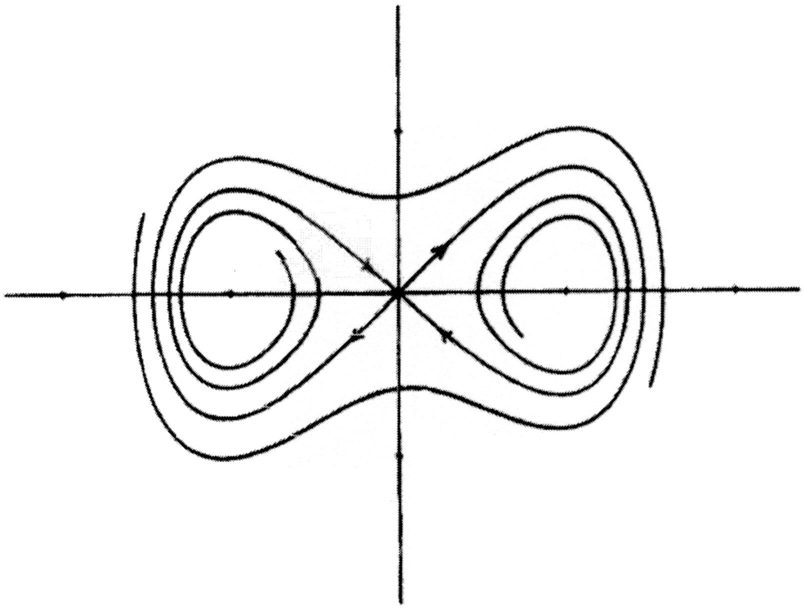


Figure 3.6 The case q + 1 > p*.

Chapter 4

Particular Cases of the Generalized Lane–Emden Equation

4.1 The Lane-Emden equation as a special case of the Generalized Lane-Emden equation

Literature has covered the Lane-Emden equation extensively

$$\Delta u + |u|^{q-1}\, u = 0 \qquad x \in \Omega \tag{4.1}$$

$$u|_{\partial\Omega} = 0, q > 1, n > 2$$

Where Ω is the unit ball in n dimensions. The Lane-Emden equation in radial form is

$$\frac{1}{r^{n-1}}(r^{n-1}u_r)_r + |u|^{q-1}\, u = 0 \tag{4.2}$$

$$u = u(r), u(1) = 0, 0 < r < 1.$$

Letting $r = e^{-t}, u_r = -e^t u_t, u_{rr} = e^{2t}(u_{tt} + u_t)$ and introducing the phase plane transformations $u = e^{\theta t}v$, $u_t = e^{\theta t}w = e^{\theta t}(v_t + \theta v)$, $u_{tt} = e^{\theta t}(\theta w + w_t)$, we obtain the autonomous system

$$w_t + \phi w + |v|^{q-1}\, v = 0 \tag{4.3}$$

$$v_t + \theta v - w = 0$$

Where $\theta = \frac{2}{q-1}$ and $\phi = 2-n+\theta$. For $q > 1$, θ is positive. Substituting $w = (v_t+\theta v)$ in (4.3) and letting $v_t = \gamma$ results in an equivalent autonomous system dependent on the variables v and v_t

$$\gamma_t + (\phi + \theta)\gamma + [\phi\theta + |v|^{q-1}]v = 0 \tag{4.4}$$

$$v_t = \gamma$$

When $\theta\phi > 0$ the origin is the only critical point of (4.4). When $\phi\theta < 0$, there exists three finite critical points, $(\pm(-\phi\theta)^{\frac{1}{q-1}}, 0)$ and $(0, 0)$.

The following lemma establishes the existence of phase plane solutions of the Lane-Emden equation for $2 < q + 1 < p^* = \frac{2n}{n-2}$ and $q > 1$.

Lemma 4.1.1. *Let u be a solution of the Lane-Emden equation in L^2, then v and w tend to zero as $t \to \infty$ for $2 < q+1 < p^* = \frac{2n}{n-2}$ and $q > 1$, where*

$|v|^{q+1}, |w|^2 \to o(e^{n-(\frac{2}{q-1})(q+1)})$ *as $t \to \infty$.*

The Jacobian matrix for the autonomous system in (4.4) is

$$\begin{pmatrix} -(\phi+\theta) & -\phi\theta - q\,|v|^{q-1} \\ 1 & 0 \end{pmatrix}$$

When evaluating the Jacobian matrix at the origin we obtain the eigenvalue $\lambda_1 = -\theta$, with corresponding eigenvector $[\theta \quad -1]$, and the eigenvalue $\lambda_1 = -\phi$, with corresponding eigenvector $[\phi \quad -1]$. It is clear that for $\theta\phi > 0$ the origin is a sink whereas for $\phi\theta < 0$ the origin is a saddle point.

Evaluating the Jacobian matrix at the point $((-\phi\theta)^{\frac{1}{q-1}}, 0)$ results in the complex eigenvalues

$$\lambda_{1,2} = \frac{-(\phi+\theta) \pm \sqrt{(\phi+\theta)^2 - 4(-(q-1)\phi\theta)}}{2}. \tag{4.5}$$

For $|\phi| < |\theta|$, the critical points are spirals with trajectories spiraling in and for $|\phi| > |\theta|$ trajectories spiral out of the critical points. Similar results are obtained for the point $(-(-\phi\theta)^{\frac{1}{q-1}}, 0)$.

The following four lemmas summarize the results obtained in phase plane for the existence of solutions of the Lane-Emden equation while relating their existence to two critical exponents also obtained in phase plane, $\hat{p} = \frac{2(n-1)}{n-2}$ and $p^* = \frac{2n}{n-2}$.

Lemma 4.1.2. *Solutions to the Lane-Emden equation (4.1), satisfying the associated boundary condition, in the phase plane approach $(0,0)$ as $t \to \infty$ along the weak attractor for $2 < q+1 < \hat{p}, \hat{p} = \frac{2(n-1)}{n-2}, \phi\theta > 0, \phi > 0,\ \phi < \theta, n > 2$.*

Lemma 4.1.3. *The autonomous system corresponding to the Lane-Emden equation does not have solutions in phase plane satisfying the associated boundary con-*

dition for $q+1 = p^*, p^* = \frac{2n}{n-2}$*,* $q > 1$*,* $\phi\theta < 0, \phi = -\theta$*. The autonomous system has a saddle origin and centers at* $(\pm(-\phi\theta)^{\frac{1}{q-1}}, 0)$ *for* $q+1 = p^*$*.*

Lemma 4.1.4. *The autonomous system in the phase plane corresponding to the Lane-Emden equation for* $\hat{p} < q+1 < p^*$*, where* $p^* = \frac{2n}{n-2}$ *and* $\hat{p} = \frac{2(n-1)}{n-2}$*,* $q > 1$*,* $\phi\theta < 0, \theta > 0, \phi < 0, \phi > -\theta$*, has a saddle origin and spiral points at* $(\pm(\phi\theta)^{1/q-1}, 0)$ *with trajectories spiraling in. Solutions satisfying the associated boundary condition start at* $(0, \delta)$ *and approach the origin along the weak attractor as* $t \to \infty$*.*

Lemma 4.1.5. *The autonomous system in the phase plane corresponding to the Lane-Emden equation (4.1) has a saddle origin and spirals at* $(\pm(\phi\theta)^{1/q-1}, 0)$ *where trajectories spiral out of the critical points for* $q+1 > p^*$*,* $p^* = \frac{2n}{n-2}$*,* $q > 1$*,* $\phi\theta < 0, \theta > 0, \phi < 0, \phi < -\theta$*. Solutions satisfying the associated boundary condition of the Lane-Emden equation (4.1) do not exist for* $q+1 > p^*$*.*

We conclude the discussion of the Lane-Emden equation by pointing to the fact that the number of critical points change from one point in the case $\phi\theta > 0$ to three critical points for $\phi\theta < 0, \phi < 0$. This change is associated with the appearance of the sub-critical exponent $\hat{p} = \frac{2(n-1)}{n-2}$. Interestingly, a corresponding change takes place at a related value in the weighted case, $\hat{p} = \frac{p(\beta+n-1)+\alpha-\beta}{n+\alpha-p}$. At this point we are not clear on whether this represents a general critical value for the trace embedding in the weighted case, but due to this strong connection with the Lane-Emden case we hypthosize that it may be related to a weighted trace embedding.

4.2 Applications to the Generalized Lane-Emden equation

In this section we consider two applications of the Generalized Lane-Emden equation from two different fields of study, Astronomy and Engineering.

The first application is given by the equation

$$\Delta u + |x|^{\beta} |u|^{q-1} u = 0 \tag{4.6}$$

$$u|_{\partial\Omega} = 0, q > 1, \beta > 0$$

The second application is given by the equation

$$-\nabla \cdot (|x|^{-ap} |\nabla u|^{p-2} \nabla u) = |x|^{-(a+1)p+c} |u|^{q-1} u \tag{4.7}$$

$$u|_{\partial\Omega} = 0, q > 1, 1 < p \leq 2, a < -1, c > 0$$

Where Ω is the unit ball in $\Re^n$.

4.2.1 The first application and phase plane

The radial form of (4.6) is given by

$$u_{rr} + \frac{n-1}{r} u_r + r^{\beta} |u|^{q-1} u = 0 \tag{4.8}$$

Using the the phase plane transformations we obtain the autonomous system

$$w_t + \phi w + |v|^{q-1} v = 0 \tag{4.9}$$

$$v_t + \theta v - w = 0$$

Where $\theta = \frac{2+\beta}{q-1}$ and $\phi = (2 - n + \theta)$. The system (4.9) can be further transformed in to an autonomous system dependent on v and v_t as follows

$$\gamma_t + (\phi + \theta)\gamma + (\phi\theta + |v|^{q-1})v = 0 \tag{4.10}$$

$$v_t = \gamma$$

For $\phi\theta > 0$, the origin is the only finite critical point of (4.10), and for $\phi\theta < 0$ there exists three finite critical points, $(0,0)$ and $(\pm(-\phi\theta)^{\frac{1}{q-1}}, 0)$.

Existence of phase plane solutions to (4.6) for $2 < q+1 < p^* = \frac{2(n+\beta)}{n-2}, \beta > 0$ and $\theta > 0$ is stated in the following lemma.

Lemma 4.2.1. *Let u be a solution of the partial differential equation (4.6) in L^2, then v and w tend to zero as $t \to \infty$ for $2 < q+1 < p^* = \frac{2(n+\beta)}{n-2}$, $\beta > 0$ and $\theta > 0$, where*

$|v|^{q+1}, |w|^2 \to o(e^{[n+\beta-\frac{2+\beta}{q-1}(q+1)]t})$ *as* $t \to \infty$

The Jacobian matrix of the autonomous system (4.10) is

$$\begin{pmatrix} -(\phi+\theta) & -\phi\theta - q\,|v|^{q-1} \\ 1 & 0 \end{pmatrix}$$

Linearization about the point $(0,0)$ gives the eigenvalues $\lambda_1 = -\theta$ and $\lambda_2 = -\phi$ with the corresponding eigenvectors $[\theta \quad -1]$ and $[\phi \quad -1]$ respectively.

For $\phi, \theta > 0$, the origin is a stable sink. Solutions to the PDE satisfying the associated asymptotic condition, $e^{\theta t}w \to 0$, approach $(0,0)$ along the weak attractor $e^{-\theta t}[\theta \quad -1]$ as $t \to \infty$. When $\phi\theta < 0$, $(\phi < 0)$, the origin is an unstable saddle with trajectories approaching the origin along the weak attractor.

Linearization about the critical point $((-\phi\theta)^{\frac{1}{q-1}}, 0)$ gives complex eigenvalues

$$\lambda_{1,2} = \frac{-(\phi+\theta) \pm \sqrt{(\phi+\theta)^2 - 4(-\phi\theta(q-1))}}{2} \tag{4.11}$$

When $|\phi| < |\theta|$ the critical points are spirals with trajectories spiraling in and for $|\phi| > |\theta|$ trajectories spiral out. For $\phi = -\theta$ the critical points $(\pm(-\phi\theta)^{\frac{1}{q-1}}, 0)$ are centers.

Phase plane results for the existence of solutions of (4.6) and the role of the phase plane critical exponents, $\hat{p} = \frac{2(n-1)+\beta}{n-2}$ and $p^* = \frac{2(\beta+n)}{n-2}$, are summarized in the following lemmas.

Lemma 4.2.2. *Solutions to the PDE in (4.6) satisfy the associated boundary condition for* $2 < q+1 < \hat{p}, \hat{p} = \frac{2(n-1)+\beta}{n-2}$, $\beta > 0$, $\phi\theta > 0$, *and* $\phi > 0$.

Lemma 4.2.3. *Solutions to the PDE in (4.6) satisfy the associated boundary condition for* $\hat{p} < q+1 < p^*, p^* = \frac{2(\beta+n)}{n-2}$, $\beta > 0$, $\phi\theta < 0$, $\phi < 0$, *and* $\phi > -\theta$.

Lemma 4.2.4. *The autonomous system in the phase plane corresponding to the partial differential equation (4.6) has no solutions satisfying the associated asymptotic condition,* $e^{\theta t}w \to 0$, *for* $\beta > 0$, $\phi\theta < 0, \phi < 0, \phi < -\theta$, $q+1 > p^*, p^* = \frac{2(n+\beta)}{n-2}$.

Lemma 4.2.5. *The autonomous system in the phase plane corresponding to the partial differential equation (4.6) has no solutions satisfying the associated asymptotic condition,* $e^{\theta t}w \to 0$, *for* $\beta > 0$, $\phi\theta < 0, \phi < 0, \phi = -\theta$, $q+1 = p^*, p^* = \frac{2(n+\beta)}{n-2}$.

4.2.2 The second application and phase plane

The partial differential equation in (4.7) is not singular for $q > 1, 1 < p \leq 2, a < -1$ and $c > 0$. The radial form of (4.7) is

$$(p-1)r^{-ap+n-1}\left|u_r\right|^{p-2}u_{rr}+(-ap+n-1)r^{-ap+n-2}\left|u_r\right|^{p-2}u_r+r^{-(a+1)p+c+n-1}\left|u\right|^{q-1}u=0 \tag{4.12}$$

The corresponding autonomous system is therefore

$$(p-1)\left|w\right|^{p-2}w_t+\phi\left|w\right|^{p-2}w+\left|v\right|^{q-1}v=0 \tag{4.13}$$

$$v_t+\theta v-w=0$$

Where $\theta = \frac{c}{q+1-p}$ and $\phi = (a+1)p - n + \theta(p-1)$.

Substituting $(\theta v + v_t)$ for w and γ for v_t in (4.13) we obtain a simpler equivalent autonomous system dependent on v and v_t as follows

$$(p-1)\left|\gamma+\theta v\right|^{p-2}\gamma_t+[\phi+\theta(p-1)]\left|\gamma+\theta v\right|^{p-2}\gamma+[\phi\theta\left|\gamma+\theta v\right|^{p-2}+\left|v\right|^{q-1}]v=0 \tag{4.14}$$

$$v_t=\gamma$$

Observe that the autonomous system in (4.14) is the same autonomous system of the Generalized Lane-Emden equation. For $\phi\theta > 0$, the origin is a sink with solutions approaching the origin along the weak attractor. When $\phi\theta < 0$, the origin is an unstable saddle with trajectories approaching along the weak attractor and $(\pm(-\phi\theta\left|\theta\right|^{p-2})^{\frac{1}{q+1-p}}, 0)$ are spirals with trajectories spiraling in for $\phi > -\theta$ and spiraling out for $\phi < -\theta$. When $\phi = -\theta$, the critical points $(\pm(-\phi\theta\left|\theta\right|^{p-2})^{\frac{1}{q+1-p}}, 0)$ are centers.

Lemma 4.2.6. *Let u be a solution of the partial differential equation (4.7) in L^p_{-ap}, then v and w tend to zero as $t \to \infty$ for $p < q+1 < p^*, p^* = \frac{p(-(a+1)p+c+n)}{n-(a+1)p}, 1 < p \leq 2$, $a < -1, \theta > 0$, where*

$|v|^{q+1}, |w|^p \to o(e^{[n-(a+1)p+c-\frac{c}{q+1-p}(q+1)]t})$ *as* $t \to \infty$

We summarize the phase plane results for the existence and non-existence of solutions for (4.7) in relation to the two critical exponents, $\hat{p} = \frac{p(-(a+1)p+c+n)-c}{n-(a+1)p}$ and $p^* = \frac{p(-(a+1)p+c+n)}{n-(a+1)p}$, in the following lemmas.

Lemma 4.2.7. *Solutions to the partial differential equation in (4.7) satisfy the associated boundary condition for $p < q+1 < \hat{p}, \hat{p} = \frac{p(-(a+1)p+c+n)-c}{n-(a+1)p}, 1 < p \leq 2$, $a < -1, c > 0, \phi\theta > 0, \phi > 0, \phi < \theta$.*

Lemma 4.2.8. *Solutions to the partial differential equation in (4.7) satisfy the associated boundary condition for $\hat{p} < q+1 < p^*, p^* = \frac{p(-(a+1)p+c+n)}{n-(a+1)p}, 1 < p \leq 2$, $a < -1, c > 0, \phi\theta < 0, \phi < 0, \phi > -\theta$.*

Lemma 4.2.9. *The autonomous system (4.14) corresponding to the partial differential equation (4.7) has no solutions satisfying the associated asymptotic condition, $e^{\theta t}w \to 0$, for $q+1 > p^*, p^* = \frac{p(-(a+1)p+c+n)}{n-(a+1)p}$, $1 < p \leq 2$, $a < -1, c > 0, \phi\theta < 0$, $\phi < 0, \phi < -\theta$.*

Lemma 4.2.10. *The autonomous system (4.14) corresponding to the partial differential equation (4.7) has no solutions satisfying the associated asymptotic condition, $e^{\theta t}w \to 0$, for $q+1 = p^*, p^* = \frac{p(-(a+1)p+c+n)}{n-(a+1)p}$, $1 < p \leq 2$, $a < -1, c > 0$, $\phi\theta < 0, \phi < 0, \phi = -\theta$.*

Appendix

Appendix A1

Weighted Sobolev Compact Embedding Theorem(Caffarelli-Kohn-Nirenberg)

Let $\Omega \subseteq \Re^n$, be an open bounded domain with C^1 boundary and $0 \in \ \Omega$, $1 < p < n$, $-\infty < \frac{-\alpha}{p} < \frac{n-p}{p}$,$1 \leq q < \frac{np}{n-p}$, $-\beta < (1-\frac{\alpha}{p})q+n(\frac{1-q}{p})$, $p < q+1 \leq \ p^*$, $p^* = \frac{p(n-bq)}{-ap+n-p}$. Then $\|u\|_{L^{q+1}_{(B^n,|x|^\beta)}} \leq C\,\|\nabla u\|_{L^p_{(B^n,|x|^\alpha)}}$ and the embedding

$W^{1,p}_{(B^n,|x|^\alpha)} \hookrightarrow L^{q+1}_{(B^n,|x|^\beta)}$ is continuous . If the upper bound for $q+1$ is strict then the embedding is compact, [42].

Proof:

First we prove that for $\Omega \subseteq \Re^n$, an open bounded domain with C^1 boundary and $0 \in \Omega$, $1 < p < n$, $-\infty < \frac{-\alpha}{p} < \frac{n-p}{p}$, the embedding $W^{1,p}_{(B^n,|x|^\alpha)} \hookrightarrow L^q_{(B^n,|x|^\beta)}$ is compact if $1 \leq q < \frac{np}{n-p}$, $-\beta < (1-\frac{\alpha}{p})q+n(\frac{1-q}{p})$. The continuity of the embedding is a direct consequence of the Caffarelli-Kohn-Nirenberg inequality which states that for $1 < p < n$ and for all $u \in C_0^\infty(\Re^n)$, there is a constant $C_{a,b} > 0$ such that

$(\int_{\Re^n} |x|^{-bq}\,|u|^q\,dx)^{p/q} \leq C_{a,b}\int_{\Re^n} |x|^{-ap}\,|Du|^p\,dx$

where $-\infty < a < \frac{n-p}{p}$, $a \leq b \leq a+1$, $q = p^* = \frac{np}{n-dp}$, $d = 1+a-b$.

Where for our problem we take $a = -\frac{\alpha}{q}$ and $b = -\frac{\beta}{q}$.

To prove the compactness, let $\{u_m\}$ be abounded sequence in $W^{1,p}_\alpha$ for any $\rho > 0$ with $B_\rho(0) \subset \Omega$ is a ball centered at the origin with radius ρ there holds $\{u_m\} \subset W^{1,p}/B_\rho(0)$. Then the classical Rellich-Kondrachov compactness theorem guarantees the existence of a convergent subsequence of $\{u_m\}$ in $L^q_{(\Omega,B_\rho(0))}$. By taking a diagonal sequence, we can assume without loss of generality that $\{u_m\}$ converges in $L^q_{(\Omega,B_\rho(0))}$ for any $\rho > 0$.

On the other hand, for any $1 \leq q < \frac{np}{n-p}$ there exists $a, b \in (a, a+1]$ such that $q < p^* = \frac{np}{n-p}$, $d = 1+a-b \in [0,1)$. From the Caffarelli-Kohn-Nirenberg inequality

above, $\{u_m\}$ is also bounded in $L^t_{(\Omega,|x|^{-bt})}$. By the holder inequality, for any $\delta > 0$, there holds

$\int_{|x|<\delta} |x|^\beta |u_m - u_j|^q \, dx \le (|x|^{(\beta+qb)t/t-q} \, dx)^{1-(q/t)}$

. $\quad \le (\int_\Omega |x|^{-bq} |u_m - u_j|^q \, dx)^{q/t}$

. $\quad \le C(\int_0^\delta r^{\frac{n-1-(-\beta-bq)t}{t-q}} dr$

. $\quad = C\delta^{\frac{n-1+(\beta+bq)t}{t-q}}$

Where $C > 0$ is a constant independent of m. Since $-\beta < (1+a)q + n(\frac{1-q}{p})$, there holds $\frac{n+(\beta+bt)t}{t-q} > 0$. Therefore, for any given $\epsilon > 0$, we fix $\delta > 0$ such that

$\int_{|x|<\delta} |x|^\beta |u_m - uj|^q \, dx \le \epsilon/2, \forall m, j \in N$

Then we choose $s \in N$ such that

$\int_{\Omega \setminus B_{\delta(0)}} |x|^\beta |u_m - u_j|^q \, dx \le C_{-\beta} \int_{\Omega \setminus B_{\delta(0)}} |u_m - u_j|^q \, dx \le \epsilon/2, \forall m, j \in N$

Where $C_{-\beta} = \delta^\beta$ if $-\beta \ge 0$ and $C_{-\beta} = (diam\Omega)^\beta$ if $\beta > 0$. Thus

$\int_\Omega |x|^\beta |u_m - u_j|^q \, dx \le \epsilon, \forall m, j \in N.$

That is $\{u_m\}$ is a Cauchy sequence in $L^t_{(\Omega,|x|^{-bt})}$, [42].

Appendix A2:

The weighted Sobolev space $W^{1,p}_{(\Omega,w)}$ is a Banach space.

Proof:

1- We first prove that $W^{1,p}_{(\Omega,w)}$ is a norm. Recall that the norm of a function $u \in W^{1,p}_{(\Omega,w)}$ is defined as

$\|u\|_{W^{1,p}_{(\Omega,w)}} = (\sum_{|\alpha|\leq 1} \int_\Omega w\,|D^\alpha u|^p\,dx)^{1/p}$ for $1 \leq p < \infty$.

Which in expanded form is written as

$\|u\|_{W^{1,p}(\Omega,w)} = (\int_\Omega w\,|u|^p\,dx + \int_\Omega w\,|Du|^p\,dx)^{1/p}$ for $1 \leq p < \infty$.

Therefore

a) $\|\lambda u\|_{W^{1,p}(\Omega,w)} = (\|\lambda u\|^p_{L^p(\Omega,w)} + \|\lambda Du\|^p_{L^p(\Omega,w)})^{1/p}$

$= (\int_\Omega w\,|\lambda u|^p\,dx + \int_\Omega w\,|\lambda Du|^p\,dx)^{1/p}$

$= (|\lambda|^p \int_\Omega w\,|u|^p\,dx + |\lambda|^p \int_\Omega w\,|Du|^p\,dx)^{1/p}$

$= |\lambda|\,(\int_\Omega |u|^p\,wdx + \int_\Omega |Du|^p\,wdx)^{1/p}$

Hence

$\|\lambda u\|_{W^{1,p}_{(\Omega,w)}} = |\lambda|\,\|u\|_{W^{1,p}_{(\Omega,w)}}$, for $1 \leq p < \infty$.

b) $\|u\|_{W^{1,p}_{(\Omega,w)}} = 0$ if and only if $u = 0$ is an obvious result.

c) Assume $u, v \in W^{1,p}_{(\Omega,w)}$. Then for $1 \leq p < \infty$, Minkowski's inequality implies

$\|u+v\|_{W^{1,p}_{(\Omega,w)}} = (\sum_{|\alpha|\leq 1} \|D^\alpha u + D^\alpha v\|^p_{L^p_{(\Omega,w)}})^{1/p}$

$\leq (\sum_{|\alpha|\leq 1} \|D^\alpha u\|^p_{L^p_{(\Omega,w)}} + \|D^\alpha v\|^p_{L^p_{(\Omega,w)}})^{1/p}$

$\leq (\sum_{|\alpha|\leq 1} \|D^\alpha u\|^p_{L^p_{(\Omega,w)}})^{1/p} + (\sum_{|\alpha|\leq 1} \|D^\alpha v\|^p_{L^p_{(\Omega,w)}})^{1/p}$

Which in turn implies that

$\|u+v\|_{W^{1,p}_{(\Omega,w)}} \leq \|u\|_{W^{1,p}_{(\Omega,w)}} + \|v\|_{W^{1,p}_{(\Omega,w)}}$.

2- It remains to show that $W^{1,p}_{(\Omega,w)}$ is complete. Assume that $\{u_m\}_{m=1}^\infty$ is a cauchy sequence in $W^{1,p}_{(\Omega,w)}$. Then for each $|\alpha| \leq 1$, $\{D^\alpha u_m\}_{m=1}^\infty$ is a cauchy sequence in $L^p_{(\Omega,w)}$ since $L^p_{(\Omega,w)}$ is complete there exists functions $u_\alpha \in L^p_{(\Omega,w)}$ such that

$D^\alpha u_m \to u_\alpha \in L^p_{(\Omega,w)}$ where $|\alpha| \leq 1$. In particular $u_m \to u_{(0,0,\ldots,0)} =: u \in L^p_{(\Omega,w)}$. We now claim that $u \in W^{1,p}_{(\Omega,w)}$, $D^\alpha u = u_\alpha$. To verify, fix $\phi \in C_c^\infty(\Omega)$ then

$\int_\Omega u D^\alpha \phi dx = lim_{m\to\infty} \int_\Omega u_m D^\alpha \phi dx$

$= lim_{m\to\infty} (-1)^{|\alpha|} \int_\Omega D^\alpha u_m \phi dx$

$= (-1)^{|\alpha|} \int_\Omega u_\alpha \phi dx$

Since $D^\alpha u_m \to D^\alpha u$ in $L^p_{(\Omega,w)}$ for all $|\alpha| \leq 1$, we see that $u_m \to u \in W^{1,p}_{(\Omega,w)}$ as required.

Appendix A3

Consider the equation, [43].

$\nabla \cdot (|x|^{\alpha} |\nabla u|^{p-2} \nabla u) = -|x|^{\beta} |u|^{q-1} u$

On the domain $\Omega_\delta = \Omega \backslash \{x \in \Re^n : |x| \leq \delta\}$.

Multiply by $x \cdot \nabla u$ and integrate both sides of the equation we have

$\int_{\Omega_\delta} \nabla \cdot (|x|^{\alpha} |\nabla u|^{p-2} \nabla u)(x \cdot \nabla u) dx = -\int_{\Omega_\delta} |x|^{\beta} |u|^{q-1} u(x \cdot \nabla u) dx \qquad$ (A.1)

Integrating by parts the left hand side gives the result

LHS $= \int_{\partial\Omega_\delta} |x|^{\alpha} |\nabla u|^{p-2} (\nabla u \cdot \nu)(x \cdot \nabla u) dS - \int_{\Omega_\delta} |x|^{\alpha} |\nabla u|^{p-2} \nabla u \cdot \nabla(x \cdot \nabla u) dx$

(A.2)

Where ν is a unit outer normal vector. Consider part I of (A.2)

$\int_{\partial\Omega_\delta} |x|^{\alpha} |\nabla u|^{p-2} (\nabla u \cdot \nu)(x \cdot \nabla u) dS =$

$\int_{\partial\Omega_\delta} |x|^{\alpha} |\nabla u|^{p} (x \cdot \nu) dS + \int_{|x|=\delta} \delta^{\alpha} |\nabla u|^{p-2} (\nabla u \cdot \nu)(x \cdot \nabla u) dS$

Part II of (A.2) is simplified as follows

$\int_{\Omega_\delta} |x|^{\alpha} |\nabla u|^{p-2} \nabla u \cdot \nabla(x \cdot \nabla u) dx$

$= \int_{\Omega_\delta} |x|^{\alpha} |\nabla u|^{p-2} (|\nabla u|^2 + (x \cdot \nabla) |\nabla u|^2) dx$

$= \int_{\Omega_\delta} |x|^{\alpha} |\nabla u|^{p} dx + \int_{\Omega_\delta} |x|^{\alpha} (x \cdot \nabla) \frac{1}{p} |\nabla u|^{p} dx$

$= \int_{\Omega_\delta} |x|^{\alpha} |\nabla u|^{p} dx + \int_{\partial\Omega_\delta} |x|^{\alpha} (x \cdot \nu) \frac{1}{p} |\nabla u|^{p} dS - (\alpha + n) \int_{\Omega_\delta} |x|^{\alpha} \frac{1}{p} |\nabla u|^{p} dx$

$= (1 - \frac{\alpha+n}{p}) \int_{\Omega_\delta} |x|^{\alpha} |\nabla u|^{p} dx + \frac{1}{p} \int_{\partial\Omega_\delta} |x|^{\alpha} (x \cdot \nu) |\nabla u|^{p} dS$

Then the LHS = I - II gives

$(1 - \frac{1}{p}) \int_{\partial\Omega_\delta} |x|^{\alpha} |\nabla u|^{p} (x \cdot \nu) dS + \int_{|x|=\delta} \delta^{\alpha} |\nabla u|^{p-2} (\nabla u \cdot \nu)(x \cdot \nabla u) dS$

$-\frac{1}{p} \int_{|x|=\delta} \delta^{\alpha} (x \cdot \nu) |\nabla u|^{p} dS - (1 - \frac{\alpha+n}{p}) \int_{\Omega_\delta} |x|^{\alpha} |\nabla u|^{p} dx \qquad$ (A.3)

Now consider the right hand side of (A.1)

R.H.S $= -\int_{\Omega_\delta} |x|^{\beta} |u|^{q-1} u(x \cdot \nabla u) dx$

$= -\int_{\Omega_\delta} |x|^{\beta} (x \cdot \nabla) \frac{|u|^{q+1}}{q+1} dx$

$= -\int_{\partial\Omega_\delta} |x|^{\beta} (x \cdot \nu) \frac{|u|^{q+1}}{q+1} dS + \int_{\Omega_\delta} \nabla \cdot (|x|^{\beta} x) \frac{|u|^{q+1}}{q+1} dx$

$= -\int_{\partial\Omega_\delta} |x|^\beta (x\cdot\nu)\frac{|u|^{q+1}}{q+1}dS + (n+\beta)\int_{\Omega_\delta} |x|^\beta \frac{|u|^{q+1}}{q+1}dx - \int_{|x|=\delta} \delta^\beta (x\cdot\nu)\frac{|u|^{q+1}}{q+1}dS$

(A.4)

On $|x| = \delta$, $x = -\delta\nu$, then $x\cdot\nu = -\delta$ and therefore $\delta^\alpha(x\cdot\nu) = -\delta^{\alpha+1}$ and $\delta^\beta(x\cdot\nu) = -\delta^{\beta+1}$, hence equations (A.1), (A.3) and (A.4) give

$(1-\frac{1}{p})\int_{\partial\Omega_\delta} |x|^\alpha |\nabla u|^p (x\cdot\nu)dS + \int_{|x|=\delta} \delta^\alpha |\nabla u|^{p-2} (\nabla u\cdot\nu)(x\cdot\nabla u)dS + \frac{1}{p}\int_{|x|=\delta} \delta^{\alpha+1} |\nabla u|^p dS -$

$(1-\frac{\alpha+n}{p})\int_{\Omega_\delta} |x|^\alpha |\nabla u|^p dx = -\frac{1}{q+1}\int_{\partial\Omega_\delta} |x|^\beta (x\cdot\nu) |u|^{q+1} dS + \frac{n+\beta}{q+1}\int_{\Omega_\delta} |x|^\beta |u|^{q+1} dx$

$+\frac{1}{q+1}\int_{|x|=\delta} \delta^{\beta+1} |u|^{q+1} dS$ (A.5)

Next we need to get rid of the boundary integrals along $|x| = \delta$ in (A.5). In fact let u be a solution of (1.1), from Caffarelli-Kohn-Nirenberg inequality

$(\int_{\Re^n} |x|^{-bq} |u|^q dx)^{\frac{p}{q}} \le C_{a,b}\int_{\Re^n} |x|^{-ap} |Du|^p dx$

and the compact embedding theorem, we know that

$\int_\Omega |x|^\alpha |\nabla u|^p dx$ and $\int_\Omega |x|^\beta |u|^{q+1} dx$

are finite, therefore by the mean value theorem there exists a sequence $\{\delta_m\}$, $\delta_m \to 0$ such that integrals

$\int_{|x|=\delta} |x|^\alpha |\nabla u|^p (x\cdot\nu)dS$ and $\int_{|x|=\delta} |x|^\beta |u|^{q+1} (x\cdot\nu)dS$

go to zero as $m \to \infty$.

Letting $m \to \infty$ in (A.5) we obtain

$\frac{p-1}{p}\int_{\partial\Omega} |x|^\alpha |\nabla u|^p (x\cdot v)dS + \frac{n+\alpha-p}{p}\int_\Omega |x|^\alpha |\nabla u|^p dx =$

$\frac{n+\beta}{q+1}\int_\Omega |x|^\beta |u|^{q+1} dx.$

Appendix A4

The Jacobian of (3.33) evaluated at zero results in the matrix on page 52; a proof using decay rates for the term $\frac{\frac{2-p}{p-1}v|v|^{q-1}(\gamma+\theta v)}{|\gamma+\theta v|^p}$

Using the phase plane transformations $w = e^{-\theta t}u_t$ where $w = \gamma + \theta v$ we have

$$\frac{2-p}{p-1} v\,|v|^{q-1}\,(\gamma + \theta v) = C_1 e^{[-\theta(q+1)-1]t} \tag{4.1}$$

And

$$|\gamma + \theta v|^p = C_2 e^{-(\theta+1)pt} \tag{4.2}$$

Hence

$$\frac{\frac{2-p}{p-1} v\,|v|^{q-1}\,(\gamma + \theta v)}{|\gamma + \theta v|^p} = C e^{[-\theta(q+1-p)+(p-1)]t} \tag{4.3}$$

The exponent Since $[-\theta(q+1-p)+(p-1)]$ simplifies to $-(\beta-\alpha+1)$, where $\theta = \frac{p+\beta-\alpha}{q+1-p}$. Hence the exponent is negative and the term approaches zero as t approaches infinity for $\theta > 0$ and $\beta - \alpha + 1 > 0$.

Appendix A5: A Three Dimensional Example with Variable Polytropic Equation of State.

The equations of motion describing a gaseous star are the equations of continuity, Euler's equation and Poisson's equation. For $n = 3$ we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \tag{4.4}$$

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\frac{1}{\rho} \nabla P - \nabla \Phi \tag{4.5}$$

$$\Delta \Phi = 4\pi G \rho \tag{4.6}$$

The condition of hydrostatic equilibrium for a spherically symmetric distribution of matter therefore becomes

$$\frac{1}{\rho}\frac{dP}{dr} = -\frac{d\Phi}{dr} \tag{4.7}$$

$$\frac{1}{r^2}\frac{d}{dr}(r^2 \frac{d\Phi}{dr}) = 4\pi G \rho \tag{4.8}$$

Where polytropic stars are characterized by an equation of state relating pressure with density,

$$P = K\rho^{\gamma}, \quad \gamma \geq 1 \tag{4.9}$$

To allow spatial variation in K, set

$$K = kr^{\delta}, \quad k > 0, \quad \delta \in (-\infty, \infty) \tag{4.10}$$

in equation (9), giving

$$\frac{k}{r^2}\frac{d}{dr}(\frac{r^2}{\rho}\frac{d}{dr}(r^{\delta}\rho^{\gamma})) + 4\pi G \rho = 0 \tag{4.11}$$

For $\gamma \neq 1$, Let $\rho = v^{\frac{1}{\gamma-1}}$ and set $\frac{4\pi G}{k} = 1$ to give

$$\frac{1}{r^2}\frac{d}{dr}(\frac{r^2}{v^{\frac{1}{\gamma-1}}}\frac{d}{dr}(r^{\delta} v^{\frac{\gamma}{\gamma-1}})) + v^{\frac{1}{\gamma-1}} = 0 \tag{4.12}$$

$$\frac{1}{r^2}\frac{d}{dr}(r^{2+\frac{\delta}{\gamma}}(\delta r^{\frac{\delta(\gamma-1)}{\gamma}-1} v + \frac{\gamma}{\gamma-1} r^{\frac{\delta(\gamma-1)}{\gamma}} \frac{dv}{dr})) + v^{\frac{1}{\gamma-1}} = 0 \tag{4.13}$$

$$\frac{1}{r^2}\frac{d}{dr}(r^{2+\frac{\delta}{\gamma}}\frac{d}{dr}(r^{\frac{\delta(\gamma-1)}{\gamma}}v)) + \frac{\gamma-1}{\gamma}v^{\frac{1}{\gamma-1}} = 0 \tag{4.14}$$

Set $u = r^{\frac{\delta(\gamma-1)}{\gamma}}v$, then

$$\frac{1}{r^2}\frac{d}{dr}(r^2 \cdot (r^{\frac{\delta}{\gamma}}u)) + \frac{\gamma-1}{\gamma}v^{\frac{-\delta}{\gamma}}u^{\frac{1}{\gamma-1}} = 0 \tag{4.15}$$

Such that $\alpha = \frac{\delta}{\gamma}, p = 2, \beta = -\frac{\delta}{\gamma}, q = \frac{1}{\gamma-1}$.

List of Symbols

R^n = n-dimensional real space.

B^n = a ball of n dimension.

∇ = gradient.

$\nabla\cdot$ = divergence.

ν = unit vector.

Ω = domain.

$\partial\Omega$ = boundary of a domain.

C^∞ = The set of functions such that $u : \Omega \to R$ is infinitely differentiable.

Given a mutiindex α, then $D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} .... \partial x_n^{\alpha_n}}$. If k is a nonnegative integer, then $D^k u(x) = \{D^\alpha u(x) : |\alpha| = k\}$ is the set of all partial derivative of order k.

Special cases:

$k = 2$, then $D^2 u =$

$$\begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2} & .... & \frac{\partial^2 u}{\partial x_1 \partial x_n} \\ .... & .... & .... \\ \frac{\partial^2 u}{\partial x_n \partial x_1} & .... & \frac{\partial^2 u}{\partial x_n^2} \end{pmatrix}$$

$k = 1$, then $Du = (u_{x_1}, ..., u_{x_n})$ is a gradient vector of u.

Definitions:

Newtonian fluid: A fluid with a linear relationship between shear stress and deformation (rate of change in velocity). Viscosity for a Newtonian fluid is a constant.

Non- Newtonian fluid: Is a fluid in which the relationship between shear stress and deformation is not linear. The viscosity of a Non- Newtonian fluid is a function of some mechanical variable like shear stress or time.

Pseudoplastics: Fluids that show a variable change in velocity with changing shear stress: their velocity decreases as the shear stress changes.

Dilatant: (also termed shear thickening) is a fluid in which viscosity increases with the rate of shear.

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